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A REPRESENTATION THEOREM ON STATIONARY GAUSSIAN PROCESSES AND SOME LOCAL PROPERTIES[1]

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Let $X(t, \omega)$, $a \leqq t \leqq b$, $\omega \in \Omega$ be a real continuous stationary Gaussian process with mean 0 and covariance R. We prove that there exist analytic functions f_n defined on $[a, b]$ and independent random variables X_n $N(0, 1)$, $n = 0, 1, 2, \cdots$, such that the series $\sum_{n=0}^{\infty} f_n(t)X_n$ converges uniformly to $X(t)$ with probability 1. Among other applications of this representation theorem, we show that if the second spectral moment is infinite and $\int_0^\delta (R(0) - R(t))^{-\frac{1}{2}}\, dt < \infty$ for some $0 < \delta \leqq b - a$, then for any given $u \in \mathbb{R}$, $P\{\omega \mid X_\omega^{-1}(u)$ is infinite$\} > 0$.

1. Introduction. Let $X(t, \omega)$, $a \leqq t \leqq b$, $\omega \in \Omega$, be a real continuous Gaussian process with mean $EX(t) \equiv 0$ and covariance $R(s, t) = E(X(s)X(t))$. It is known (see, for instance, Dudley (1973), Theorem 0.3) that for every orthonormal basis $\{X_n\}$ of the linear span of $\{X(t) \mid t \in [a, b]\} \subset L^2(\Omega, P)$, the series $\sum (X(t), X_n)X_n$ converges uniformly to $X(t)$ with probability 1. In particular, the Karhunen–Loève expansion of a Gaussian process is just a special case of this orthogonal series.

For stationary processes, we show another special case of this orthogonal series where the functions $f_n(t) = (X(t), X_n)$ are analytic. The way to do this is to exploit the congruence between $L^2(X(t) \mid t \in [a, b])$ and the reproducing kernel Hilbert space $H(R)$.

With this representation we prove some local properties of real continuous Gaussian stationary processes with mean 0, covariance $R(h)$, spectral measure μ and second spectral moment $\lambda_2 = \int_{-\infty}^{\infty} \lambda^2\, d\mu(\lambda) = \infty$. In particular we show that all four Dini derivates are infinite almost everywhere with probability 1. This implies the nondifferentiability almost everywhere with probability 1 shown by Geman and Horowitz (1973).

Next, we apply this representation and these local properties to the following problem in level crossings.

Let u be any real number and define $C(u) = \#\{t \mid X(t) = u\}$, where $\#$ denotes "cardinality of," and is interpreted as ∞ when the set is infinite. $C(u)$ is a random variable and it is known that $EC(u) < \infty$ if and only if $\lambda_2 < \infty$. (See, for instance, Cramér–Leadbetter ((1967), page 195). Dudley ((1973), Section 8.2) asks if it can happen that $C(u) < \infty$ almost surely and $EC(u) = \infty$.

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We answer this question with "no" for processes satisfying the following condition: there exists $0 < \delta \leqq b - a$ such that $\int_0^\delta (R(0) - R(t))^{-\frac{1}{2}}\, dt < \infty$.

Kahane (1968), page 146), Orey (1970), Berman (1972) approached this problem by considering the Hausdorff dimension of the set $\{t: X_\omega(t) = u\}$. Kahane studied Gaussian–Fourier series satisfying the above condition and showed that in certain cases, $\dim\{t: X_\omega(t) = u\} > 0$ for a u-set of positive measure, with a positive probability. Orey and Berman showed that for some ergodic stationary Gaussian processes $X(t)$, $t \geqq 0$, with $\sigma^2(t) = E|X(t) - X(0)|^2 \sim C|t|^\alpha$ for some constant C and some α; $0 < \alpha < 2$, $\dim\{t: X_\omega(t) = u\} = 1 - \alpha/2$ for all u, almost surely.

We only use this condition to assure the almost sure absolute continutity of the occupation time distribution (defined in Section 4) of a certain process depending on $X(t)$. If Orey's conjecture is true for processes with differentiable mean (in fact, C^∞ mean is enough), then this condition is not needed and the proof works for all stationary Gaussian process with $\lambda_2 = \infty$.

