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A CENTRAL LIMIT THEOREM FOR THE NUMBER OF ZEROS OF A STATIONARY GAUSSIAN PROCESS

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Carson Systems

Using a device which approximates stationary Gaussian processes by M-dependent processes, we find conditions on the covariance function to insure that the number of zero crossings, after centering and rescaling, has an asymptotically normal distribution. This device is then used to obtain central limit theorems for integrals of functions of stationary Gaussian processes.

1. Introduction. Central limit theorems for dependent random variables have been a topic of considerable recent interest. General results have required strong restrictions on the degree of dependence. Under the assumption of M-dependence, Diananda (1955) gave a central limit theorem for identically distributed sequences with no assumptions about the existence of moments higher than the second. In studying M-dependent sequences with M tending to infinity, Berk (1973) had to make further assumptions about moments. Rozanov (1960) obtained central limit theorems for additive random functions under ϕ-mixing conditions. Again assumptions about moments higher than the second were required (see also Ibragimov (1962)).

In this paper we specialize to additive functionals of stationary Gaussian processes. Major attention is focused on a limit theorem for the number of zeros of such processes. This example serves to illustrate our general techniques. Following an idea of Malevich (1969), we approximate the underlying Gaussian process $X(t)$ (and its derivatives when they exist) by an M-dependent process $X_M(t)$. One can then establish the convergence of a functional of $X_M(t)$ as $M \to \infty$ to the corresponding functional of $X(t)$ under dependence conditions which only involve the covariance function for $X(t)$.

In studying zeros, no finite moments above the second are needed if it is assumed that the covariance function and its second derivative are square integrable. More specifically let $X(t)$ be a separable zero mean stationary Gaussian process with $\rho(t) = EX(0)X(t)$. Assume that $\rho''(t)$ exists and take $\rho(0) = 1$ and $-\rho''(0) = \lambda_2$. Now X must be sample continuous and we may consider $N_X(T)$ the number of times that X crosses zero in time interval $[0, T]$ (cf. Cramér and Leadbetter (1967, pages 191 ff), referred to hereafter as CL). Write $\phi(t) = \rho''(t) - \rho''(0) = \frac{1}{2}E(X'(0) - X'(t))^2$ for half the increment variance of $X'(t)$. We

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obtain the following result which was first given by Malevich (1969) under hypotheses on the spectrum of X which are stronger than our conditions.

THEOREM 1. *If*

(A1) $$\rho,\ \rho'' \in L_2$$

(A2) $$\int_0^\varepsilon \phi(t)/t\,dt < \infty \qquad \textit{for some} \quad \varepsilon > 0$$

(A3) $$\lim_{T\to\infty} T^{-1}\,\mathrm{Var}\,(N_X(T)) = \sigma_2 > 0\,,$$

then $T^{-\frac{1}{2}}(N_X(T) - EN_X(T)) \to_{\mathscr{L}} \mathscr{N}(0, \sigma)$ *as* $T \to \infty$ *where*

(1) $$\sigma^2 = \frac{1}{\pi}\left[(\lambda_2)^{\frac{1}{2}} + \int_0^\infty \left[\frac{E(|X'(0)X'(t)|\,|\,X(0) = X(t) = 0)}{(1-\rho^2(t))^{\frac{1}{2}}} - (E|X'(0)|)^2\right]dt\right].$$

DISCUSSION. Any central limit theorem for continuous processes must have some mixing condition at infinity and some local condition. The former is supplied here by A1. A1 also guarantees the almost everywhere existence of a spectral density function which we denote by $f^2(\lambda)$. Our local condition A2 is the weakest possible since Geman (1972) has shown that A2 is necessary and sufficient for the variance of the number of zeros to be finite. It will become apparent from the proof of the theorem that A1 and A2 alone guarantee that $\lim T^{-1}\,\mathrm{Var}\,N_X(T)$ exists and is finite. Thus we can replace A3 with

(A3)′ $$\liminf_{T\to\infty} T^{-1}\,\mathrm{Var}\,(N_X(T)) > 0\,.$$

After some preliminaries in Section 2, Theorem 1 is proven in Section 3. In Section 4, we study integrals of functions of stationary Gaussian processes. We obtain a general central limit theorem which is applicable, for example, to the time spent above a level and the Z_n-exceedance measures.

