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EXISTENCE AND UNIQUENESS OF COUNTABLE ONE-DIMENSIONAL MARKOV RANDOM FIELDS

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For a countable set S and strictly positive matrix $Q = (Q(x, y))_{x,y \in S}$ let $\mathscr{G}(Q)$ be the set of all probability measures μ on $\Omega \equiv S^{\mathbb{Z}}$, strictly positive on cylinder sets, and with the following "two-sided Markov property": $\mu\{\omega_n = x \mid \omega_l, l \neq n\} = [Q^2(x, z)]^{-1}Q(y, x)Q(x, z)$ a.e. on the set $\{\omega_{n-1} = y, \omega_{n+1} = z\}$. In other words, for every $\mu \in \mathscr{G}(Q)$, the conditional distribution of ω_n given all other ω_l depends on ω_{n-1} and ω_{n+1} only, and "behaves as if $\{\omega_n\}_{n \in \mathbb{Z}}$ is a Markov chain with transition probability matrix Q." $\mathscr{G}_0(Q)$ denotes the set of those $\mu \in \mathscr{G}(Q)$ which are in addition translation invariant. We establish a conjecture of Spitzer's [9] that either $\mathscr{G}_0(Q) = \varnothing$ or $\mathscr{G}_0(Q)$ consists of one element only, which is then necessarily a stationary Markov chain on Ω. We also give a condition for $\mathscr{G}(Q) = \varnothing$.

1. Introduction and statement of results. Let S be a countable set and Ω the product space $S^{\mathbb{Z}}$ ($\mathbb{Z}$ = the integers). Denote a generic point of Ω by ω and its nth coordinate by ω_n. For $-\infty < a \leqq b < \infty$ $\mathscr{F}_a^{\,b}$ is the smallest σ-field containing all sets of the form $\{\omega_n = x_n, a \leqq n \leqq b\}$.

$$\mathscr{F}^{\,b} = \bigvee_{a \leqq b} \mathscr{F}_a^{\,b}, \qquad \mathscr{F}_a = \bigvee_{b \geqq a} \mathscr{F}_a^{\,b} \qquad \text{and} \qquad \mathscr{F} = \bigvee_b \mathscr{F}^{\,b}$$

= smallest σ-field containing all cylinder sets. In [9] Spitzer investigated the structure of the Markov random fields on Ω (see also Föllmer [1]). This is the class of probability measures μ on $\mathscr{F}$ which assign strictly positive probability to all cylinder sets and for which the conditional distribution of ω_n given $\mathscr{F}^{\,n-1} \vee \mathscr{F}_{n+1}$ depends on the values of ω_{n-1} and ω_{n+1} only, but not on any of the other ω_l or n. In other words, there should exist constants $f_{x,z}(y)$ $(x, y, z \in S)$ such that

$$(1.1) \qquad \mu\{\omega_n = y \mid \mathscr{F}^{\,n-1} \vee \mathscr{F}_{n+1}\} = f_{x,z}(y) \qquad \text{a.e. on} \quad \{\omega_{n-1} = x, \omega_{n+1} = z\}\,.$$

As explained in [9] (see Theorem 1; also [5], Theorem 4.1 or [8], Theorem 3.22) there must then exist a strictly positive matrix $Q = (Q(x, y))_{x,y \in S}$ which plays the role of a transition probability matrix in the sense that

$$(1.2) \qquad \mu\{\omega_n = y \mid \mathscr{F}^{\,n-1} \vee \mathscr{F}_{n+1}\} = \frac{Q(x, y)Q(y, z)}{Q^2(x, z)} \qquad \text{a.e. on} \quad \{\omega_{n-1} = x, \omega_{n+1} = z\}\,.$$

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This leads to the following problem (compare [1] and [9]): Let Q be a matrix on $S \times S$ for which

$$Q(x, y) > 0\,, \qquad x, y \in S \tag{1.3}$$

$$Q^2(x, y) = \sum_{z \in S} Q(x, z)Q(z, y) < \infty\,, \qquad x, y \in S\,. \tag{1.4}$$

Define $\mathscr{G}(Q)$ to be the class of all probability measures on $\mathscr{F}$ which satisfy (1.2), and let $\mathscr{G}_0(Q)$ be the set of translation invariant elements of $\mathscr{G}(Q)$, i.e., the set of $\mu \in \mathscr{G}(Q)$ for which

$$\mu\{\omega_{a+n} = x_n,\ 0 \leqq n \leqq b\}$$

is independent of a for all $0 \leqq b < \infty$ and $x_n \in S$. What is the structure of $\mathscr{G}(Q)$ and $\mathscr{G}_0(Q)$? In particular when are these sets nonempty, and if so, how many elements do they contain? If $\# \mathscr{G}(Q) > 1$ we say that phase transition occurs.[2] The most interesting situation is where $\mathscr{G}(Q)$ contains at least two elements μ_1 and μ_2 which cannot be obtained from each other by translation and Spitzer gives examples of this kind in [9]. Spitzer called two matrices Q and Q' equivalent ($Q \sim Q'$) if there exists some strictly positive vector $v : S \rightarrow \mathbb{R}^+$ and a constant $\lambda > 0$ such that

$$Q'(x, y) = \frac{Q(x, y)v(y)}{\lambda v(x)}\,, \qquad x, y \in S\,.$$

One easily sees that if $Q \sim Q'$ then $\mathscr{G}(Q) = \mathscr{G}(Q')$. Spitzer [9] also showed that, conversely, $\mathscr{G}(Q) \cap \mathscr{G}(Q') \neq \emptyset$ implies $Q \sim Q'$ and hence $\mathscr{G}(Q) = \mathscr{G}(Q')$. Thus $\mathscr{G}(Q)$ depends only on the equivalence class of Q. If $Q \sim P$ for some positive recurrent stochastic matrix P then clearly $P^\nu \in \mathscr{G}_0(Q) = \mathscr{G}_0(P)$ where P^ν is the distribution of the stationary doubly infinite Markov chain with transition probability matrix P. That is,

$$P^\nu\{\omega_{a+n} = x_n,\ 0 \leqq n \leqq b\} = \nu(x_0) \prod_{i=0}^{b-1} P(x_i, x_{i+1})\,, \tag{1.5}$$

where ν is the unique invariant probability measure of P ($\nu P = \nu$). Our main result is Theorem 1 below which confirms a conjecture of Spitzer's [9] that $\mathscr{G}_0(Q)$ never can have any other elements. *Throughout this paper all matrices have all entries strictly positive, and have finite second powers.* The stochastic matrices are therefore automatically irreducible and spheriodic.

