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TEUGELS' RENEWAL THEOREM AND STABLE LAWS

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Let $\{S_n\}$, $n = 1, 2, \cdots$ denote the partial sums of a sequence of independent, identically distributed nonnegative random variables with common distribution function F having finite mean μ, and let $H(t) = \sum_{n=1}^{\infty} P(S_n \leqq t)$. Further, let F be nonarithmetic. It is shown in this paper that as $t \to \infty$ $H(t) - t/\mu$ is regularly varying if and only if F belongs to the domain of attraction of a stable law with exponent α, $1 < \alpha \leqq 2$.

1. Introduction. Let $\{X_n\}$, $n = 1, 2, \cdots$ be a sequence of independent, identically distributed nonnegative random variables with common distribution function (df) F and finite mean μ. Further, F is assumed to be nonarithmetic. Define $S_n = \sum_{k=1}^{n} X_k$, $N(t) = \max(n: S_n \leqq t)$, $t \geqq 0$, $H(t) = EN(t) = \sum_{k=1}^{\infty} P(S_k \leqq t)$ and $F_2(x) = \mu^{-1} \int_0^x \{1 - F(y)\}\, dy$. H is called the renewal function and F_2 is the stationary df.

When F has finite variance σ^2 then from Smith's key renewal theorem we have

$$\lim_{t\to\infty} \{H(t) - t/\mu\} = (\sigma^2 - \mu^2)/2\mu^2 \tag{1.1}$$

while if F has infinite variance Teugels (1968) established that as $t \to \infty$

$$H(t) - t/\mu \sim t^{2-\alpha} L(t)/\mu^2(\alpha - 1)(2 - \alpha) \tag{1.2}$$

assuming

$$1 - F(t) \sim t^{-\alpha} L(t) \qquad \text{as} \quad t \to \infty \tag{1.3}$$

where $1 < \alpha < 2$ and L is some slowly varying (s.v.) function at infinity. He also assumes a supplementary condition (see (8) in [6]) on $H(t)/t$ which we show to be unnecessary. A remark to this effect has also been made by Erickson (1970). Further, Teugels did not consider the case $\alpha = 2$.

In Section 2 we show that $H(t) - t/\mu$ will be regularly varying with exponent θ, $0 \leqq \theta < 1$ if and only if $F \in D(2 - \theta)$, where by $F \in D(\alpha)$ we mean that F belongs to the domain of attraction of stable law with exponent α.

2. Renewal theorems when $F \in D(\alpha)$, $1 < \alpha \leqq 2$. From Doeblin's necessary and sufficient conditions (see Feller (1966)) it is easy to deduce that $F \in D(\alpha)$, $\alpha < 2$ if it satisfies (1.3); while if $\alpha = 2$, that is, the limit law is normal, then $U(t) = \int_0^t y^2\, dF(y)$ is s.v. at ∞ or equivalently (Feller (1966))

$$\lim_{t\to\infty} U(t)^{-1} t^2 \{1 - F(t)\} = 0\,. \tag{2.1}$$

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LEMMA 2.1. *$F \in D(\alpha)$, $1 < \alpha \leqq 2$, if and only if for some* s.v. *function $L_\alpha(t)$, as $t \to \infty$*

$$B(t) = \int_0^t \{1 - F_2(y)\}\, dy \sim t^{2-\alpha} L_\alpha(t) . \tag{2.2}$$

PROOF. First let $1 < \alpha < 2$. If F satisfies (1.3), then by Theorem 1, VIII. 9 (Feller (1966)), as $t \to \infty$ we have $1 - F_2(t) \sim t^{1-\alpha}L(t)/(\alpha - 1)\mu$ and $B(t) \sim t^{2-\alpha}L(t)/(\alpha - 1)(2 - \alpha)\mu$ from which (2.2) follows upon setting $L_\alpha(t) = L(t)/(\alpha - 1)(2 - \alpha)\mu$.