2. Preliminaries. $X(t)$ has spectral density $f^2(\lambda)$ so that we can realize X via the spectral representation

$$X(t) = \int \cos \lambda t\, f(\lambda)\, dB(\lambda)$$

where $dB(\lambda)$ is Gaussian white noise and $f(\lambda) \geqq 0$. (All integrals will be taken from $-\infty$ to ∞ unless specific limits are given.) We wish to determine when

(2) $$Z(T) \equiv T^{-\frac{1}{2}}(N_X(T) - EN_X(T))$$

converges in law to a normal distribution as $T \to \infty$. To this end, we define for each positive M, the process

$$X_M(t) = \int \cos \lambda t\, (f^2 * P_M)^{\frac{1}{2}}\, dB(\lambda)\,.$$

Then $\{X(t), X_M(t), X'(t), X_M'(t)\}$ are jointly Gaussian and stationary. Here $*$ denotes convolution and $P_M(\lambda) = MK_1(\sin M\lambda/M\lambda)^4$ with K_1 chosen so that $\int P_M(\lambda)\,d\lambda \equiv 1$. If we define $\rho_M(t) = EX_M(0)X_M(t)$, then we have that $\rho_M(t) = \rho(t)\hat{P}_M(t)$ where $\hat{P}_M(t) = \mathscr{F}(P_M(\lambda))$ is the Fourier transform of $P_M(\lambda)$. One can verify that

(a) $\hat{P}_M(t)$ is piecewise cubic,
(b) $\hat{P}_M(t) = 1 - (K_2/M^2)t^2 + O(|t|^3)$ for t near zero where $K_2 = K_1 \int u^2(\sin u/u)^4\,du$,
(c) $\hat{P}_M(t) = 0$ for $|t| > 4M$.

From this it follows that $\rho(0) = \rho_M(0)$ and $\rho_M(t) = 0$ for $|t| > 4M$ so that $N_{X_M}(t)$ is an M-dependent process.

Now define $Z_M(T) \equiv T^{-\frac{1}{2}}(N_{X_M}(T) - EN_{X_M}(T))$. In order to establish that $Z(T)$ is asymptotically normal, it suffices to show that

(3A) $\quad Z_M(T) \rightarrow_{L_2} Z(T) \quad$ uniformly in $\quad T > T_0 \quad$ as $\quad M \rightarrow \infty$,

(3B) $\quad Z_M(T) \rightarrow_{\mathscr{L}} \mathscr{N}(0, \sigma_M) \quad$ for each $\quad M \quad$ as $\quad T \rightarrow \infty$, and

(3C) $\quad \sigma_M \rightarrow \sigma$.

We obtain (3C) immediately from (3A) and (3B). To verify (3A) it is enough to show that given any $\varepsilon > 0$, we can take M_0 so large that for all $M > M_0$ and all $T > T_0$ we have

(4) $$T^{-1}[E(N_{X_M}(T) - N_X(T))^2 - (E(N_{X_M}(T) - N_X(T)))^2] < \varepsilon\,.$$

Once this is proven A3 implies that

(5) $$\lim_{T\to\infty} T^{-1} \operatorname{Var}(N_{X_M}(T)) = \sigma_M^2 > 0$$

for M large enough. This condition allows us to apply the results in Diananda (1955) to obtain the conclusion (3B). Thus it remains to show the truth of (4).

First let us digress briefly in order to show that $X_M(t) \rightarrow_{L_2} X(t)$ and $X_M'(t) \rightarrow_{L_2} X'(t)$ uniformly in t.

LEMMA 1. *If $f \geqq 0$ and $f_n \geqq 0$, then $f_n^2 \rightarrow f^2$ in L_1 implies that $f_n \rightarrow f$ in L_2.*

LEMMA 2. *Let $\{I_M(x)\}$ be an approximate identity,* i.e. *each $I_M(x)$ is a probability density function and $\int_{|x|>\varepsilon} I_M(x)\,dx \rightarrow 0$ as $M \rightarrow \infty$ for all $\varepsilon > 0$. If $f \in L_p$, $1 \leqq p < \infty$, then $I_M * f \rightarrow f$ in L_p.*

PROOF. Cf. Loomis (1953, page 124).