THEOREM 1. *For all Q satisfying* (1.3) *and* (1.4) *one has either $\mathscr{G}_0(Q) = \emptyset$ or $\# \mathscr{G}_0(Q) = 1$. The latter case occurs if and only if $Q \sim P$ for some positive recurrent stochastic matrix P. Moreover, in this case P is unique and $\mathscr{G}_0(Q) = \{P^\nu\}$ with P^ν as in* (1.5).

REMARK 1. Define

$$\begin{aligned}\mathscr{G}_1(Q) = \{\mu \in \mathscr{G}(Q) :\ &\text{for all } \varepsilon > 0 \text{ there exists a finite set} \\ &S_\varepsilon^1 \text{ with } \liminf_{n\to\infty} \mu\{\omega_n \notin S_\varepsilon^1\} \leqq \varepsilon \text{ and} \\ &\liminf_{n\to-\infty} \mu\{\omega_n \notin S_\varepsilon^1\} \leqq \varepsilon\}\,,\end{aligned} \tag{1.6}$$

[2] $\# A$ denotes the number of elements in A.

$$\mathscr{G}_2(Q) = \{\mu \in \mathscr{G}(Q) : \text{for some } \eta > 0 \text{ there exists a finite set } S^2 \text{ with } \mu\{\omega_{-n} \in S^2, \omega_n \in S^2\} \geqq \eta \text{ for all } n \geqq 0\}. \tag{1.7}$$

Clearly $\mathscr{G}_0 \subset \mathscr{G}_1 \cap \mathscr{G}_2$ and we shall actually prove the stronger result that either $\mathscr{G}_2(Q) = \mathscr{G}_0(Q) = \emptyset$ or Q is equivalent to a positive recurrent stochastic P and $\mathscr{G}_1(Q) = \mathscr{G}_0(Q) = \{P^\nu\}$. Note that if $\mathscr{G}_2(Q) = \emptyset$, then $\mathscr{G}(Q)$ can only contain measures μ for which $\min(|\omega_n|, |\omega_{-n}|)$ tends to infinity in probability as $n \to \infty$. (Here $|x| = k$ if $x = s_k$, where $s_0, s_1, s_2 \cdots$ is an arbitrary fixed ordering of S.)

Spitzer, [9], end of Section 3, showed that if $S = \mathbb{Z}$ and Q is equivalent to the transition probability matrix P of a random walk, i.e., $P(x+z, y+z) = P(x, y)$, then $\mathscr{G}(Q) = \emptyset$. This result is generalized considerably in

THEOREM 2. *Assume Q satisfies* (1.3) *and* (1.4), *but is not equivalent to a positive recurrent stochastic matrix P. If there exist $\delta_0 > 0$ and $m_0 > 1$ such that*

$$\textstyle\sum_{n=1}^{m_0} Q^n(x, x) \geqq \delta_0, \qquad x \in S, \tag{1.8}$$

then $\mathscr{G}(Q) = \emptyset$.

The above theorem shows for instance that $\mathscr{G}(P) = \emptyset$ for any substochastic P which is not positive recurrent but has its diagonal elements bounded away from zero. More complicated arguments than in Section 3 show that (1.8) can be weakened to

$$Q^{n_0+m_0}(x, y) \geqq \delta_0 Q^{n_0}(x, y), \qquad x, y \in S, \tag{1.9}$$

for some $n_0 \geqq 0$, $m_0 \geqq 1$. However, this still leaves open the general problem of finding necessary and sufficient conditions for $\mathscr{G}(Q) \neq \emptyset$. We note in passing, that the arguments which establish $\mathscr{G}(Q) = \emptyset$ from (1.9) also can be used to show that the still weaker condition

$$\textstyle\sum_y \min\{P^{n_0+m_0}(x, y), P^{n_0}(x, y)\} \geqq \delta_0, \qquad x \in S \tag{1.10}$$

implies the strong ratio limit property for an irreducible, aperiodic, recurrent stochastic matrix P (compare [3] and also Remark 4 below).

2. The structure of $\mathscr{G}_0(Q)$. Before starting the proof of Theorem 1 we note that, by the assumption of the strict positivity of Q, the right hand side of (1.2) is strictly positive for all $x, y, z \in S$. Thus if μ satisfies (1.2) and if $x_n \in S$ are such that

$$\mu\{\omega_n = x_n, a \leqq n \leqq b\} > 0,$$

then also for any $y_{a+1} \in S$

$$\begin{aligned}\mu\{\omega_a &= x_a, \omega_{a+1} = y_{a+1}, \omega_n = x_n, a+2 \leqq n \leqq b\} \\ &= \mu\{\omega_a = x_a, \omega_n = x_n, a+2 \leqq n \leqq b\} \frac{Q(x_a, y_{a+1})Q(y_{a+1}, x_{a+2})}{Q^2(x_a, x_{a+2})} > 0.\end{aligned}$$

