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A CRITERION FOR TIGHTNESS FOR A SEQUENCE OF MARTINGALES

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An improved result is presented, showing that if the finite-dimensional distributions of a sequence of martingales converge, and if for each time t the variables are uniformly integrable, then weak convergence follows (in either C or D) provided the limiting process satisfies a certain condition; this condition is satisfied by the Wiener process.

1. Introduction and summary. Various authors (Brown (1971), Loynes (1970), McLeish (1974)) have shown that in discussing weak convergence of martingales the necessary proof of tightness is much simpler than in the general case. The essential parts of the argument, have, however, been rather concealed among the details, and it seems worth separating them out; moreover Brown and Loynes both assumed the existence of second moments (McLeish assumed a different kind of condition). Here an improved result is presented, requiring a condition on first moments only; that the limit process need not be the Wiener process is also true.

We deal with processes $X, X_1, X_2, \cdots$ in $C = C[0, 1]$ or $D = D[0, 1]$ and on the latter we put the Skorokhod J_1-topology. Each process X_n is assumed to be a (separable) martingale.

The limit process X will be assumed to have the following property:

A. Given $\varepsilon, \eta > 0 \; \exists \; \delta, 0 < \delta < 1$, for which the following is true: for every t of the form $t = i\delta$, where i is an integer, the smallest solution $\lambda \; (= \lambda(t))$ of the following inequality

(a) $\varepsilon P[|X(t + \delta) - X(t)| > \lambda] \leqq E[|X(t + \delta) - X(t)| I(|X(t + \delta) - X(t)| \geqq \lambda)]$
satisfies

(b) $\min [P(|X(t + \delta) - X(t)| \geqq \lambda), \varepsilon^{-1} E[|X(t + \delta) - X(t)| I(|X(t + \delta) - X(t)| \geqq \lambda)]] \leqq \eta\delta$.

(In most cases, and in particular if the distribution of $X(t + \delta) - X(t)$ has no atoms, (a) can be strengthened to equality, and both terms of the minimum in (b) are equal; unfortunately this would exclude the 0 function.)

In this, t is restricted to $[0, 1]$, $t + \delta$ is to be replaced by 1 if $t + \delta > 1$, and I is the indicator function of the associated set. It follows from A that (if X is separable) X is in C with probability 1; this is most easily seen from the proof of the theorem below and Theorem 15.5 of Billingsley (1968). Much the most important limit is of course the Wiener process.

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LEMMA. *The Wiener process satisfies condition* A. Clearly any process obtained from one satisfying A by distorting the time-scale nonrandomly and not too unreasonably will also satisfy A. If $Y(t)$ is the Ornstein–Uhlenbeck process (with decay parameter β) normalised to 0 at $t = 0$, then $e^{\beta t} Y(t)$ is such a distortion of the Wiener process; that this process satisfies A, and that this is also true if Y is the stationary O–U process, is an observation I owe to my colleague David Grey. Any process for which $P[|X(t+\delta) - X(t)| \geqq x] \sim \phi(\delta) x^{-\alpha}$, where $\alpha > 1$, for $x \geqq c$ and every $c > 0$, and $\phi(\delta) = o(\delta)$ also satisfies A.

The proofs of the lemma and the theorem are contained in Section 2.

THEOREM. *In either C or D, assume*

(i) *for each n, X_n is a martingale*;
(ii) $X_n \to_{\mathrm{fd.}} X$ (i.e., *finite-dimensional distributions converge*);
(iii) *for each t, $E|X_n(t)| \to E|X(t)|$*;
(iv) *X satisfies condition* A.

Then $X_n \Rightarrow_{\mathscr{D}} X$.

It may be noted that from (i) and (iii) it follows that X is a martingale.

If X_n is to be a martingale, then $EX_n(t)$ must exist; whether condition (iii) can be dropped is not obvious, but to avoid it would seem to require a better martingale inequality than that used below. The result is somewhat disappointing, in that X is necessarily in C, but the problems of really dealing with D seem very difficult.

A very special case is the result that if X_n are martingales and $E|X_n(1)| \to 0$, $X_n \Rightarrow 0$; this is of course an obvious consequence of the martingale inequality. A less trivial application is to the 'autoregressive process': $Y_i = \alpha Y_{i-1} + U_i$ for $i \geqq 1$, where $Y_0 = 0$ and U_i are independent and identically distributed with mean 0 and variance 1. Take $\alpha = 1 - \beta n^{-1}$, and define $X_n(t) = n^{-\frac{1}{2}} \alpha^{-m} Z_m$ with $m = [nt]$: then it follows easily that the conditions of the theorem are satisfied (second moments are easily shown to be bounded, implying (iii)) and so $X_n \Rightarrow X$, where $e^{-\beta t} X$ is the O–U process starting from 0. (The final step, to conclude that $Y_n \Rightarrow e^{-\beta t} X$, where $Y_n(t) = n^{-\frac{1}{2}} Z_{[nt]}$, is also straightforward, but not relevant to the present paper.)

For the following corollary, let W_{μ,σ^2} denote a Wiener process with drift μ and scale parameter σ^2. (The idea of the corollary, and its proof, are due to W. J. Hall.)