Let now (2.2) hold. Since for any $a > 1$, $\lim_{t\to\infty} \int_t^{at} \{1 - F_2(y)\}\, dy/B(t) = a^{2-\alpha} - 1$ and $\int_t^{at} \{1 - F_2(y)\}\, dy \leqq (a - 1)t\{1 - F_2(t)\}$, we have $(a^{2-\alpha} - 1)/(a - 1) \leqq \liminf_{t\to\infty} t\{1 - F_2(t)\}/B(t)$. Letting $a \to 1$ we get $2 - \alpha \leqq \liminf_{t\to\infty} t\{1 - F_2(t)\}/B(t)$. Taking $a < 1$, it is not difficult to see that $(1 - a^{2-\alpha})/(1 - a) \geqq \limsup_{t\to\infty} t\{1 - F_2(t)\}/B(t)$ and hence that $2 - \alpha \geqq \limsup_{t\to\infty} t\{1 - F_2(t)\}/B(t)$ upon letting $a \to 1$. Combining the two, we have, as $t \to \infty$

$$\mu\{1 - F_2(t)\} = \int_t^\infty \{1 - F(y)\}\, dy \sim \mu(2 - \alpha)t^{1-\alpha}L_\alpha(t) . \tag{2.3}$$

Proceeding exactly as before it can be shown that (2.3) implies $1 - F(t) \sim \mu(\alpha - 1)(2 - \alpha)t^{-\alpha}L_\alpha(t)$ as $t \to \infty$ which is (1.3) with $L(t) = \mu(\alpha - 1)(2 - \alpha)L_\alpha(t)$.

Let now $\alpha = 2$. If F satisfies (2.1), then for given $\varepsilon > 0$ there exists a $T = T(\varepsilon)$ such that for all $t > T$, $1 - F(t) \leqq \varepsilon U(t)/t^2$ and hence $t \int_t^\infty \{1 - F(y)\}\, dy/\mu U(t) \leqq \varepsilon t \int_t^\infty y^{-2}U(y)\, dy/\mu U(t)$. By Theorem 1, VIII. 9 (Feller (1966)), the right side tends to ε/μ as $t \to \infty$. Since ε is arbitrary we get $\lim_{t\to\infty} t\{1 - F_2(t)\}/U(t) = 0$. In view of this and the equations

$$B(t) = \frac{1}{\mu} \int_0^t y\{1 - F(y)\}\, dy + t\{1 - F_2(t)\} \tag{2.4}$$

$$U(t) = 2 \int_0^t y\{1 - F(y)\}\, dy - t^2\{1 - F(t)\} \tag{2.5}$$

we see that (2.2) holds with $L_2(t) = U(t)/2\mu$.

To prove the converse, notice that for any $a < 1$, (2.2) implies $\lim_{t\to\infty} \int_{at}^t \{1 - F_2(y)\}\, dy/L_2(t) = 0$. Since $\int_{at}^t \{1 - F_2(y)\}\, dy \geqq (1 - a)t\{1 - F_2(t)\}$ we conclude that $\lim_{t\to\infty} t\{1 - F_2(t)\}/L_2(t) = 0$ while from (2.4) it follows that $\lim_{t\to\infty} \int_0^t y\{1 - F(y)\}\, dy/\mu L_2(t) = 1$. Thus for any $a < 1$ $\lim_{t\to\infty} \int_{at}^t y\{1 - F(y)\}\, dy/\int_0^t y\{1 - F(y)\}\, dy = 0$. Again, since $\int_{at}^t y\{1 - F(y)\}\, dy \geqq (1 - a^2)\{1 - F(t)\}t^2/2$ it is clear that $\lim_{t\to\infty} t^2\{1 - F(t)\}/\int_0^t y\{1 - F(y)\}\, dy = 0$. This and (2.5) together imply (2.1). The proof of the lemma is complete.

THEOREM 2.1. *Let $Q(t)$ be a nonnegative, nonincreasing bounded function of $t \geqq 0$ and let $0 \leqq \theta < 1$. Then as $t \to \infty$*

$$\int_0^t Q(t - y)\, dH(y) \sim t^\theta L_\theta(t)/\mu \tag{2.6}$$

if and only if as $t \to \infty$

$$\int_0^t Q(y)\, dy \sim t^\theta L_\theta(t) , \qquad 0 \leqq \theta < 1 \tag{2.7}$$

where L_θ is some s.v. *function with $L_0(t) \to \infty$ as $t \to \infty$.*

PROOF. If (2.7) holds then it is possible to find a T_0 such that for all $t > T_0$