Iteration of this argument shows that any $\mu \in \mathscr{G}(Q)$ assigns strictly positive measure to $\{\omega_a = x_a, \omega_n = y_n, a < n < b, \omega_b = x_b\}$ for any y_n, and hence to all cylinder sets. The proof of [8], Theorem 3.22 (see also [5], Theorem 4.1 or [9], Theorem 1) now shows that if $\mathscr{G}(Q) \neq \emptyset$, then necessarily

$$Q^n(x, y) < \infty , \qquad x, y \in S,\ n = 1, 2, \cdots , \tag{2.1}$$

and for any $\mu \in \mathscr{G}(Q)$

$$\begin{aligned} &\mu\{\omega_n = x_n, a < n < b \mid \mathscr{F}^a \vee \mathscr{F}_b\} \\ &\quad = \frac{Q(y, x_{a+1}) \prod_{n=a+1}^{b-2} Q(x_n, x_{n+1}) Q(x_{b-1}, z)}{Q^{b-a}(y, z)} \end{aligned} \tag{2.2}$$

on the set $\{\omega_a = y, \omega_b = z\}$. *From now on we therefore also assume* (2.1) *for all our matrices.* It is simple to see ([10], Lemma 1 or [7], Theorem 6.1) that for such a Q all the power series $\sum_{n=0}^{\infty} Q^n(x, y) w^n$ have a common finite radius of convergence, say $R = R(Q)$. Our first lemma takes care of the case $R(Q) = 0$.

LEMMA 1. *If* $R(Q) = 0$, *then* $\mathscr{G}_0(Q) = \mathscr{G}_2(Q) = \emptyset$.

PROOF. Assume $\mu \in \mathscr{G}_2(Q)$. Then, by (2.2)

$$\begin{aligned} &\mu\{\omega_{-n} = x, \omega_0 = y, \omega_n = z\} \\ &\quad = \mu\{\omega_{-n} = x, \omega_n = z\}\mu\{\omega_0 = y \mid \mathscr{F}^{-n} \vee \mathscr{F}_n\} \\ &\quad = \mu\{\omega_{-n} = x, \omega_n = z\}\{Q^{2n}(x, z)\}^{-1} Q^n(x, y) Q^n(y, z) . \end{aligned} \tag{2.3}$$

Now let η and S^2 be as in definition (1.7) and let S^* be such that $\mu\{\omega_0 \notin S^*\} \leqq \eta/2$. Summation of (2.3) over $x, z \in S^2$ and $y \in S^*$ then yields

$$\begin{aligned} \frac{\eta}{2} &\leqq \mu\{\omega_{-n} \in S^2, \omega_0 \in S^*, \omega_n \in S^2\} \\ &\leqq \sum_{x,z \in S^2} \sum_{y \in S^*} \{Q^{2n}(x, z)\}^{-1} Q^n(x, y) Q^n(y, z) . \end{aligned}$$

It follows that for each $n \geqq 0$ there exist $y_n \in S^*$ and $x_n, z_n \in S^2$ such that

$$Q^n(x_n, y_n) Q^n(y_n, z_n) \geqq \eta\{2\# S^*(\# S^2)^2\}^{-1} Q^{2n}(x_n, z_n) .$$

Now let s be any fixed element of S. Then

$$\begin{aligned} Q^{n+2}(s, s) &\geqq Q(s, x_n) Q^n(x_n, y_n) Q(y_n, s) \\ &\geqq \min_{x \in S^2} Q(s, x) \cdot \min_{y \in S^*} Q(y, s) Q^n(x_n, y_n) \end{aligned}$$

and similarly for $Q^n(y_n, z_n)$. Also

$$Q^{2n}(x_n, z_n) \geqq Q(x_n, s) Q^{2n-2}(s, s) Q(s, z_n) .$$

Since the minima of $Q(\cdot, \cdot)$ over finite sets are strictly positive it follows that there exists a $C_0 > 0$ for which

$$\{Q^{n+2}(s, s)\}^2 \geqq C_0 Q^{2n-2}(s, s) . \tag{2.4}$$

Define a sequence of integers n_k by

$$n_0 = 7 , \qquad n_{k+1} = 2n_k - 6 ,$$

and write

$$\gamma_k = Q^{n_k}(s, s) .$$

Then

$$\frac{n_k}{2^k} = \frac{n_{k-1}}{2^{k-1}} - \frac{3}{2^{k-1}} = \frac{n_{k-2}}{2^{k-2}} - \frac{3}{2^{k-1}} - \frac{3}{2^{k-2}} = \cdots$$
$$= n_0 - 3 \sum_{l=0}^{k-1} 2^{-l} \to 1 , \qquad k \to \infty .$$

On the other hand, (2.4) implies

$$\gamma_k \leqq C_0^{-1}\gamma_{k-1}^2 \leqq C_0^{-1-2}\gamma_{k-2}^4 \leqq \cdots \leqq C_0^{-(2^k-1)}\gamma_0^{2^k} ,$$

so that

$$\limsup_{k\to\infty} \{Q^{n_k}(s, s)\}^{1/n_k} = \limsup_{k\to\infty} \{\gamma_k\}^{1/n_k} \leqq C_0^{-1}\gamma_0 < \infty . \tag{2.5}$$

However, it is known ([11], Theorem A or [7], proof of Theorem 6.1) that

$$R(Q) = \lim_{n\to\infty} \{Q^n(s, s)\}^{-1/n}$$

which together with (2·5) contradicts $R(Q) = 0$. □

REMARK 2. $R(Q) = 0$ does *not* imply $\mathscr{G}(Q) = \emptyset$. That is, one can construct examples with $\mathscr{G}(Q) \neq \emptyset$ even though the powers of Q increase so rapidly that $\{Q^n(x, x)\}^{1/n} \to \infty$.

In view of Lemma 1 we may restrict ourselves to $0 < R(Q) < \infty$, and for such matrices we have the following proposition of Vere-Jones ([10], [11]).

