The Annals of Probability
1976, Vol. 4, No. 4, 680–683

THE STUDENT t-DISTRIBUTION FOR ODD DEGREES OF FREEDOM IS INFINITELY DIVISIBLE

BY EMIL GROSSWALD

Temple University

Let P_n be the nth Bessel polynomial. Kelker (1971) showed that the Student t-distribution of $k = 2n + 1$ degrees of freedom is infinitely divisible if and only if $\varphi_n(x) = P_{n-1}(x^{\frac{1}{2}})/P_n(x^{\frac{1}{2}})$ is completely monotonic. Kelker and Ismail proved that φ_n is indeed completely monotonic for some small values of n and conjectured that this is always the case. This conjecture is proved here by a twofold application of Bernstein's theorem and the use of some special properties of the zeros of the Bessel polynomials. The same conclusion follows for $Y_k = (\chi_k^2)^{-1}$, where χ_k^2 is a chi-square variable with k degrees of freedom.

1. Introduction. Let y_n be the nth Bessel polynomial (denoted by BP in what follows) in the normalization of Krall and Frink [9] (see also [5]) and set $P_n(x) = x^n y_n(1/x)$; this is the normalization adopted by Burchnall and Chaundy [3], Burchnall [2], Kelker [8] and Ismail and Kelker [7], and it will be used in the present paper. After P. Lévy [10] and then Gnedenko and Kolmogorov [4] had obtained necessary and sufficient conditions for the infinite divisibility of distributions, it was shown by Kelker [8] and Ismail and Kelker [7] that, for $k = 2n + 1$, both the Student t-distribution of k degrees of freedom and $Y_k = (\chi_k^2)^{-1}$ (χ_k^2 = a chi-square variable with k degrees of freedom) are infinitely divisible, provided that the function

$$\varphi_n(x) = P_{n-1}(x^{\frac{1}{2}})/P_n(x^{\frac{1}{2}}) \tag{1}$$

is completely monotonic on $(0, \infty)$.

The complete monotonicity of φ_n has been proved for small values of n([6], [7] and [8]) and has been conjectured to hold for all n (see [6] and [8]). The main purpose of this paper is to prove this conjecture.

I am grateful to Professor S. Kotz, who brought this problem to my attention and with whom I had many stimulating conversations.

2. Notations. The nth BP is denoted by P_n and its zeros by $\alpha_1, \alpha_2, \cdots, \alpha_n$. The Laplace transform is denoted by $\mathscr{L}$, so that $\varphi_n = \mathscr{L}(G_n)$ stands for $\varphi_n(x) = \int_0^\infty G_n(t)e^{-tx}\,dt$ and the inverse Laplace transform by $\mathscr{L}^{-1}$, so that we write $G_n = \mathscr{L}^{-1}(\varphi_n)$. The integer n is considered fixed; therefore, to simplify notations, the dependence on n will generally not be indicated and we shall write simply φ and G for φ_n and G_n, etc.

Received September 30, 1975.

AMS 1970 *subject classifications.* Primary 62E10; Secondary 26A48, 33A45, 44A10.

Key words and phrases. Student t-distribution, infinite divisibility, complete monotonicity, Laplace transform, Bernstein's theorem.

Integrals of the form $\int_{-\alpha t^{\frac{1}{2}}}^{\infty} u^{-2k} e^{-u^2}\, du$ will occur where $\operatorname{Re} \alpha < 0$. The exact path of integration is not important, but for definiteness we shall consider the integral taken along a line parallel to the real axis, starting from the point $-\alpha t^{\frac{1}{2}}$ (recall: $\operatorname{Re}(-\alpha t^{\frac{1}{2}}) > 0$) and going towards the right.

3. Main results. With the above notations the main results of the present paper can be formulated as follows:

THEOREM 1. *For every integer $n \geqq 1$, the function φ defined by* (1) *is completely monotonic.*

COROLLARY 1. *The t-distributions with odd degrees of freedom are infinitely divisible; the same holds for Y_k, with k odd.*

In order to prove Theorem 1 it is sufficient to show that the condition of Bernstein's theorem (see [11], pages 160–161) is satisfied, namely that $\varphi(x) = \mathscr{L}(G)$, with $G(t) \geqq 0$ on $0 < t < \infty$. Instead of this simple inequality, we shall prove a stronger statement, namely:

THEOREM 2. *For every integer $n \geqq 1$, the function*

$$G(t) = \mathscr{L}^{-1}(\varphi)(t) = (\pi t)^{-\frac{1}{2}} - 2\pi^{-\frac{1}{2}} \sum_{j=1}^{n} e^{\alpha_j^2 t} \int_{-\alpha t^{\frac{1}{2}}}^{\infty} e^{-u^2}\, du \tag{2}$$

is positive and completely monotonic.