With the help of these two lemmas we see that $(P_M * f^2)^{\frac{1}{2}} \rightarrow f$ in L_2 and thus $X_M(t) \rightarrow X(t)$ in L_2 uniformly in t. To show

(6) $$X_M'(t) \rightarrow X'(t) \quad \text{in} \quad L_2 \quad \text{uniformly in} \quad t$$

we need to show $\lambda \cdot (P_M * f^2)^{\frac{1}{2}} \rightarrow \lambda f$ in L_2. But

$$\|\lambda \cdot (P_M * f^2)^{\frac{1}{2}}\|_2 = \|f \cdot (\lambda^2 * P_M)^{\frac{1}{2}}\|_2 \leqq K_3\|\lambda f\|_2 < \infty$$

since

(7) $$(\lambda^2 * P_M)^{\frac{1}{2}}(\mu) < K_4\mu \qquad \text{for} \quad \mu \geqq \mu_0 > 0\,.$$

Thus $\lambda \cdot (P_M * f^2)^{\frac{1}{2}} \in L_2$ and $X_M'(t) \in L_2$ for all t. In view of Lemma 1, to prove (6), we need only show that $\|\lambda^2[(f^2 * P_M) - f^2]\|_1$ goes to zero as $M \rightarrow \infty$. But this equals

$$\int \lambda^2|(f^2 * P_M) - f^2|\,d\lambda \leqq C^2 \int_{-C}^{C} |(f^2 * P_M) - f^2|\,d\lambda + 2\left[\int_C^\infty f^2(\lambda^2 * P_M)\,d\lambda + \int_C^\infty \lambda^2 f^2\,d\lambda\right].$$

Taking C large, the last terms go to zero independent of M (see (7)). Then for fixed C, we make the first term small by letting $M \to \infty$.

We shall also make use of the following:

LEMMA 3. *Let* $\rho_{M,0}(t) = E(X(0)X_M(t))$ *and* $\rho_{M,M}(t) = E(X_M(0)X_M(t))$. *Then* $\rho_{M,0}$ *and* $\rho_{M,M}$ *are covariance functions and under hypothesis* (A1) *we have that*

(a) $\rho_{M,0}, \rho''_{M,0}, \rho_{M,M}, \rho''_{M,M} \in L_2$ *and*

(b) $\rho_{M,0} \to_{L_2} \rho$, $\rho''_{M,0} \to_{L_2} \rho''$, $\rho_{M,M} \to_{L_2} \rho$, $\rho''_{M,M} \to_{L_2} \rho''$.

PROOF. We have

$$\rho_{M,0}(t) = E(X(0)X_M(t)) = \int e^{i\lambda t} f(\lambda)(f^2 * P_M)^{\frac{1}{2}}\, d\lambda$$

so that by the Plancherel identity

$$\begin{aligned} \|\rho_{M,0}\|_2 &= 2\pi \|f(f^2 * P_M)^{\frac{1}{2}}\|_2 = 2\pi \|f^2(f^2 * P_M)\|_1^{\frac{1}{2}} \\ &\leqq 2\pi \|f^2\|_2^{\frac{1}{2}} \|f^2 * P_M\|_2^{\frac{1}{2}} \leqq 2\pi \|f^2\|_2^{\frac{1}{2}} \|f^2\|_2^{\frac{1}{2}} \|P_M\|_1^{\frac{1}{2}} \\ &= 2\pi \|f^2\|_2 = \|\rho\|_2 . \end{aligned}$$

Further we have that

$$\begin{aligned} \|\rho_{M,0} - \rho\|_2 &= 2\pi \|f[(f^2 * P_M)^{\frac{1}{2}} - f]\|_2 \\ &= 2\pi \|f^2[(f^2 * P_M)^{\frac{1}{2}} - f]^2\|_1^{\frac{1}{2}} \\ &\leqq 2\pi \|f^2\|_2^{\frac{1}{2}} \|[(f^2 * P_M)^{\frac{1}{2}} - f]^2\|_2^{\frac{1}{2}} \\ &\leqq (2\pi)^{\frac{1}{2}} \|\rho\|_2^{\frac{1}{2}} \|(f^2 * P_M) - f^2\|_2^{\frac{1}{2}} \to 0 \end{aligned}$$

by Lemma 2. Since $\rho \in L_2$, we have that $\rho_{M,0} \in L_2$ and $\rho_{M,0} \to_{L_2} \rho$.