PROPOSITION 1. *If $R(Q) > 0$ then there exists a substochastic matrix P such that*

$$Q \sim P \qquad and \qquad R(P) = 1 . \tag{2.6}$$

Exactly one of the following three cases must occur:

i) (positive recurrent case) *P is positive recurrent. In this case there is only one P satisfying* (2.6).

ii) (null recurrent case) *P is null recurrent. In this case there is only one P satisfying* (2.6).

iii) (transient case) *P is transient. This automatically includes all cases where any row sum of P is strictly less than one.*

PROOF. Vere-Jones ([10], Theorem II; [11], Section 4; see also [7], Chapter 6) showed that there exists a strictly positive vector $v: S \to \mathbb{R}_+$ satisfying $R(Q)(Qv)(x) \leqq v(x)$, $x \in S$. Thus,

$$P(x, y) = R\,\frac{Q(x, y)v(y)}{v(x)} \tag{2.7}$$

defines a substochastic matrix, equivalent to Q. If Q' is given by

$$Q'(x, y) = \frac{Q(x, y)w(y)}{\lambda w(x)}$$

for some $w(\cdot) > 0$, $\lambda > 0$, then

$$R(Q') = \lim_{n\to\infty} \{(Q')^n(x, y)\}^{-1/n} = \lambda R(Q) .$$

Thus by (2.7) $R(P) = 1$. Moreover if P' is another matrix equivalent to Q, with $R(P') = R(P) = 1$ then necessarily

$$P'(x, y) = \frac{P(x, y)w(y)}{w(x)}$$

for some $w(\cdot) > 0$. If P' is in addition substochastic then necessarily

$$Pw(x) \leqq w(x) , \tag{2.8}$$

and it is well known ([2], Proposition 6.3 or [11], Corollary 4.2) that the only positive solution of (2.8) for a given irreducible recurrent matrix P is the constant vector. Thus, in cases (i) and (ii) we must have $P' = P$. Also if P and P' are two equivalent irreducible substochastic matrices with $R(P) = R(P') = 1$, then it is impossible that one is recurrent and the other transient, or one positive recurrent and the other null recurrent. □

COROLLARY. *An irreducible positive recurrent stochastic matrix P cannot be equivalent to an irreducible null recurrent stochastic matrix P'.*

PROOF. If P is recurrent, then

$$\textstyle\sum_n P^n(x, x) = \infty .$$

Consequently

$$R(P) = \lim_{n\to\infty} \{P^n(x, x)\}^{-1/n} \leqq 1 .$$

Since $R(P) \geqq 1$ for any stochastic matrix, it follows that $R(P)$ must equal one, and the same holds for $R(P')$. We can now apply Proposition 1 with $Q = P$. □

Theorem 1 will now follow from the next two lemmas.

LEMMA 2. *In the transient and null recurrent case $\mathscr{G}_0(Q) = \mathscr{G}_2(Q) = \varnothing$.*

PROOF. By Proposition 1 we may and shall assume that Q is substochastic with $R(Q) = 1$. Let

$$\bar{\Omega} = \textstyle\prod_{n=0}^{\infty} S \tag{2.9}$$

and denote a generic point of $\bar{\Omega}$ by $\bar{\omega} = \{\bar{\omega}_n\}_{n\geqq 0}$. Also let ν_x be the unique subprobability measure on $\bar{\Omega}$ which satisfies

$$\nu_x\{\bar{\omega}_0 = x\} = 1 , \tag{2.10}$$

$$\nu_x\{\bar{\omega}_n = x_n, 0 \leqq n \leqq a\} = \textstyle\prod_{n=0}^{a-1} Q(x_n, x_{n+1}) \qquad \text{if} \quad x_0 = x . \tag{2.11}$$

Under $\nu_x\{\bar{\omega}_n\}_{n\geqq 0}$ is a Markov chain (possibly with a finite life time) with transition probability matrix Q, and starting out at x. $\bar{E}_x$ denotes the expectation operator with respect to ν_x. It operates on functions on $\bar{\Omega}$. Similarly for functions on Ω, E_μ denotes the expectation operator with respect to μ.

Assume now that $\mu \in \mathscr{G}_2(Q)$ and let η, S^2 be as in definition (1.7). Consider the following random variables on Ω, respectively $\bar{\Omega}$,

$$\lambda_n = \lambda_n(\omega) = \#\{k \in [-n, n] : \omega_k \in S^2\} ,$$
$$\lambda_n(x) = \lambda_n(x, \omega) = \#\{k \in [-n, n] : \omega_k = x\} ,$$
$$\bar{\lambda}_n(x) = \bar{\lambda}_n(x, \bar{\omega}) = \#\{k \in [0, n] : \bar{\omega}_k = x\} .$$

Then

$$E_\mu \lambda_n = \sum_{k=-n}^{+n} \mu\{\omega_k \in S^2\}$$
$$\geqq 2 \sum_{k=1}^{n} \mu\{\omega_{-k} \in S^2, \omega_k \in S^2\} \geqq 2n\eta .$$

Since

$$\lambda_n = \sum_{x \in S^2} \lambda_n(x) ,$$

there exists an $x \in S^2$ and sequence $n_k \uparrow \infty$ for which

$$E_\mu \lambda_{n_k}(x) \geqq \frac{2\eta}{\# S^2} n_k = \delta n_k , \quad \text{say.} \tag{2.12}$$

Let x be fixed in this way and introduce the random times

$$\sigma_n = \sigma_n(\omega) = \min\{k \geqq -n : \omega_k = x\} ,$$
$$\tau_n = \tau_n(\omega) = \max\{k \leqq n : \omega_k = x\} .$$

Then, since $\omega_k \neq x$ for $-n \leqq k < \sigma_n$ and $\tau < k \leqq n$, one has

$$E_\mu \lambda_n(x) = \sum_{-n \leqq s \leqq t \leqq n} \mu\{\sigma_n = s, \tau_n = t\} \times \sum_{k=s}^{t} \mu\{\omega_k = x \mid \sigma_n = s, \tau_n = t\} . \tag{2.13}$$