2. The representation theorem.

THEOREM 1. *Let $X(t, \omega)$, $a \leqq t \leqq b$, $\omega \in \Omega$, be a real continuous stationary Gaussian process with mean* 0 *and covariance $R(h)$. Then there exist analytic functions f_n defined on $[a, b]$ and independent random variables $X_n N(0, 1)$, $n = 0, 1, 2, \cdots$, such that the series $\sum_{n=0}^\infty f_n(t) X_n$ converges uniformly to $X(t)$ with probability* 1.

PROOF. By Krein's theorem, there exists a bounded symmetric positive measure μ on $\mathbb{R}$ (in general, not unique) such that $R(h) = \int_{-\infty}^\infty e^{ixh}\, d\mu(x)$, $0 \leqq h \leqq b - a$. In particular, $R(h)$ can be extended to $\mathbb{R}$ still as a continuous covariance function of a stationary Gaussian process defined on $\mathbb{R}$.

R being real, we have

$$R(s, t) = R((s - t)) = \int_{-\infty}^\infty [\cos(sx) + \sin(sx)][\cos(tx) + \sin(tx)]\, d\mu(x)\,, \qquad -\infty < s, t < \infty\,.$$

By Theorem 4D (Parzen (1959)), the reproducing kernel Hilbert space $H(R)$ can be represented by the space of functions g, defined on $\mathbb{R}$, of the form $g(t) = \int_{\mathbb{R}} g^*(x)[\cos(tx) + \sin(tx)]\, d\mu(x)$ for some (necessarily unique) function g^* in the Hilbert subspace $L^2(\cos(tx) + \sin(tx); t \in \mathbb{R})$ of $L^2(\mathbb{R}, \mathscr{B}, \mu)$ spanned by the functions $\{\cos(tx) + \sin(tx); t \in \mathbb{R}\}$. The norm of g is given by $||g||^2 = \int_{\mathbb{R}} |g^*(x)|^2\, d\mu(x)$.

As $L^2(\cos(tx) + \sin(tx); t \in \mathbb{R}) = L^2(\mathbb{R}, \mathscr{B}, \mu)$ and the functions $g^*(x) \in L^2(\mathbb{R}, \mathscr{B}, \mu)$ of compact support are dense in $L^2(R, \mathscr{B}, \mu)$, so the set $\{g \in H(R) \mid g^*$ has compact support$\}$ is dense in $H(R)$. And, every function in this set is analytic.

By Theorem 6C (Parzen (1959), the reproducing kernel Hilbert space of R, restricted to the interval $[a, b] \times [a, b]$, consists of all functions h defined on $[a, b]$ which are restrictions of functions g belonging to $H(R)$. Furthermore $||h|| \leqq ||g||$ (the norms taken in their respective spaces).

Denoting also by $H(R)$, the reproducing kernel Hilbert space of R restricted to $[a, b] \times [a, b]$, it follows, then, that the set $\{g \in H(R) \mid g \text{ is analytic}\}$ is dense in $H(R)$.

$H(R)$ is a separable Hilbert space since the process is continuous in quadratic mean (Parzen (1959), Theorem 2C). So, there exists an orthonormal basis of analytic functions f_n of $H(R)$.

As $H(R)$ is congruent to $L^2(X(t); t \in [a, b])$, $X(t) = \sum_{n=0}^{\infty} f_n(t) X_n$ in $L^2(X(t), t \in [a, b])$. Therefore, by a theorem (see Dudley (1973), Theorem 0.3), the result follows.

REMARK 1. With the same hypothesis as in the theorem, but with $t \in \mathbb{R}$, the same argument shows the existence of analytic functions f_n defined on $\mathbb{R}$ and independent random variables X_n with law $N(0, 1)$, belonging to $L^2(X(t); t \in \mathbb{R})$, $n = 0, 1, 2, \cdots$ such that $X(t) = \sum_{n=0}^{\infty} f_n(t) X_n$ in $L^2(X(t); t \in \mathbb{R})$.