The other cases follow in analogous fashion.

We need one more fact before proving Theorem 1.

LEMMA 4. *Let* X, Y *be a mean zero Gaussian pair of random variables and let*

$$\rho = \operatorname{Cor}(X, Y) = \frac{EXY - EXEY}{(E(X - EX)^2 E(Y - EY)^2)^{\frac{1}{2}}} .$$

Then

$$0 \leqq \operatorname{Cor}(|X|, |Y|) \leqq \rho^2 .$$

PROOF. Without loss of generality take $EX^2 = EY^2 = 1$, so that $EXY = \rho$. Using the Hermite polynomial expansion of the bivariate normal density we have

$$f(\rho) \equiv E|XY| = \frac{1}{2\pi} \int\int \sum_{n=0}^{\infty} \frac{\rho^n}{n!} |xy| H_n(x) H_n(y) \exp\left[-\frac{x^2 + y^2}{2}\right] dx\, dy$$

where $\{H_n(x)\}$ are the Hermite polynomials. Since $f(0) = E|X|E|Y|$ and since $\int |x| H_n(x) e^{-(x^2/2)}\, dx = 0$ when n is odd, we have

$$E|XY| - E|X|E|Y| = \frac{1}{2\pi} \sum_{n=2,\, n \text{ even}}^{\infty} \frac{\rho^n}{n!} \left[\int |x| H_n(x) e^{-x^2/2}\, dx\right]^2 . \tag{8}$$

Thus Cor $(|X|, |Y|) \geqq 0$. Also (8) is less than or equal

$$\rho^2\left[\frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{1}{n!} [\int |x| H_n(x) e^{-x^2/2}\, dx]^2\right]$$

$$= \rho^2(f(1) - f(0)) = \rho^2[EX^2 - (E|X|)^2] .$$

Hence we have that Cor $(|X|, |Y|) \leqq \rho^2$.

3. Proof of Theorem 1. Now to demonstrate (4) directly. Clearly one can let T tend to infinity through the integers. Let $\tau = 2^{-n}$ for some large n to be specified later and for integer T write

$$N_X(T) = \sum_{i=0}^{2^n T-1} N_X(i)$$

and

$$N_{X_M}(T) = \sum_{i=0}^{2^n T-1} N_{X_M}(i)$$

where $N_X(i)$ is the number of crossings in time interval $[i2^{-n}, (i+1)2^{-n})$ for the process X. Then (4) equals

$$\text{(9)} \qquad T^{-1} \sum \sum_{|j-i|\leqq 1, i=0}^{2^n T-1} \operatorname{Cov}(N_X(i) - N_{X_M}(i), N_X(j) - N_{X_M}(j)) + T^{-1} \sum \sum_{|j-i|\geqq 2, i=0}^{2^n T-1} \operatorname{Cov}(N_X(i) - N_{X_M}(i), N_X(j) - N_{X_M}(j)) .$$

The first term covers a region around the diagonal and by stationarity is less than

$$\text{(10)} \qquad \frac{3}{\tau} E(N_X(\tau) - N_{X_M}(\tau))^2 + \frac{3}{\tau} |E(N_X(\tau) - N_{X_M}(\tau))| .$$

As $M \to \infty$ the second part of (10) vanishes and the first part is less than

$$\text{(11)} \qquad \frac{9}{\tau} [E(N_X(\tau) - \chi_\tau)^2 + E(\chi_\tau - \chi_\tau^M)^2 + E(N_{X_M}(\tau) - \chi_\tau^M)^2]$$

where

$$\chi_\tau = 1 \quad \text{if} \quad X(0)X(\tau) < 0$$
$$= 0 \quad \text{otherwise}$$

and similarly for χ_τ^M. Now

$$E(N_X(\tau) - \chi_\tau)^2 \leqq \sum_{k=2}^{\infty} k^2 P(N_X(\tau) = k)$$
$$\leqq 2 \sum_{k=0}^{\infty} (k^2 - k) P(N_X(\tau) = k) = 2E(N_X^2(\tau) - N_X(\tau)) .$$