But $\{\sigma_n = s, \tau_n = t\} \in \mathscr{F}^s \vee \mathscr{F}_t$ and $\omega_s = \omega_t = x$ on this set, so that by (2.2)

$$\mu\{\omega_k = x \mid \sigma_n = s, \tau_n = t\} = \frac{Q^{k-s}(x, x) Q^{t-k}(x, x)}{Q^{t-s}(x, x)}$$
$$= \nu_x\{\bar{\omega}_{k-s} = x \mid \bar{\omega}_{t-s} = x\} .$$

Consequently,

$$\begin{aligned} &\sum_{k=s}^{t} \mu\{\omega_k = x \mid \sigma_n = s, \tau_n = t\} \\ &= \bar{E}_x\{\bar{\lambda}_{t-s}(x) \mid \bar{\omega}_{t-s} = x\} \\ &\leqq \frac{\delta}{4}(t - s + 1) + (t - s + 1)\nu_x\left\{\bar{\lambda}_{t-s}(x) > \frac{\delta}{4}(t - s) \,\middle|\, \bar{\omega}_{t-s} = x\right\} \\ &\leqq \frac{\delta}{4}(t - s + 1) + (t - s + 1)\frac{1}{Q^{t-s}(x, x)} \nu_x\left\{\bar{\lambda}_{t-s}(x) > \frac{\delta}{4}(t - s)\right\} . \end{aligned} \tag{2.14}$$

Of course, always

$$\sum_{k=s}^{t} \mu\{\omega_k = x \mid \sigma_n = s, \tau_n = t\} \leqq (t - s + 1) . \tag{2.15}$$

Combining (2.13)—(2.15) we obtain

$$\begin{aligned} E_\mu \lambda_n(x) \leqq \frac{\delta}{4} n &+ \sum_{-n \leqq s \leqq t \leqq n, t-s \geqq (\delta/4)n} \mu\{\sigma_n = s, \tau_n = t\} \\ &\times \left(\frac{\delta}{4}(t - s + 1) + (t - s + 1)\frac{1}{Q^{t-s}(x, x)}\right. \\ &\left.\times \nu_x\left\{\bar{\lambda}_{t-s}(x) > \frac{\delta}{4}(t - s)\right\}\right) . \end{aligned}$$

This together with (2.12) requires

$$\frac{\delta}{16} Q^m(x, x) \leqq \nu_x \left\{\bar{\lambda}_m(x) > \frac{\delta}{4} m\right\}$$

for infinitely many m. However,

$$\lim_{m\to\infty} \{Q^m(x, x)\}^{1/m} = \{R(Q)\}^{-1} = 1 ,$$

and we shall complete the proof by showing

$$\text{(2.16)} \qquad \limsup_{m\to\infty} \left(\nu_x \left\{\bar{\lambda}_m(x) > \frac{\delta}{4} m\right\}\right)^{1/m} < 1 ,$$

thereby deriving a contradiction from the assumption $\mu \in \mathscr{G}_2(Q)$.

To prove (2.16) we consider the Markov chain governed by ν_x and define ρ_l, $l \geqq 0$, as the successive times $\bar{\omega}_k$ visits x. That is, $\rho_0 = 0$ and

$$\rho_{l+1} = \min \{k > \rho_l : \bar{\omega}_k = x\} (= \infty \text{ if no such } k \text{ exists}) .$$

Then

$$\left\{\bar{\lambda}_m(x) > \frac{\delta}{4} m\right\} = \{\rho_{(\delta/4)m} \leqq m\} .$$

Moreover, under ν_x, the conditional distribution of $\rho_{l+1} - \rho_l$, given $\rho_l < \infty$ and $\rho_0, \rho_1, \cdots, \rho_l$, is the same as the distribution of ρ_1. In particular, if Q is transient

$$\begin{aligned}\beta \equiv \nu_x\{\rho_{l+1} < \infty \mid \rho_l < \infty\} &= \nu_x\{\rho_1 < \infty\} \\ &= \nu_x\{\bar{\omega}_k = x \text{ for some } k \geqq 1\} < 1 ,\end{aligned}$$

and consequently

$$\nu_x\{\rho_{(\delta/4)m} \leqq m\} \leqq \nu_x\{\rho_{(\delta/4)m} < \infty\} = \beta^{(\delta/4)m} .$$

This already proves (2.16) in the transient case. If Q is null recurrent all the ρ_l are fiinite with ν_x-probability one, but now the $\rho_{l+1} - \rho_l$ are independent identically distributed with

$$\bar{E}_x\{\rho_{l+1} - \rho_l\} = \bar{E}_x \rho_1 = \infty .$$

(2.16) now follows from standard exponential estimates. In fact, let

$$\begin{aligned} Z_l &= \rho_l - \rho_{l-1} && \text{if } \rho_l - \rho_{l-1} \leqq A , \\ &= 0 && \text{otherwise,} \end{aligned}$$

where A is chosen so large that

$$\bar{E}_x Z_l \geqq \frac{8}{\delta} .$$

Then

$$\nu_x\{\rho_{(\delta/4)m} \leqq m\} \leqq \nu_x\{\textstyle\sum_{l=1}^{(\delta/4)m} (Z_l - \bar{E}_x Z_l) \leqq -m\}$$

which decreases exponentially in m by Bernstein's inequality since $0 \leqq Z_l \leqq A$ (see [6], Theorem VII. 4.1). □

LEMMA. 3. *If $Q \sim P$ for a positive recurrent stochastic matrix P, then $\mathscr{G}_1(Q) = \mathscr{G}_0(Q) = \{P^\nu\}$, with P^ν as in* (1.5).