COROLLARY. *Let X_n be processes in either C or D, satisfying*

(i) *$EX_n(t) \equiv \mu_n(t)$ is monotone nondecreasing for every n*;
(ii) *$X_n - \mu_n$ is a martingale for every n*;
(iii) *$X_n(t)$ are uniformly integrable for each t*;
(iv) $X_n \to_{\mathrm{fd.}} W_{\mu,\sigma^2}$.

Then $X_n \Rightarrow_{\mathscr{D}} W_{\mu,\sigma^2}$.

Conditions (i) (in which nondecreasing can be changed to nonincreasing) and (ii) imply that X_n is a submartingale of a very simple type; whether they can be relaxed usefully is not known. The proof is straightforward.

The following final result indicates an often convenient way of showing that condition (iii) is satisfied.

PROPOSITION. *If* $U_n \Rightarrow_{\mathscr{D}} U$ *and* $\sup \operatorname{Var} U_n < \infty$, *then* $\sup EU_n^2 < \infty$, $E|U_n|^\alpha \to E|U|^\alpha < \infty$ *for* $0 \leqq \alpha < 2$, *and* $EU_n \to EU$.

This form is likely to be more useful than any other, but generalization is possible: e.g., if $Eg(|U_n - EU_n|)$ is bounded, where g is nondecreasing and $g(x)/x \to \infty$, then $E|U_n|^\alpha \to E|U|^\alpha$ for $0 \leqq \alpha \leqq 1$ and $EU_n \to EU$.

PROOF. Suppose EU_n is not bounded: then there exists a subsequence along which $|EU_n| \to \infty$. Since $\{|U_n| \leqq A\} \subset \{|U_n - EU_n| \geqq |EU_n| - A\}$ when $|EU_n| > A$, Chebyshev's inequality shows that $P[|U_n| \leqq A] \to 0$ along this subsequence; a contradiction. Everything follows.

2. Proofs.

(1) PROOF OF LEMMA. In the present case $X(t + \delta) - X(t) = \delta^{\frac{1}{2}}N$, where N is a standard normal deviate. Its distribution has no atoms, and it follows easily that if δ is so small that $\delta^{\frac{1}{2}}E|N| < \varepsilon$ we may assume equality in A(a); thus we seek a solution of $\varepsilon P[|N| > \lambda\delta^{-\frac{1}{2}}] = 2\delta^{\frac{1}{2}}(\lambda\delta^{-\frac{1}{2}})$. Using the approximation

$$\frac{2\phi(A)}{A(1 + A^{-2})} < P[|N| \geqq A] < \frac{2\phi(A)}{A}$$

it is easy to see that if $\delta^{\frac{1}{2}} < \frac{1}{2}\varepsilon$ there exists a solution such that $\frac{1}{2}(\varepsilon + (\varepsilon^2 - 4\delta)^{\frac{1}{2}}) < \lambda < \varepsilon$. Then, if $x = \varepsilon\delta^{-\frac{1}{2}}$

$$\frac{1}{\delta x}\phi(\lambda\varepsilon^{-\frac{1}{2}}) \leqq \frac{1}{\varepsilon^2} x\phi\left(\frac{x + (x^2 - 4)^{\frac{1}{2}}}{2}\right) \to 0$$

as $x \to \infty$, so that δ can be chosen small enough for A(b) to be satisfied as well.

(2) PROOF OF THEOREM. Assume that all the conditions are satisfied, and from now on let δ be fixed so that A holds; for convenience write $A_n = \{\sup_{t \leqq s < t+\delta} |X_n(s) - X_n(t)| \geqq \varepsilon\}$, and $Z_n = |X_n(t + \delta) - X_n(t)|$, and similarly $Z = |X(t + \delta) - X(t)|$.

The main part of the proof consists in showing that $P(A_n) < 2\eta\delta$ for sufficiently large n, for a single t, by using the standard submartingale inequality (Doob (1953), page 353) $P(A_n) \leqq \varepsilon^{-1} \int_{A_n} Z_n$.

Define ζ_n as a solution (e.g., the smallest) of the inequalities $P(Z_n > \zeta_n) \leqq P(A_n) \leqq P(Z_n \geqq \zeta_n)$: then $P(Z_n > \zeta_n) \leqq \varepsilon^{-1}E|Z_n I(A_n)| \leqq \varepsilon^{-1}E[Z_n I(Z_n \geqq \zeta_n)]$. Let $\zeta = \liminf \zeta_n$: then there exists a subsequence of ζ_n converging to ζ, and it is easy to show that $P[Z > \zeta| \leqq \varepsilon^{-1}E|ZI(Z \geqq \zeta)]$. Thus $\zeta \geqq \lambda$, where λ is defined by A(a).

Consequently $\limsup P(A_n) \leqq \limsup P[Z_n \geqq \zeta_n| \leqq P|Z \geqq \lambda]$ and also

$\limsup P(A_n) \leqq \limsup \varepsilon^{-1}E[Z_n I(Z_n \geqq \zeta_n)] \leqq \varepsilon^{-1}E[ZI(Z \geqq \lambda)]$, and the promised result follows.

The proof of the theorem is now completed by applying Theorems 8.3 and 15.5 of Billingsley; inspection of the proof shows that only (finitely many) values of t, of the form $i\delta$, need be considered.

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