Formula (2) is essentially known (see [7]) and also follows easily from a slight generalization of 29.3.37 in [1], by taking into account the known property $\operatorname{Re} \alpha_j < 0$ for $j = 1, 2, \cdots, n$ (see [12] and [13], or Lemma 1 in Section 4). One verifies (see Section 5 for details) on hand of (2) that $G(t) = (\pi t)^{-\frac{1}{2}} - n + o(1) > 0$ for $t \to 0^+$; and that

$$G(t) = \{\pi^{\frac{1}{2}} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1) t^{n+\frac{1}{2}}\}^{-1}(1 + O(t^{-\frac{1}{2}})) > 0 \quad \text{for} \quad t \to +\infty$$

Hence, Theorem 1 follows already from (2) and the simple monotonicity of $G(t)$.

In order to prove the complete monotonicity of G, we invoke once more Bernstein's theorem. Let $\Phi = \mathscr{L}^{-1}(G)$. Then one first obtains, by use of 29.3.4 and 29.3.114 in [1] and of $\operatorname{Re} \alpha_j < 0$, that

$$\begin{aligned} &\mathscr{L}\{(\pi x^{\frac{1}{2}})^{-1} + \pi^{-1} \textstyle\sum_{j=1}^{n} \alpha_j x^{-\frac{1}{2}} (x + \alpha_j^2)^{-1}\} \\ &\qquad = (\pi t)^{-\frac{1}{2}} - 2\pi^{-\frac{1}{2}} \textstyle\sum_{j=1}^{n} e^{\alpha_j^2 t} \int_{-\alpha_j t^{\frac{1}{2}}}^{\infty} e^{-u^2}\, du = G(t)\ . \end{aligned} \tag{3}$$

Next, by the uniqueness theorem of Laplace transforms it follows that

$$\psi(x) \equiv \pi x^{\frac{1}{2}} \Phi(x) = 1 + \sum_{j=1}^{n} \alpha_j (x + \alpha_j^2)^{-1}\ . \tag{4}$$

The proof of Theorem 2, hence that of Theorem 1, is now reduced to the proof of:

THEOREM 3. *If $\alpha_j (j = 1, 2, \cdots, n)$ are the zeros of the nth BP P_n, and ψ is defined by* (4), *then*

$$\psi(x) \geqq 0 \qquad on \quad 0 \leqq x < \infty\ . \tag{5}$$

4. Lemmas. In this section some needed, known results concerning the zeros of BP's are quoted for ease of reference.

LEMMA 1 (see [12] and [13]). $\operatorname{Re} \alpha_j < 0$.

LEMMA 2 (see [5]). *All zeros of $P_n(x)$ are complex and occur as complex conjugate pairs, except for one single negative zero, in case n is odd.*

LEMMA 3 (see [2], [3]; also [7]).

$$1 + \sum_{j=1}^{n} \alpha_j^{-1} = \sum_{j=1}^{n} \alpha_j^{-(2k+1)} = 0 \qquad \text{for} \quad k = 1, 2, \cdots, n-1 .$$

LEMMA 4 (see [7]). $\sum_{j=1}^{n} \alpha_j^{-(2n+1)} = (-1)^n (1 \cdot 3 \cdot 5 \cdots (2n-1))^{-2}$.

5. Some proofs. On account of Bernstein's theorem, Theorem 1 is an immediate consequence of Theorem 2. In order to prove Theorem 2, we start from formula (2) (this is known from [7]). Next we verify that

$$\lim_{t \to 0^+} \{G(t) - (\pi t)^{-\frac{1}{2}}\} = -2\pi^{-\frac{1}{2}} \lim_{t \to 0^+} \sum_{j=1}^{n} e^{\alpha_j^2 t} \int_{-\alpha_j t^{\frac{1}{2}}}^{\infty} e^{-u^2} du = n ,$$

because $\int_0^\infty e^{-u^2} du = \pi^{\frac{1}{2}}/2$.

To find the asymptotic value of $G(t)$ for $t \to \infty$, one integrates repeatedly by parts, obtains for integral $k \geqq 1$ that

$$\int_{-\alpha t^{\frac{1}{2}}}^{\infty} e^{-u^2} du = \frac{e^{-\alpha^2 t}}{-2\alpha t^{\frac{1}{2}}} \left(1 - \frac{1}{2\alpha^2 t} + \cdots + (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{(2\alpha^2 t)^k}\right)$$
$$+ (-1)^{k+1} \frac{1 \cdot 3 \cdots (2k+1)}{2^{k+1}} \int_{-\alpha t^{\frac{1}{2}}}^{\infty} \frac{e^{-u^2} du}{u^{2k+2}} ,$$

sets $k = n$, substitutes the result in (2), estimates the integral (one may use the indications of Section 2) crudely, and obtains by Lemma 3 and Lemma 4 that[1] $G(t) = (\pi^{\frac{1}{2}} \cdot 2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1) t^{n+\frac{1}{2}})^{-1} + O(t^{-n-1})$. It follows that $G(t) \geqq 0$ for $t \to 0^+$ and for $t \to +\infty$. In order to complete the proof of Theorem 2, it is now sufficient to apply Bernstein's theorem and show that for $0 \leqq x < \infty$, $\Phi(x) = \mathscr{L}^{-1}(G) \geqq 0$.