$$Q(t) \leqq 2t^{\theta-1}L_\theta(t) . \tag{2.8}$$

For any $\varepsilon > 0$ write

$$\int_0^t Q(t-y)\,dH(y) = \{\int_0^{[\varepsilon t]} + \int_{[\varepsilon t]}^{[t]} + \int_{[t]}^t\}Q(t-y)\,dH(y) = I_\varepsilon(t) + J_\varepsilon(t) + K(t)$$

where $[x]$ denotes the greatest integer not exceeding x. Then

$$I_\varepsilon(t) \leqq Q(t-[\varepsilon t])H([\varepsilon t]) \leqq 2(t(1-\varepsilon))^{\theta-1}L_\theta(t(1-\varepsilon))H(\varepsilon t)$$

where the last inequality follows from (2.8). If $A(t) = \int_0^t Q(y)\,dy$ then by the elementary renewal theorem and (2.7) we conclude that

$$\limsup_{t\to\infty} I_\varepsilon(t)/A(t) \leqq 2\varepsilon(1-\varepsilon)^{\theta-1}/\mu . \tag{2.9}$$

Since $K(t) \leqq Q(0)\{H(t) - H(t-1)\}$, and as $t \to \infty$ $t^\varepsilon L(t) \to \infty$ for every $\varepsilon > 0$ and for any s.v. function L (see Feller (1966)) we get, using Blackwell's theorem,

$$\lim_{t\to\infty} K(t)/A(t) = 0 . \tag{2.10}$$

To estimate $J_\varepsilon(t)$, we notice that

$$\sum_{k=[\varepsilon t]}^{[t]-1} Q(t-k)\{H(k+1) - H(k)\} \leqq J_\varepsilon(t) \leqq \sum_{k=[\varepsilon t]}^{[t]-1} Q(t-k-1)\{H(k+1) - H(k)\}$$

since Q is nonincreasing. Using Blackwell's theorem it is possible to choose a $T_1 = T_1(\varepsilon)$ such that for all $t > T_1$, $1/\mu - \varepsilon \leqq H(k+1) - H(k) \leqq 1/\mu + \varepsilon$ uniformly in k, $[\varepsilon t] \leqq k \leqq [t] - 1$. Hence we have for $t > T_1$

$$(1/\mu - \varepsilon) \sum_{k=[\varepsilon t]}^{[t]-1} Q(t-k) \leqq J_\varepsilon(t) \leqq (1/\mu + \varepsilon)\{\sum_{k=[\varepsilon t]}^{[t]-1} Q(t-k) + Q(t-[t]) - Q(t-[\varepsilon t])\} . \tag{2.11}$$

Further,

$$\int_{t-[t]}^{t-[\varepsilon t]} Q(y)\,dy - Q(t-[t]) + Q(t-[\varepsilon t]) \leqq \sum_{k=[\varepsilon t]}^{[t]-1} Q(t-k) \leqq \int_{t-[t]}^{t-[\varepsilon t]} Q(y)\,dy$$

and hence we obtain

$$\int_0^{t(1-\varepsilon)} Q(y)\,dy - \int_0^1 Q(y)\,dy - Q(0) + Q(t(1-\varepsilon)+1) \leqq \sum_{k=[\varepsilon t]}^{[t]-1} Q(t-k) \leqq \int_0^{t(1-\varepsilon)+1} Q(y)\,dy .$$

Dividing throughout by $A(t)$ and using (2.7) we get

$$\lim_{t\to\infty} \sum_{k=[\varepsilon t]}^{[t]-1} Q(t-k)/A(t) = (1-\varepsilon)^\theta . \tag{2.12}$$

Diving (2.11) by $A(t)$ and using (2.12) we then have

$$(1/\mu - \varepsilon)(1-\varepsilon)^\theta \leqq \liminf_{t\to\infty} J_\varepsilon(t)/A(t) \leqq \limsup_{t\to\infty} J_\varepsilon(t)/A(t) \leqq (1/\mu + \varepsilon)(1-\varepsilon)^\theta . \tag{2.13}$$

Since $J_\varepsilon(t) \leqq \int_0^{[t]} Q(t-y)\,dH(y)$, from (2.13) we have $(1/\mu - \varepsilon)(1-\varepsilon)^\theta \leqq \liminf_{t\to\infty} \int_0^{[t]} Q(t-y)\,dH(y)/A(t)$ while from (2.9) and (2.13) we obtain

$$\limsup_{t\to\infty} \int_0^{[t]} Q(t-y)\,dH(y)/A(t) \leqq \limsup_{t\to\infty} J_\varepsilon(t)/A(t) + \limsup_{t\to\infty} I_\varepsilon(t)/A(t) \leqq (1/\mu + \varepsilon)(1-\varepsilon)^\theta + 2\varepsilon(1-\varepsilon)^{\theta-1}/\mu .$$