PROOF. Let $x_{-k}, \cdots, x_k$ be any elements of S, and S_ε^1 as in definition (1.6). Let $n_l \downarrow -\infty$ $m_l \uparrow +\infty$ be sequences of integers for which

$$\mu\{\omega_{n_l} \notin S_\varepsilon^1\} \leqq 2\varepsilon\,, \qquad \mu\{\omega_{m_l} \notin S_\varepsilon^1\} \leqq 2\varepsilon\,. \tag{2.17}$$

Then, again by (2.2),

$$\begin{aligned}\mu\{\omega_n &= x_n, -k \leqq n \leqq k\} \\ &= \textstyle\sum_{y,z\in S_\varepsilon^1} \mu\{\omega_{n_l} = y, \omega_{m_l} = z\} \dfrac{1}{Q^{m_l-n_l}(y, z)} \\ &\quad \times Q^{-k-n_l}(y, x_{-k}) \textstyle\prod_{r=-k}^{k-1} Q(x_r, x_{r+1}) Q^{m_l-k}(x_k, z) \\ &\quad + \theta(\mu\{\omega_{n_l} \notin S_\varepsilon^1\} + \mu\{\omega_{m_l} \notin S_\varepsilon^1\})\end{aligned} \tag{2.18}$$

for some $0 \leqq \theta \leqq 1$. The right hand side of (2.18) is unchanged if Q is replaced by an equivalent matrix; in particular we may replace Q by P. But P is ergodic with invariant measure ν so that, by the ergodic theorem for Markov chains ([2], Theorem 6.38)

$$\lim_{l\to\infty} P^{-k-n_l}(y, x_{-k}) = \nu(x_{-k})\,,$$
$$\lim_{l\to\infty} P^{m_l-k}(x_k, z) = \lim_{l\to\infty} P^{m_l-n_l}(y, z) = \nu(z)$$

for each fixed y, z. Letting $l \to \infty$ in (2.18) with Q replaced by P, and taking into account (2.17), we therefore obtain

$$\begin{aligned}|\mu\{\omega_n = x_n, -k \leqq n \leqq k\} &- \nu(x_{-k}) \textstyle\prod_{r=-k}^{k-1} P(x_r, x_{r+1})| \\ &\leqq \limsup_{l\to\infty} \mu\{\omega_{n_l} \notin S_\varepsilon^1\} + \limsup_{l\to\infty} \mu\{\omega_{m_l} \notin S_\varepsilon^1\} \\ &\leqq 4\varepsilon\,.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and

$$P^\nu\{\omega_n = x_n, -k \leqq n \leqq k\} = \nu(x_{-k}) \textstyle\prod_{r=-k}^{k-1} P(x_r, x_{r+1})$$

one has $\mu = P^\nu$ as desired. □

REMARK 3. Note that one does not necessarily have $\mathscr{G}_2 = \mathscr{G}_1(Q) = \{P^\nu\}$ under the conditions of Lemma 3. Indeed, Spitzer [9], Section 2 constructed examples of positive recurrent stochastic matrices P for which $\mathscr{G}(P)$ contains a measure $\mu \neq P^\nu$.[3] For such an example the convex combination $\frac{1}{2}\mu + \frac{1}{2}P^\nu$ belongs to $\mathscr{G}_2(P)$, even though it clearly differs from P^ν.

3. A condition for $\mathscr{G}(Q) = \emptyset$. In this section we prove Theorem 2. Assume that (1.8) holds. Then, as explained in [3], there exists for each x an $n \leqq m_0$ such that $Q^n(x, x) \geqq \delta_0 m_0^{-1}$ and

$$\begin{aligned}Q^{m_0!}(x, x) &\geqq \{Q^n(x, x)\}^{m_0!n^{-1}} \geqq \{\delta_0/m_0\}^{m_0!n^{-1}} \\ &\geqq \{\delta_0/m_0\}^{m_0!} = \delta\,, \quad \text{say.}\end{aligned}$$

[3] See also Section 2 of the forthcoming article "An example of phase transition in countable one dimensional Markov random fields" by T. Cox in *J. Appl. Probability*.

For the remainder of the proof we restrict ourselves to the case $m_0 = 1$, so that

$$Q(x, x) \geqq \delta > 0\,, \qquad x \in S\,. \tag{3.1}$$

This entails no loss of generality because if $\mu \in \mathscr{G}(Q)$, then the induced measure governing the sequence $\{\omega_{lm_0!}\}_{l \in \mathbb{Z}}$ belongs to $\mathscr{G}(Q^{m_0!})$ and $Q^{m_0!}$ has all diagonal elements at least δ. Moreover, one easily sees that if Q is in the transient or null recurrent case then so is $Q^{m_0!}$. Similarly $R(Q) = 0$ implies $R(Q^{m_0!}) = 0$.

Assume then that (3.1) holds and that $\mu \in \mathscr{G}(Q)$. Let x be an arbitrary fixed element of S. Since μ is a probability measure which assigns positive probability to all cylinder sets there exists a finite set $S_0 = S_0(x)$ such that

$$\mu\{\omega_2 \in S_0, \omega_0 = x\} \geqq \tfrac{15}{16}\mu\{\omega_0 = x\}\,. \tag{3.2}$$

Let

$$A_n = A_n(x) = \{y \in S : Q^{n+1}(y, x) \leqq CQ^n(y, x)\}$$

where the constant C will be fixed soon. Then, by (2.2)

$$\begin{aligned} 1 &\geqq \mu\{\omega_{-n} \notin A_n, \omega_1 = x, \omega_2 \in S_0\} \\ &= \textstyle\sum_{y \notin A_n} \sum_{z \in S_0} \mu\{\omega_{-n} = y, \omega_2 = z\} \dfrac{Q^{n+1}(y, x)Q(x, z)}{Q^{n+2}(y, z)} \\ &\geqq C \min_{z \in S_0} \dfrac{Q(x, z)}{Q^2(x, z)} \textstyle\sum_{y \notin A_n} \sum_{z \in S_0} \mu\{\omega_{-n}=y, \omega_2=z\} \dfrac{Q^n(y, x)Q^2(x, z)}{Q^{n+2}(y, z)} \\ &= C \min_{z \in S_0} \dfrac{Q(x, z)}{Q^2(x, z)} \mu\{\omega_{-n} \notin A_n, \omega_0 = x, \omega_2 \in S_0\}\,. \end{aligned} \tag{3.3}$$