3. Some differentiability properties.

LEMMA (Zero-one law). *Let X_n, $n = 0, 1, 2, \cdots$, be $N(0, 1)$ independent random variables. Let f_n, $n = 0, 1, 2, \cdots$, be differentiable real functions defined on $[a, b]$ such that for every $t \in [a, b]$, $\sum_{n=0}^{\infty} f_n^2(t) < \infty$. Suppose $X(t) = \sum_{n=0}^{\infty} f_n(t) X_n$ is continuous with probability 1. Then for fixed t,*

$$P\{\omega : \limsup_{h \downarrow 0} [X(t+h) - X(t)]/h = +\infty\} = 0 \quad or \quad 1 .$$

PROOF.

$$\begin{aligned} P\{\omega &: \limsup_{h \downarrow 0} [X(t+h) - X(t)]/h = +\infty] \\ &= P\{\omega : \limsup_{h \downarrow 0} \textstyle\sum_{n=0}^{\infty} X_n [f_n(t+h) - f(t)]/h = +\infty\} \\ &= P\{\omega : \limsup_{h \downarrow 0} \textstyle\sum_{n=j}^{\infty} X_n [f_n(t+h) - f(t)]/h = +\infty\} \end{aligned}$$

since the f_n's are differentiable.

So the event $\{\omega : \limsup_{h \downarrow 0} \sum_{n=0}^{\infty} X_n [f_n(t+h) - f_n(t)]/h = +\infty\}$ is a tail event. So by Kolmogorov's zero-one law, its probability is 0 or 1.

THEOREM 2. *Let $X(t, \omega)$, $a \leqq t \leqq b$, $\omega \in \Omega$, be a real, continuous, stationary Gaussian process with mean 0 and covariance $R(h)$. Then if the second spectral moment λ_2 is infinite, for fixed t, $P\{\omega : \limsup_{h \downarrow 0} [X(t+h) - X(t)]/h = +\infty\} = 1$.*

PROOF. By the zero-one law, it's enough to prove that

$$P\{\omega : \limsup_{h \downarrow 0} [X(t+h) - X(t)]/h < \infty\} < 1 .$$

Fix $\beta > 0$ and consider the events

$$A_n(\beta) = \{\omega : -\infty < X(t+h) - X(t) < \beta h \ \text{ if } \ 0 < h \leqq 1/n\} .$$

The $A_n(\beta)$'s form an increasing sequence and $A(\beta) = U_n A_n(\beta)$ contains $\{\omega : \limsup_{h \downarrow 0} [X(t+h) - X(t)]/h \leqq \beta/2\}$.

$$\begin{aligned} P(A_n(\beta)) &\leqq P\{\omega : -\infty < X(t + 1/n) - X(t) < \beta/n\} \\ &= \Phi(\beta/n[2(R(0) - R(1/n))]^{\frac{1}{2}}) \end{aligned}$$

since $X(t+1/n)-X(t)$ is $N(0, 2(R(0)-R(1/n)))$ where

$$\Phi(x)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x}\exp(-x^2/2)\,dx\,;$$

which implies $P(A(\beta))=\lim_{n\to\infty}P(A_n(\beta))\leqq\frac{1}{2}$ since $\lambda_2=\lim_{h\to 0}2[R(0)-R(h)]/h^2=\infty$.

As this is true for every $\beta>0$, we get the result.

Let us denote $D^{\pm}X_\omega(t)=\limsup_{h\to 0^{\pm}}[X_\omega(t+h)-X_\omega(t)]/h$ and $D_{\pm}X_\omega(t)=\liminf_{h\to 0^{\pm}}[X_\omega(t+h)-X_\omega(t)]/h$.

COROLLARY. *With $X(t)$ as in Theorem 2, for each t fixed,*

$$(*)\qquad D^{+}X_\omega(t)=D^{-}X_\omega(t)=-D_{+}X_\omega(t)=-D_{-}X_\omega(t)=+\infty\quad \text{a.s.}$$

PROOF. By symmetry of the process, we can apply Theorem 2 to $-X(t)$, $X(-t)$ and $-X(-t)$ obtaining the result.

REMARK 2. Since the Dini derivates $D^+X_\omega(t)$, $D^-X_\omega(t)$, $D_+X_\omega(t)$ and $D_-X_\omega(t)$ are jointly measurable (due to the continuity in t of $X(t,\omega)$), it follows from Fubini's theorem and the corollary, that with probability one, (*) holds for λ-a.e. $t\in[a,b]$. In particular, $X(t,\omega)$ is nondifferentiable a.e. a.s.