In Section 3 it was shown that $\pi x^{\frac{1}{2}} \Phi(x) = \psi(x) = 1 + \sum_{j=1}^{n} \alpha_j (x + \alpha_j^2)^{-1}$, so that Theorem 2, hence also Theorem 1, will be completely proved, as soon as we show that

$$\psi(x) \geqq 0 \tag{5}$$

holds on $0 \leqq x < \infty$, as asserted by Theorem 3.

6. Proof of Theorem 3. In Section 3 we saw that (4) is an immediate consequence of (3), while (3) is obtained routinely from known results (29.3.4 and 29.3.114 in [1]). It only remains to prove (5). This is done by the following steps.

(a) It follows from (4), Lemma 1 and Lemma 2, that $\psi(x) = p(x)/q(x)$, where

[1] By taking $k = n + 1$, it may be shown that the error term is, in fact, only $O(t^{-n-\frac{3}{2}})$, but this will not be needed here.

$q(x) = \prod_{j=1}^{n}(x + \alpha_j^2)$ is a real-valued polynomial of exact degree n, without positive zeros and with at most one negative zero, and $p(x)$ is also a real-valued polynomial of exact degree n.

(b) One observes that $q(x) > 0$ for sufficiently large x; hence, as $q(x) \neq 0$ for $x \geqq 0$, it follows that $q(x) > 0$ for $0 \leqq x < \infty$. Also $p(x) > 0$ for sufficiently large x, as follows, e.g., from (4), because $\lim_{x\to\infty} \psi(x) = 1$.

(c) We now prove that $p(x) = 0$ has a root of order n at $x = 0$.

(d) Once (c) is proved, it is clear that the polynomial $p(x)$ of degree n cannot vanish for $x \neq 0$, so that in particular, by (b), $p(x) > 0$ for $0 < x < \infty$. It now follows from (a), (b) and (d) that $\psi(x) = p(x)/q(x) > 0$ for $0 < x < \infty$, thus proving (5). The proof of (c) will therefore finish the proof of Theorem 3 and with it that of Theorems 2 and 1.

In order to prove (c), i.e., that p has a zero of order n at $x = 0$, it is sufficient to show directly that $\psi^{(m)}(0) = 0$ for $m = 0, 1, \cdots, (n-1)$; indeed, $q(0)$ is finite, so that the order of the zero $x = 0$ is the same for $\psi(x)$ as for $p(x)$. However, $\psi(0) = 1 + \sum_{j=1}^{n} \alpha_j^{-1} = 0$ and $\psi^{(m)}(0) = \sum_{j=1}^{n} \alpha_j^{-(2m+1)} = 0$ by Lemma 3, so that (c) is proved and with this the proofs of all three theorems are complete.

7. Secondary results. The results concern new properties of the BP's are easily obtained from some of the preceding considerations, and may be the subject of a separate publication.

REFERENCES

[1] ABRAMOWITZ, M. and SEGUN, I. A. (1966). *Handbook of Mathematical Functions*, 5th ed. Dover, New York.

[2] BURCHNALL, J. L. (1951). The Bessel polynomials. *Canad. J. Math.* **31** 62–68.

[3] BURCHNALL, J. L. and CHAUNDY, T. W. (1931). Commutative ordinary differential operators, II. The identity $P^n = Q^m$. *Proc. Roy. Soc. Ser. A* **134** 471–485.

[4] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1954). *Limit Distribution for Sums of Independent Random Variables.* Addison-Wesley, Cambridge, Mass.

[5] GROSSWALD, E. (1951). On some algebraic properties of the Bessel polynomials. *Trans. Amer. Math. Soc.* **71** 197–210.

[6] ISMAIL, M. E. H. and KELKER, D. N. (1975). The Bessel polynomials and the Student t-distribution. *Notices Amer. Math. Soc.* **22** A-131.

[7] ISMAIL, M. E. H. and KELKER, D. N. (1976). The Bessel polynomials and the Student t-distribution. *SIAM J. Math. Anal.* **7** 82–91.

[8] KELKER, D. N. (1971). Infinite divisibility and various mixtures of the normal distribution. *Ann. Math. Statist.* **42** 802–808.

[9] KRALL, H. L. and FRINK, O. (1949). A new class of orthogonal polynomials, the Bessel polynomials. *Trans. Amer. Math. Soc.* **65** 100–115.

[10] LÉVY, P. (1937). *Théorie de L'Addition des Variables Aléatoires.* Gauthier-Villars, Paris.

[11] WIDDER, D. V. (1941). *The Laplace Transform.* Princeton Univ. Press.

[12] WIMP, J. (1965). On the zeros of a confluent hypergeometric function. *Proc. Amer. Math. Soc.* **16** 281–283.

[13] WRAGG, A. and UNDERHILL, C. (1976). Remarks on the zeros of Bessel polynomial. *Amer. Math. Monthly* **83** 122–126.

DEPARTMENT OF MATHEMATICS
TEMPLE UNIVERSITY
PHILADELPHIA, PENNSYLVANIA 19122