Now fix $C = C(x)$ such that

$$C \min_{z \in S_0} \frac{Q(x, z)}{Q^2(x, z)} \geqq 16(\mu\{\omega_0 = x\})^{-1}\,.$$

Then (3.2) and (3.3) show

$$\begin{aligned} \mu\{\omega_{-n} \notin A_n, \omega_0 = x\} &\leqq \mu\{\omega_0 = x, \omega_2 \notin S_0\} \\ &\quad + \mu\{\omega_{-n} \notin A_n, \omega_0 = x, \omega_2 \in S_0\} \leqq \tfrac{1}{8}\mu\{\omega_0 = x\}\,. \end{aligned} \tag{3.4}$$

We now claim that for all $\varepsilon > 0$ and $r \in \mathbb{Z}$ there exists an $n_0 = n_0(\varepsilon, r, x)$ such that

$$Q^{n+r}(y, x)Q^{n-r}(x, z) \geqq Q^n(y, x)Q^n(x, z) - \varepsilon Q^{2n}(y, z) \tag{3.5}$$

for all $y \in A_n$, $z \in S$ and $n \geqq n_0$. Before proving (3.5) we show that it will imply Theorem 2. Indeed, take $\varepsilon = \frac{1}{8}\mu\{\omega_0 = x\}$. Then for any r and $n \geqq n_0$ (again by (2.2))

$$\begin{aligned} \mu\{\omega_r = x\} &\geqq \textstyle\sum_{y \in A_n, z \in S} \mu\{\omega_{-n} = y, \omega_n = z\} \dfrac{Q^{n+r}(y, x)Q^{n-r}(x, z)}{Q^{2n}(x, z)} \\ &\geqq -\varepsilon + \textstyle\sum_{y \in A_n, z \in S} \mu\{\omega_{-n} = y, \omega_n = z\} \dfrac{Q^n(y, x)Q^n(x, z)}{Q^{2n}(y, z)} \\ &\geqq -\varepsilon - \mu\{\omega_{-n} \notin A_n, \omega_0 = x\} + \mu\{\omega_0 = x\} \geqq \tfrac{3}{4}\mu\{\omega_0 = x\}\,. \end{aligned}$$

Thus, μ has to satisfy for all $r \in \mathbb{Z}$

$$\mu\{\omega_r = x\} \geqq \tfrac{3}{4}\mu\{\omega_0 = x\} ,$$

and this can be proved for each x. If now S_1 is a large finite set for which $\mu\{\omega_0 \in S_1\} > \frac{7}{8}$, then for all r

$$\mu\{\omega_r \in S_1\} \geqq \tfrac{3}{4} \textstyle\sum_{x \in S_1} \mu\{\omega_0 = x\} \geqq \tfrac{3}{4} \cdot \tfrac{7}{8} = \tfrac{21}{32} ,$$

and a fortiori for $n \geqq 0$

$$\mu\{\omega_{-n} \in S_1, \omega_n \in S_1\} \geqq 1 - 2 \cdot \tfrac{11}{32} > 0 .$$

Thus, any $\mu \in \mathscr{G}(Q)$ necessarily lies in $\mathscr{G}_2(Q)$ and we already know that $\mathscr{G}_2(Q) = \varnothing$ from Lemmas 1 and 2. Theorem 2 has therefore been reduced to (3.5).

We turn to the proof of (3.5). By (3.1) $Q = \delta I + T$ for some positive matrix T, and consequently for any $t \in \mathbb{Z}$, $n \geqq |t|$

$$\begin{aligned} Q^{n-t}(y, x)Q^{n+t}(x, z) &= (\delta I + T)^{n-t}(y, x)(\delta I + T)^{n+t}(x, z) \\ &= \textstyle\sum_{m \geqq 0} \delta^m \sum_{k+l=m} \binom{n-t}{k}\binom{n+t}{l} T^{n-t-k}(y, x)T^{n+t-l}(x, z) \\ &\leqq Q^{2n}(y, z) . \end{aligned} \tag{3.6}$$

We show first that for any p there exists an $n_1 = n_1(p, \varepsilon, x)$ such that for $n \geqq n_1$, $y \in A_n$, $z \in S$

$$\begin{aligned} &\textstyle\sum_{m \leqq p} \delta^m \sum_{k+l=m} \binom{n}{k}\binom{n}{l} T^{n-k}(y, x)T^{n-l}(x, z) \\ &\qquad \leqq \textstyle\sum_{k \leqq p} \delta^k \binom{n}{k} T^{n-k}(y, x)Q^n(x, z) \leqq \varepsilon Q^{2n}(y, z) . \end{aligned} \tag{3.7}$$

Indeed, for $y \in A_n$, we have by the definition of A_n,

$$\begin{aligned} \textstyle\sum_{k \leqq p} \delta^k \binom{n}{k} T^{n-k}(y, x) &\leqq \frac{p + 1}{\delta(n + 1)} \textstyle\sum_{k \leqq p} \delta^{k+1} \binom{n+1}{k+1} T^{n+1-(k+1)}(y, x) \\ &\leqq \frac{p + 1}{\delta(n + 1)} Q^{n+1}(y, x) \leqq \frac{C(p + 1)}{\delta(n + 1)} Q^n(y, x) . \end{aligned}$$