4. On level crossings. Let $f(t)$, $a\leqq t\leqq b$, be a real-valued measurable function.

DEFINITION. f is said to be T_1 if $f^{-1}(u)$ is a finite set for almost every $u\in\mathbb{R}$.

DEFINITION. $t\in[a,b]$ is an oscillation point of f if for every $\varepsilon>0$, there exist $t_1, t_2\in(t-\varepsilon, t)$ such that $f(t_1)>f(t)>f(t_2)$ or there exist $t_3, t_4\in(t, t+\varepsilon)$ such that $f(t_3)>f(t)>f(t_4)$.

Let λ be Lebesgue measure on $\mathbb{R}$. For every measurable set $E\subset\mathbb{R}$, define $\nu(E)=\lambda(f^{-1}(E))$. As it is easily verified, ν is a measure on $\mathbb{R}$. ν is called the *occupation time distribution* (O.T.D.) of f. If ν is absolutely continuous with respect to λ, we write $\nu\ll\lambda$.

Let $A(f)=\{t\in[a,b]\mid D^+f(t)=-D_+f(t)=+\infty$ or $D^-f(t)=-D_-f(t)=+\infty\}$. Let $B=f(A(f))$. We remark that every $t\in A(f)$ is an oscillation point of f.

LEMMA. *Let $f(t)$, $a\leqq t\leqq b$, be a real-valued continuous function. Suppose f is not T_1 and $\lambda(A(f))=b-a$. Then $\lambda(B)>0$.*

PROOF. Let $E=\{u\in\mathbb{R}\mid f^{-1}(u)$ is infinite$\}$, $A=\{t\in[a,b]\mid t$ is an oscillation point of $f\}$, $M=\{t\in[a,b]\mid t$ is a point of relative maximum (minimum) and an accumulation point of points of relative maximum (minimum) with the same value of $f\}$ and $N=\{t\in[a,b]\mid$at least one Dini derivate is finite$\}$.

As $A(f)=(A(f)\cap N)\cup(A(f)\cap N^c)$, $A\sim A(f)=\{t\in A\mid t\notin A(f)\}\subset A\cap N\subset N$. By the Denjoy relations (Saks (1964), page 271), $\lambda(A(f)\cap N)=0$, so $b-a\geqq\lambda(N^c)\geqq\lambda(A(f)\cap N^c)=\lambda(A(f))=b-a$. Thus, by Theorem (4.6) (Saks (1964), page 271), $\lambda(f(N))=0$ which implies $\lambda(f(A\sim A(f)))=0$.

$\lambda(E\sim B)\leqq\lambda(f(A\sim A(f))+\lambda(f(M))=0$ since f is continuous and $E=f(A\cup M)=f(A)\cup f(M)$.

As $B \subset E$, $\lambda(B) = \lambda(E) > 0$ since f is not T_1.

THEOREM 4. *Let $X(t, \omega)$, $a \leqq t \leqq b$, $\omega \in \Omega$, be a real continuous Gaussian process with mean 0 and covariance $R(s, t)$. Suppose $X(t)$ can be represented almost surely as $X(t) = f(t)X_0 + Y(t)$ where $f(t)$ is differentiable and strictly positive on $[a, b]$, X_0 is $N(0, 1)$ and X_0 is independent of the family $\{Y(t); t \in [a, b]\}$. Suppose also that $\lambda(A(X_\omega)) = b - a$ almost surely. Then if $R(t, t) > 0$ for $t > a$ and*

$$\int_a^b \int_a^b [R(s, s)R(t, t) - R(s, t)^2]^{-\frac{1}{2}}\, ds\, dt < \infty ,$$

for any given $u \in R$, $X^{-1}(u)$ is infinite with positive probability.

PROOF. Let $u \in \mathbb{R}$ and define $Z(t) = (u - X(t))/f(t)$. $Z(t)$ is a real continuous Gaussian process with mean $m(t) = u/f(t)$ and covariance $\eta(s, t) = R(s, t)/f(s)f(t)$. Furthermore, $\eta(t, t) > 0$ for $t > a$ and

$$\int_a^b \int_a^b [\eta(s, s)\eta(t, t) - \eta(s, t)^2]^{-\frac{1}{2}}\, ds\, dt < \infty .$$

Then by Theorem 2 of Orey (1970), Z_ω has an absolutely continuous O.T.D. with respect to Lebesgue measure.