Letting $\varepsilon \to 0$, we have $\lim_{t\to\infty} \int_0^{[t]} Q(t-y)\,dH(y)/A(t) = 1/\mu$ from which (2.6) follows if we use (2.10); and the sufficiency of (2.7) is proved.

Suppose (2.6) holds. Writing $\int_0^t Q(t-y)\,dH(y) = \{\int_0^{[t]} + \int_{[t]}^t\} Q(t-y)\,dH(y) = I_1(t) + K(t)$ we notice, as in the proof of (2.10), that $K(t)/t^\theta L_\theta(t) \to 0$ as $t \to \infty$ and hence that

$$\lim_{t\to\infty} I_1(t)/t^\theta L_\theta(t) = 1/\mu\,. \tag{2.14}$$

If $\varepsilon > 0$ is sufficiently small, by Blackwell's theorem we obtain for some $K = K(\varepsilon)$

$$\begin{aligned}\textstyle\sum_{k=0}^{K} Q(t-k)\{H(k+1) - H(k) - (1/\mu - \varepsilon)\} + (1/\mu - \varepsilon) \sum_{k=0}^{[t]-1} Q(t-k) \\ \leqq I_1(t) \leqq \textstyle\sum_{k=0}^{K} Q(t-k-1)\{H(k+1) - H(k) - (1/\mu + \varepsilon)\} \\ + (1/\mu + \varepsilon) \textstyle\sum_{k=0}^{[t]-1} Q(t-k-1)\,.\end{aligned}$$

Dividing these inequalities by $t^\theta L_\theta(t)$ and taking the limit first as $t \to \infty$ and then as $\varepsilon \to 0$ and using (2.14) we get $\lim_{t\to\infty} \sum_{k=1}^{[t]-1} Q(t-k)/t^\theta L_\theta(t) = 1$. This is equivalent to $\lim_{t\to\infty} \int_0^t Q(y)\,dy/t^\theta L_\theta(t) = 1$ and the proof of the theorem is complete.

A direct application of this theorem is the following:

THEOREM 2.2. *As $t \to \infty$, $H(t) - t/\mu \sim t^\beta L_\beta(t)/\mu$, $0 \leqq \beta < 1$ where L_β is some* s.v. *function with $L_0(t) \to \infty$ as $t \to \infty$ if and only if $F \in D(2-\beta)$ such that when $\beta = 0$ F has infinite variance.*

PROOF. If $Q(t) = 1 - F_2(t)$ then by Lemma 11 of Teugels (1968) we have

$$H(t) - t/\mu + F_2(t) = \int_0^t Q(t-y)\,dH(y)\,. \tag{2.15}$$

Since $F_2(t) \to 1$ as $t \to \infty$ the proof of the theorem follows from Lemma 2.1 and Theorem 2.1.

REMARK 1. The connection between F and L_β is given by $\int_0^t \{1 - F_2(y)\}\,dy \sim t^\beta L_\beta(t)$, $0 \leqq \beta < 1$, as $t \to \infty$. From the proof of the theorem it is clear that if $F \in D(\alpha)$, $1 < \alpha \leqq 2$, we always have $H(t) - t/\mu \sim \int_0^t \{1 - F_2(y)\}\,dy/\mu$ as $t \to \infty$. In particular, as seen from the proof of Lemma 2.1, if F satisfies (1.3) with $1 < \alpha < 2$, this reduces to (1.2) which is Teugels' renewal theorem.

REMARK 2. The proof of Theorem 2.2 makes use of the fact that $L_0(t) \to \infty$ as $t \to \infty$ when $F \in D(2)$. Should it tend to a finite limit, in which case F has finite variance, the asymptotic value as given by the theorem, taking into account the fact that $F_2(t) \to 1$ as $t \to \infty$ in (2.15), will be the limit in (1.1). But then we have the proof of this case by Takacs (see discussion in Smith (1958)).