Using the expression for the second factorial moment of the number of zeros in an interval (CL, page 209) we see that

$$\frac{1}{\tau} E(N_X(\tau) - \chi_\tau)^2 \leqq \frac{4}{\tau} \int_0^\tau (\tau - t) \frac{E(|X'(0)X'(t)| \mid X(0) = X(t) = 0)}{2\pi(1 - \rho^2(t))^{\frac{1}{2}}} dt .$$

Further estimation shows this is bounded by a constant times

$$\int_0^\tau \phi(t)/t\, dt$$

where $\phi(t) = \rho''(t) - \rho''(0)$. (See CL, page 210.) In view of assumption A2, this goes to zero as $\tau \to 0$. Similarly we can show that the third term in (11) is $o(\tau)$.

The second term of (11) is $P(\chi_\tau \neq \chi_\tau{}^M)$ and this goes to zero for fixed τ as $M \to \infty$ since $X_M(t) \to_{L_2} X(t)$.

It remains to bound the second sum in (9). By an appropriate modification of the development in CL (pages 202 ff), we can show that for $|i-j| \geqq 2$

$$E(N_X(i)N_{X_M}(j)) = \int_{i2^{-n}}^{(i+1)2^{-n}} ds \int_{j2^{-n}}^{(j+1)2^{-n}} dt \frac{E(|X'(s)X_M'(t)|\,|X(s) = X_M(t) = 0)}{2\pi(1-\rho_{M,0}^2(s-t))^{\frac{1}{2}}}$$

where $\rho_{M,0}(t) = E(X(0)X_M(t))$. It then follows that

$$\begin{aligned} &\operatorname{Cov}(N_X(i), N_{X_M}(j)) \\ &\quad = \int_{i2^{-n}}^{(i+1)2^{-n}} ds \int_{j2^{-n}}^{(j+1)2^{-n}} dt \left[\frac{E(|X'(s)X_M'(t)|\,|(X(s) = X_M(t) = 0)}{2\pi(1-\rho_{M,0}^2(t-s))^{\frac{1}{2}}} \right. \\ &\qquad \left. - \frac{E|X'(s)|E|X_M'(t)|}{2\pi}\right]. \end{aligned} \tag{12}$$

Since $X_M(t) \to_{L_2} X(t)$ and $X_M'(t) \to_{L_2} X'(t)$ uniformly in t so that $N_{X_M}(i) \to_{L_2} N_X(i)$ uniformly in i, it is necessary only to consider terms in (9) with $|i-j| > 2^n T_0$ for some large T_0 chosen independent of $M > M_0$. Since $E(N_{X_M}(i)N_X(j)) = E(N_X(i)N_{X_M}(j))$, the summand in (9) can be written as

$$\operatorname{Cov}(N_X(i), N_X(j)) - 2\operatorname{Cov}(N_X(i), N_{X_M}(j)) + \operatorname{Cov}(N_{X_M}(i), N_{X_M}(j)).$$

We shall only estimate the middle term above; the others are similar, but easier.

Invoking stationarity and using (12), our proof is complete if we show that as $T_0 \to \infty$

$$\int_{T_0}^{\infty} \frac{\operatorname{Cov}(|Y_1(t)|, |Y_2(t)|)}{(1-\rho_{M,0}^2(t))^{\frac{1}{2}}} dt \tag{13A}$$

and

$$\int_{T_0}^{\infty} \left| \frac{E|Y_1(t)|E|Y_2(t)|}{(1-\rho_{M,0}^2(t))^{\frac{1}{2}}} - E|X'(0)|E|X_M'(t)| \right| dt \tag{13B}$$

both go to zero uniformly in $M > M_0$. Here $Y_1(t)$ and $Y_2(t)$ are the variables $X'(0)$ and $X_M'(t)$ conditioned on $X(0) = X_M(t) = 0$. By the Riemann-Lebesgue lemma $\rho_{M,0}(t)$ tends to zero for large t so that $(1-\rho_{M,0}^2(t))^{\frac{1}{2}}$ is bounded away from zero on the integrating set. Using Lemma 4 (13A) is bounded by a constant times

$$\int_{T_0}^{\infty} [\operatorname{Cov}(Y_1(t), Y_2(t)]^2 dt. \tag{14}$$

Making use of the underlying multivariate normal distribution of $\{X(0), X'(0), X_M(t), X_M'(t)\}$ we find that (14) equals

$$\int_{T_0}^{\infty} \left[-\rho_{M,0}''(t) + \frac{\rho_{M,0}(t)(\rho_{M,0}'(t))^2}{1-\rho_{M,0}^2(t)}\right]^2 dt. \tag{15}$$

The uniform convergence to zero of this expression follows from Lemma 3.