Thus, as soon as $C(p + 1)\delta^{-1}(n + 1)^{-1} \leqq \varepsilon$, the second member of (3.7) is at most $\varepsilon Q^n(y, x)Q^n(x, z) \leqq \varepsilon Q^{2n}(y, z)$. Next we show that for all $\varepsilon > 0$ we can choose p so large that for some $n_2 = n_2(\varepsilon)$

$$\begin{aligned} &\textstyle\sum_{m > p} \delta^m \sum_{k+l=m, |k/m - \frac{1}{2}| > \varepsilon} \binom{n}{k}\binom{n}{l} T^{n-k}(y, x)T^{n-l}(x, z) \\ &\qquad \leqq \varepsilon Q^{2n}(y, z) \quad \text{for all } \ y, z \in S , \quad n \geqq n_2 . \end{aligned} \tag{3.8}$$

To prove (3.8) split the same over k, l into the two pieces $k/m > \frac{1}{2} + \varepsilon$ and $k/m < \frac{1}{2} - \varepsilon$. We restrict ourselves to the first piece. Choose t such that

$$\left(\frac{1 - \varepsilon}{1 + \varepsilon}\right)^t \leqq \frac{\varepsilon}{2} . \tag{3.9}$$

Now let $p \geqq (2/\varepsilon)t$. Then for $m \geqq p$ and $m \geqq k \geqq m(\frac{1}{2} + \varepsilon)$, $l = m - k$, one has $(l+t)(k-t+1)^{-1} \leqq (1-\varepsilon)(1+\varepsilon)^{-1}$. Thus, for $n \geqq t$

$$\binom{n}{k}\binom{n}{l} = \frac{(n-t+1)\cdots n}{(n+1)\cdots(n+t)} \frac{(l+1)\cdots(l+t)}{(k-t+1)\cdots k} \binom{n-t}{k-t}\binom{n+t}{l+t}$$
$$\leqq \left(\frac{l+t}{k-t+1}\right)^t \binom{n-t}{k-t}\binom{n+t}{l+t} \leqq \frac{\varepsilon}{2}\binom{n-t}{k-t}\binom{n+t}{l+t}.$$

Consequently, for $n \geqq t$

$$\sum_{m>p} \delta^m \sum_{k+l=m, k>m(\frac{1}{2}+\varepsilon)} \binom{n}{k}\binom{n}{l} T^{n-k}(y, x)T^{n-l}(x, z)$$
$$\leqq \frac{\varepsilon}{2} \sum_{m>p} \delta^m \sum_{k+l=m} \binom{n-t}{k-t}\binom{n+t}{l+t} T^{(n-t)-(k-t)}(y, x)T^{(n+t)-(l+t)}(x, z)$$
$$\leqq \frac{\varepsilon}{2} \{Q^{2n}(y, z)\} \quad (\text{see } (3.6)),$$

which proves (3.8). (3.6)—(3.8) together show that for given $\varepsilon > 0$ there exist $p_1 = p_1(\varepsilon)$ and $n_3 = n_3(p, \varepsilon, x)$ such that $p \geqq p_1$ and $n \geqq n_3$ and all $y \in A_n$, $z \in S$

$$(3.10) \qquad Q^n(y, x)Q^n(x, z) - 2\varepsilon Q^{2n}(y, z)$$
$$\leqq \sum_{m>p} \delta^m \sum_{k+l=m, |k/m-\frac{1}{2}|\leqq\varepsilon} \binom{n}{k}\binom{n}{l} T^{n-k}(y, x)T^{n-l}(x, z).$$

Lastly for $|km^{-1} - \frac{1}{2}| \leqq \varepsilon$ and $l = m - k$ and $m \geqq p \geqq (2/\varepsilon)|r|$ and $n \geqq |r|$ one has

$$\binom{n+r}{k+r}\binom{n-r}{l-r} = \frac{(n+1)\cdots(n+r)}{(n-r+1)\cdots n} \frac{(l-r+1)\cdots l}{(k+1)\cdots(k+r)} \binom{n}{k}\binom{n}{l}$$
$$\geqq \left(\frac{l-r+1}{k+r}\right)^r \binom{n}{k}\binom{n}{l} \geqq \left(\frac{1-3\varepsilon}{1+3\varepsilon}\right)^{|r|} \binom{n}{k}\binom{n}{l}.$$

Thus for $y \in A_n$, $z \in S$

$$Q^{n+r}(y, x)Q^{n-r}(x, z)$$
$$\geqq \sum_{m>p} \delta^m \sum_{k+l=m, |k/m-\frac{1}{2}|\leqq\varepsilon} \binom{n+r}{k+r}\binom{n-r}{l-r} T^{n+r-(k+r)}(y, x)T^{n-r-(l-r)}(x, z)$$
$$\geqq \left(\frac{1-3\varepsilon}{1+3\varepsilon}\right)^{|r|} Q^n(y, x)Q^n(x, z) - 2\varepsilon Q^{2n}(y, z),$$

as soon as $p \geqq p_1(\varepsilon) + (2/\varepsilon)|r|$, $n \geqq |r| + n_3(p, \varepsilon, x)$. Since ε is arbitrary this proves (3.5) and thereby Theorem 2.

REMARK 4. (1.8) is exactly the sufficient condition given by Kingman and Orey [3] for the strong ratio limit property (SRLP), and the above proof has some resemblance to the proof of the SRLP. As we already pointed out in the introduction one can even prove that the SRLP for an irreducible, aperiodic, recurrent stochastic matrix P holds under the weaker condition (1.10). We doubt that (1.10) is enough to conclude $\mathscr{S}(P) = \emptyset$. However, we do know that the SRLP itself is not enough to guarantee $\mathscr{S}(P) = \emptyset$. For example, one can construct strictly positive, null recurrent, stochastic, *reversible* P with a non-empty $\mathscr{S}(P)$, and one knows that any such P has the SRLP (by [4], Theorem 3).

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