The following theorem establishes that (1.1) holds if and only if the variance of F is finite.

THEOREM 2.3. *As $t \to \infty$, $H(t) - t/\mu \to C$ where C is some finite constant if and only if the variance of F is finite.*

PROOF. We need only prove the "only if" part. Suppose therefore that $H(t) - t/\mu \to C$. For the special Q of Theorem 2.2 we have from (2.15),

$\int_0^t Q(t-y)\,dH(y) \to C+1$. We now show that this implies $\int_0^\infty Q(y)\,dy < \infty$. From the inequality

$$\int_0^{[t]} Q([t]-y)\,dH(y) \geqq \sum_{k=0}^{[t]-1} Q([t]-k)\{H(k+1)-H(k)\}$$

and Blackwell's theorem we obtain $\limsup_{t\to\infty} \sum_{k=0}^{[t]-1} Q([t]-k) \leqq \mu(C+1)$ using the fact that $Q(t) \to 0$ as $t \to \infty$. This implies $\int_0^\infty Q(y)\,dy < \infty$ since $\int_0^{[t]} Q(y)\,dy \leqq \sum_{k=0}^{[t]-1} Q([t]-k)+1$. But $\int_0^\infty \{1-F_2(y)\}\,dy$ is finite if and only if the variance of F is finite. The proof is complete.

REMARK 3. It is trivial to see that the elementary renewal theorem holds if and only if the mean of F is finite. We have as $t \to \infty$, $H(t) \sim (\sin \pi\alpha/\pi\alpha)t^\alpha/L(t)$, $0 < \alpha < 1$, for some s.v. function L if and only if $1-F(t) \sim t^{-\alpha}L(t)$, that is, if and only if $F \in D(\alpha)$, $0 < \alpha < 1$ (see Feller (1966)), while Theorems 2.2 and 2.3 above establish the connection between the domains of attraction of stable law with exponent α, $1 < \alpha \leqq 2$ and the asymptotic value of $H(t) - t/\mu$. The case $\alpha = 1$ is interesting. From Theorem 5 of Erickson (1970) we have $H(t) \sim t/\int_0^t \{1-F(y)\}\,dy$ as $t \to \infty$ if and only if $\int_0^t \{1-F(y)\}\,dy$ is s.v. See also Theorem 1 of Teugels (1968). A random variable with df F satisfying this property is said to be relatively stable, and the class of relatively stable positive random variables is wider than the class of random variables for which either the mean is finite or the random variable belongs to the domain of attraction of a stable law with exponent 1 (Rogozin (1971)). Hence, in this case, from the asymptotic value of $H(t)$ one cannot say whether $F \in D(1)$.

The following is an improvement of Corollary 2 of Teugels (1968) in that it not only removes his supplementary condition (see (8) in [6]) but also gives the estimate when $\alpha = 2$.

THEOREM 2.4. *If* $F \in D(\alpha)$, $1 < \alpha \leqq 2$, *then as* $t \to \infty$ $V(N(t)) \sim 2(\alpha-1)t\int_0^t \{1-F_2(y)\}\,dy/(3-\alpha)\mu^2$.

PROOF. The proof is on lines similar to that of Theorem 6 of Smith (1954) and hence is omitted.

Notice that if F satisfies (1.3) then the above estimate simplifies to $2t^{3-\alpha}L(t)/\mu^3(2-\alpha)(3-\alpha)$, which obtains Teugels' Corollary 2.

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REFERENCES

[1] ERICKSON, K. B. (1970). Strong renewal theorems with infinite mean. *Trans. Amer. Math. Soc.* **151** 263-291.

[2] FELLER, W. (1966). *An Introduction to Probability Theory and its Applications* **2**. Wiley, New York.

[3] ROGOZIN, B. A. (1971). The distribution of the first ladder moment and height and fluctuation of a random walk. *Theor. Probability Appl.* **16** 575-595.

[4] SMITH, W. L. (1954). Asymptotic renewal theorems. *Proc. Roy. Soc. Edinburgh Sect. A* **64** 9-48.

[5] SMITH, W. L. (1958). Renewal theory and its ramifications. *J. Roy. Statist. Soc. Ser. B* **20** 243–302.

[6] TEUGELS, J. L. (1968). Renewal theorems when the first or the second moment is infinite. *Ann. Math. Statist.* **39** 1210–1219.

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