Again using Lemma 3 to see that $\rho_{M,0} \to_{L_2} \rho$, (13B) will go to zero if the following two expressions vanish as $T_0 \to \infty$:

(16A) $$\int_{T_0}^{\infty} |E|Y_1| - E|X'(0)||\, dt$$

and

(16B) $$\int_{T_0}^{\infty} |E|Y_2| - E|X_M'(t)||\, dt\,.$$

Evaluating (16B) we find it equals

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{T_0}^{\infty} \left|\left(-\rho''_{M,0}(0) - \frac{(\rho'_{M,0}(t))^2}{1-\rho^2_{M,0}(t)}\right)^{\frac{1}{2}} - (-\rho''_{M,0}(0))^{\frac{1}{2}}\right| dt\,.$$

Since $\rho_{M,0}(t)$ and $\rho'_{M,0}(t)$ tend to zero as $t \to \infty$, this is less than

(17) $$(-\rho''_{M,0}(0))^{-\frac{1}{2}} \int_{T_0}^{\infty} \frac{(\rho'_{M,0}(t))^2}{1-\rho^2_{M,0}(t)}\, dt\,.$$

From Lemma 3 $\rho_{M,0} \to_{L_2} \rho$ and $\rho''_{M,0} \to_{L_2} \rho''$, implying that $\rho'_{M,0} \to_{L_2} \rho'$ and thus (17) tends uniformly to zero. Expression (16A) follows accordingly, completing the proof of Theorem 1.

REMARKS ON ASSUMPTION A3. (i) The regularity assumption A3 is the only condition that is not directly related to the covariance structure of $X(t)$. This is unfortunate, for it appears to be difficult to determine exactly when this condition holds. We can however establish the following sufficient condition:

LEMMA 5. *Suppose* $EX(t)^2 = 1$, $EX'(t)^2 = \lambda_2$. *Under assumptions* A1 *and* A2 *of Theorem* 1 *and the additional condition that*

(18) $$\int_0^{\infty} \frac{\rho'(t)^2}{1-\rho^2(t)}\, dt < \frac{\pi}{2}(\lambda_2)^{\frac{1}{2}}$$

we have that $\lim_{T\to\infty} T^{-1} \operatorname{Var}(N_X(T)) = \beta > 0$.

PROOF.

(19) $$\begin{aligned}\lim_{T\to\infty} & T^{-1} \operatorname{Var} N_X(T) \\ &= \lim_{T\to\infty} T^{-1}(E(N_X(N_X-1)) - (EN_X)^2 + EN_X) \\ &= \lim_{T\to\infty} \frac{1}{\pi}\left\{(\lambda_2)^{\frac{1}{2}} + 2\int_0^T \left(1-\frac{t}{T}\right)\right. \\ &\qquad \left.\times \left[\frac{E(|X'(0)X'(t)|\,|\,X(0)=X(t)=0)}{2(1-\rho^2(t))^{\frac{1}{2}}} - \frac{\lambda_2}{\pi}\right] dt\right\}.\end{aligned}$$

Under assumptions A1 and A2, the analysis above shows that this limit exists and is finite, and permits us to write (19) as

(20) $$\frac{1}{\pi}\left[(\lambda_2)^{\frac{1}{2}} + 2\int_0^{\infty}\left(\frac{E(|X'(0)X'(t)|\,|\,X(0)=X(t)=0)}{2(1-\rho^2)^{\frac{1}{2}}} - \frac{\lambda_2}{\pi}\right)dt\right].$$