It follows easily from the definition of a T_1-function and the fact that Z_ω is continuous that Z_ω is not T_1 a.s. since $A(Z_\omega) = A(X_\omega) \Rightarrow \lambda(A(Z_\omega)) = b - a > 0$ a.s.

Let $g_\omega(t) = (u - Y(t, \omega))/f(t) = Z(t, \omega) + X_0(\omega)$. $g_\omega(t)$ is not T_1 a.s. and $\lambda(A(g_\omega)) = b - a$ a.s.

We can consider the probability space Ω as being a product space $\Omega_0 \times \Omega_1$ where X_0 is defined on Ω_0, $X_0(\omega_0, \omega_1) = X_0(\omega_0)$, $Y(t)$ is defined on Ω_1 for every t, $Y(t, \omega_0, \omega_1) = Y(t, \omega_1)$ and $P = P_0 \times P_1$, since X_0 is independent of the family $\{Y(t); t \in [a, b]\}$.

Let $B(\omega_1) = g_{\omega_1}(A(g_{\omega_1}))$. By the lemma, $\lambda(B(\omega_1)) > 0$ a.s. ω_1.

Let $\Omega_u = \{\omega = (\omega_0, \omega_1) \mid X_\omega^{-1}(u)$ is infinite$\}$. Ω_u is a measurable set (Cramér–Leadbetter (1967), page 195). We want to show

$$P(\Omega_u) = \int_{\Omega_1} \int_{\Omega_0} 1_{\{\omega=(\omega_0,\omega_1) \mid X_\omega^{-1}(u) \text{ is infinite}\}}\, dP_0(\omega_0)\, dP_1(\omega_1) > 0 .$$

For fixed ω_1, $\{\omega_0 \mid X_0 \in B(\omega_1)\} \subset \{\omega_0 \mid X^{-1}_{(\omega_0,\omega_1)}(u)$ is infinite$\}$ since $X_0(\omega_0) \in B(\omega_1) \Rightarrow$ there exists $t \in A(g_{\omega_1})$ such that $X_0(\omega_0) = g_{\omega_1}(t)$. So $u = f(t)X_0(\omega_0) + Y(t, \omega_1) = X(t, \omega_0, \omega_1)$. But $t \in A(g_{\omega_1}) \Rightarrow t \in A(Y_{\omega_1}) \Rightarrow t \in A(X_{(\omega_0,\omega_1)}) \Rightarrow X^{-1}_{(\omega_0,\omega_1)}(u)$ is infinite.

So $P(\Omega_u) \geqq \int_{\Omega_1} P_0(X_0 \in B(\omega_1))\, dP_1(\omega_1) > 0$ since $P_0(X_0 \in B(\omega_1)) > 0$ a.s. ω_1 because X_0 is $N(0, 1)$ and $\lambda(B(\omega_1)) > 0$ a.s. ω_1.

We now apply this to the stationary case.

THEOREM 5. *Let $X(t, \omega)$, $a \leqq t \leqq b$, $\omega \in \Omega$ be a real continuous stationary Gaussian process with mean 0, covariance $R(h)$ and second spectral moment infinite, and suppose there exists $0 < \delta \leqq b - a$ such that $\int_0^\delta (R(0) - R(t))^{-\frac{1}{2}}\, dt < \infty$. Then for any given $u \in R$, $P\{\omega \mid X_\omega^{-1}(u)$ is infinite$\} > 0$.*

PROOF. Let $[c, d]$ be such that $0 < d - c \leqq \delta$, $R(0) + R(t) > 0$ for $0 \leqq t \leqq d - c$ and the function f_0 given by Theorem 1 is strictly positive on $[c, d]$. There is no loss of generality in considering f_0 strictly positive on some $[c, d] \subset [a, b]$ since if f_0 is strictly negative on $[a, b]$, we can always replace it by $-f_0$ and X_0 by $-X_0$.

It follows that $\int_c^d \int_c^d (R(0)^2 - R(s-t)^2)^{-\frac{1}{2}}\, ds\, dt < \infty$. So in view of Theorems 1, 2 and Remark 2, we can apply Theorem 4 for $X(t)$, $c \leqq t \leqq d$. So $P\{\omega : X_\omega^{-1}(u)$ is infinite$\} > 0$.

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