Thus we need only verify that (20) is greater than zero. By Lemma 4 it is greater than

$$\frac{1}{\pi}\left[(\lambda_2)^{\frac{1}{2}} + 2\int_0^\infty\left(\frac{(E(|X'(0)|\,|\,X(0)=X(t)=0))^2}{2(1-\rho^2)^{\frac{1}{2}}} - \frac{\lambda_2}{\pi}\right)dt\right]$$

$$= \frac{1}{\pi}\left[(\lambda_2)^{\frac{1}{2}} + \frac{2}{\pi}\int_0^\infty\left(\frac{\operatorname{Var}(X'(0)\,|\,X(0), X(t))}{(1-\rho^2)^{\frac{1}{2}}} - \lambda_2\right)dt\right].$$

This expression is greater than zero when

$$(\lambda_2)^{\frac{1}{2}} + \frac{2}{\pi}\int_0^\infty\left(\frac{\lambda_2 - \rho'^2/(1-\rho^2)}{(1-\rho^2)^{\frac{1}{2}}} - \lambda^2\right)dt > 0$$

which will occur when

$$\int_0^\infty \frac{\rho'^2}{1-\rho^2}\,dt < \frac{\pi}{2}(\lambda_2)^{\frac{1}{2}}.$$

(ii) Observe that (18) is not an asymptotic condition and cannot be satisfied by looking only at the tails of the covariance function. In a general context this is characteristic of assumption A3. For example, a 1-dependent stationary sequence of random variables with covariance $(1, -\frac{1}{2}, 0, 0, \cdots)$ has partial sums, all of whose variances are unity.

EXAMPLE. The simplest example which satisfies our conditions is the stationary Gaussian process for which $(X(t), X'(t))$ forms a vector Markov process. In normalized form it has covariance function (cf. Wong (1966))

$$\rho(t) = \tfrac{3}{2}[\exp[-3^{-\frac{1}{2}}t](1 - \tfrac{1}{3}\exp[-2(3^{-\frac{1}{2}})t])] \qquad t \geqq 0. \tag{21}$$

With the help of the substitution $x = \exp[-3^{-\frac{1}{2}}t]$, we find that

$$\int_0^\infty \frac{\rho'^2}{1-\rho^2}\,dt = \frac{3(3)^{\frac{1}{2}}}{2}\ln\tfrac{4}{3} \cong .7474 < \frac{\pi}{2}.$$

Thus, for this process, (18) holds with considerable margin.

4. Integrals of functions of Gaussian paths. In this section we briefly study central limit theorems for processes of the form

$$Z(T) = \int_0^T g(X(t))\,dt$$

when $X(t)$ is a stationary Gaussian process. Useful examples of such processes are afforded by the time spent above a fixed level and the Z_n-exceedance measures (cf. CL, page 212 ff). By our previous approximation methods we establish

THEOREM 2. *Let $X(t)$ be a stationary Gaussian process with covariance function $\rho(t)$ and let g be a real-valued function satisfying $Eg(X(t))^2 < \infty$. Set $Z(T) = \int_0^T g(X(t))\,dt$. If $\rho \in L_1$, then $T^{-\frac{1}{2}}(Z(T) - EZ(T)) \to \mathscr{N}(0, \sigma)$. Further if $g(x) - g(0)$ is not an odd function, then $\sigma > 0$.*

PROOF. Assume $EX(t) = 0$ and $EX^2(t) = 1$ and write $\phi(x, y, \rho)$ for the bivariate normal density with correlation ρ. Now

$$\begin{aligned}\sigma^2 &= \lim T^{-1} \operatorname{Var} Z(T) \\ &= 2 \int_0^\infty dt \int dx \int dy g(x) g(y)[\phi(x, y, \rho) - \phi(x, y, 0)] .\end{aligned}$$

Expanding $\phi(x, y, \rho)$ in Hermite polynomials $H_n(x)$, a use of Fubini's theorem yields

$$\sigma^2 = \frac{1}{\pi} \sum_{n=1}^\infty (n!)^{-1} \left(\int_0^\infty (\rho(t))^n \, dt\right)\left(\int g(x) H_n(x) e^{-(x^2/2)} \, dx\right)^2 . \tag{22}$$

Since $|\rho(t)| < 1$, we have $||\rho^n||_1 < ||\rho||_1$ for all $n \geqq 1$. Thus $\sigma^2 < \infty$ when $\rho \in L_1$. Now notice that

$$\begin{aligned}\int_0^\infty (\rho(t))^n \, dt &= [\mathscr{F}(\rho^n)](0) \\ &= [\underbrace{\mathscr{F}(\rho) * \cdots * \mathscr{F}(\rho)}_{n \text{ times}}](0)\end{aligned}$$

where $*$ denotes convolution and $\mathscr{F}$ is the Fourier transform. Since $\mathscr{F}(\rho) \geqq 0$, we see that

$$\int_0^\infty (\rho(t))^n \, dt \geqq 0 \qquad n \geqq 1$$

with equality possible only when n is odd and when the spectral density $f^2(\lambda)$ is identically zero in a neighborhood of $\lambda = 0$. From this we see that $\sigma > 0$ except when $g(x) - g(0)$ is odd and $\int \cos \lambda t \rho(t) \, dt = 0$ for all t in some neighborhood of zero.

To establish the theorem, we approximate $X(t)$ by an M-dependent process $X_M(t)$ just as in Section 3. By Diananda's (1955) results we have $T^{-\frac{1}{2}}(Z_M(T) - EZ_M(T)) \to \mathscr{N}(0, \sigma_M)$. As before we need only show that

$$T^{-1} E(Z_M(T) - Z(T))^2 \to 0 \tag{23}$$

for $M \to \infty$ uniformly in $T > T_0$. Expanding (23) we find that it equals

$$2 \int_0^T \left(1 - \frac{t}{T}\right) E([g(X(t)) - g(X_M(t))][g(X(0)) - g(X_M(0))]) \, dt . \tag{24}$$

We now establish the convergence $g(X_M(t)) \to_{L_2} g(X(t))$. Since $E(g(X(t)))^2 < \infty$, we expand g in Hermite polynomials $g(x) = \sum_{n=0}^\infty a_n H_n(x)$ and let $g_N(x) = \sum_{n=0}^N a_n H_n(x)$. Then

$$\begin{aligned}||g(X_M) - g(X)||_2 &\leqq ||g(X_M) - g_N(X_M)||_2 \\ &\quad + ||g_N(X_M) - g_N(X)||_2 + ||g_N(X) - g(X)||_2 \\ &= 2 \sum_{n=N+1}^\infty a_n^2 + ||g_N(X_M) - g_N(X)||_2 .\end{aligned}$$

The first term goes to zero with increasing N. To bound the second term, it is enough to consider $g_N(X) = X^k$, $k \leqq N$. Then

$$\begin{aligned}||X_M^k - X^k||_2 &\leqq ||(X_M - X) \sum_{j=0}^k X_M^j X^{k-j}||_2 \\ &\leqq (k+1) E(X^{2k}) ||X_M - X||_2 \to 0\end{aligned}$$

which establishes the mean square convergence. Thus, the integrand in (24) can be made uniformly small, and as before, we need only estimate the tails. Without loss of generality assume $E(g(X(t))) \equiv 0$. One can easily show that as $T_0 \to \infty$

$$\int_{T_0}^{\infty} E(g(X(t))g(X(0)))\, dt \to 0$$

and

$$\int_{T_0}^{\infty} E(g(X_M(t))g(X_M(0)))\, dt \to 0 \qquad \text{uniformly in} \quad M > M_0 .$$

It remains to show that the cross terms are small. By analogy with (22), this can be reduced to showing that as $T_0 \to \infty$

$$(25) \qquad \int_{T_0}^{\infty} |\rho_{M,0}(t)|^n\, dt \to 0 \qquad \text{uniformly in} \quad M \geqq M_0 \quad \text{and} \quad n \geqq 1 .$$

Since $|\rho_{M,0}(t)| \leqq 1$, it is enough to consider $n = 1$. But

$$\int_{T_0}^{\infty} |\rho_{M,0}(t)|\, dt \leqq ||\rho_{M,0} - \rho||_1 + \int_{T_0}^{\infty} |\rho(t)|\, dt .$$

Since $\rho, \rho_{M,0} \in L_1$ and $\rho_{M,0}(t) \to \rho(t)$ pointwise, by dominated convergence the first term is small for large M; the second term vanishes as $T_0 \to \infty$.

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