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ERGODIC THEOREMS FOR AN INFINITE PARTICLE SYSTEM WITH BIRTHS AND DEATHS

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Let $p(x, y)$ be an irreducible symmetric transition function for a Markov chain on a countable set S. Let η_t be the infinite particle system on S with simple exclusion interaction modified to allow the spontaneous creation and destruction of particles in the system. A complete characterization of the invariant probability measures for this system is obtained in the case where the exponential rates of creation and destruction are independent of the configuration of the system.

Furthermore, if $\mathscr{M}$ is the set of probability measures on the state space of η_t and $S(t)$ is the semigroup on $\mathscr{M}$ determined by

$$S(t)\mu(A) = \int P^{\eta}[\eta_t \in A]\, d\mu(\eta)$$

theorems concerning the weak convergence of $S(t)\mu$ to the invariant measures of η_t are proved.

1. Introduction. Let $p(x, y)$ be an irreducible symmetric probability transition function for a Markov chain on a countable set S and let $X = \{0, 1\}^S$ with the product topology. Let $C(X)$ be the space of continuous real valued functions on X with the sup norm and let $\mathscr{F}$ be the functions in $C(X)$ which depend on only finitely many coordinates. Following Spitzer (1970) an infinite particle system on S with simple exclusion interaction is defined to be the strong Markov process with state space X whose infinitesimal generator is given for $f \in \mathscr{F}$ by

$$\Gamma f(\gamma) = \sum_{\gamma(x)=1,\gamma(y)=0} p(x, y)[f(\gamma_{xy}) - f(\gamma)]\,,$$

where

$$\begin{aligned} \gamma_{xy}(u) &= \gamma(u) && \text{if} \quad u \neq x\,, \quad u \neq y \\ &= \gamma(y) && \text{if} \quad u = x \\ &= \gamma(x) && \text{if} \quad u = y\,. \end{aligned}$$

Each $\gamma \in X$ is interpreted as a configuration of particles on S where site x is occupied by a single particle if $\gamma(x) = 1$ and is vacant if $\gamma(x) = 0$ and where γ_{xy} represents a transition between x and y if $\gamma_{xy} \neq \gamma$. With this interpretation the simple exclusion process is a system of indistinguishable particles moving on S in the following manner. A particle at site x waits a random time (exponentially distributed with mean one) then chooses to jump to y with probability $p(x, y)$. If site y is vacant the jump is executed; otherwise the jump is suppressed. The particles move independently except for the interaction which suppresses jumps to occupied sites. A proof that the closure of Γ is the generator of a unique strongly continuous semigroup $U(t)$ of contractions on $C(X)$ and hence of a

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strong Markov process γ_t on X is given by Liggett (1972). We define this γ_t to be the simple exclusion process on X with one particle motion determined by $p(x, y)$.

Let $\mathscr{M}$ be the set of probability measures on X. Then $U(t)$ can be used to define a semigroup on $\mathscr{M}$ by

$$U(t)\mu(A) = \int_X U(t)1_A(\gamma)\, d\mu(\gamma) = \int_X P^\gamma[\gamma_t \in A]\, d\mu(\gamma) ,$$

hence $U(t)\mu$ is the distribution of the system at time t when the system has initial distribution μ. A probability measure μ is said to be invariant for the process γ_t if $U(t)\mu = \mu$ for all t. Let J be the set of invariant probability measures for γ_t. Then Spitzer (1974) and Liggett (1973, 1974) have characterized J and have given necessary and sufficient conditions on the initial distribution, μ, to have weak convergence of $U(t)\mu$ to an extreme point of J. (See Theorem (6.1).) In this paper we modify the simple exclusion process to allow the spontaneous creation and destruction of particles at sites in S and investigate the effects this modification has on the invariant probability measures of the model. For example suppose $S = Z^d$ and $p(x, y)$ is the transition function for a simple random walk on Z^d, i.e., $p(x, y) = \frac{1}{2}d$ if $|x - y| = 1$. Spitzer and Liggett have shown that the extreme invariant measures of γ_t are exactly the collection of product measures $\{\nu_\alpha\}_{0 \leqq \alpha \leqq 1}$ where $\nu_\alpha\{\eta \mid \eta(x) = 1\} = \alpha$. Let $F \subset Z^d$ be a finite set and suppose whenever site $x \in F$ is empty a particle is created at site x at an exponential rate $\beta(x) > 0$. Suppose also that the new particles satisfy the same laws of motion as the original particles in the system. In the case $d = 1$ or $d = 2$ the only invariant measure for the system is pointmass on $\eta \equiv 1$ (i.e., all sites are occupied) and from any initial distribution the distribution of the system at time t converges to pointmass on $\eta \equiv 1$. When $d > 2$ the mechanism of creation of particles at finitely many sites is not sufficient to eliminate the effect of the initial distribution. In this case there is a one parameter family $\{\theta_\alpha\}_{0 \leqq \alpha \leqq 1}$ of extreme invariant probability measures where each θ_α is asymptotically like the product measure ν_α.

The process described above for $d \leqq 2$ is ergodic in the sense that there exists a unique invariant probability measure ν such that from any initial distribution the distribution of the process converges to ν. A main result of this paper gives necessary and sufficient conditions for the ergodicity of the modified simple exclusion process when the rates of creation and destruction depend only on the site of the particle created or destroyed.

Further results include a characterization of the invariant measures when the modified simple exclusion process is nonergodic and the rates of creation and destruction depend only on the site of the created and destroyed particle (e.g., case $d > 2$ above). In this case necessary and sufficient conditions are given for the convergence of the system to an extreme invariant measure. The results in the nonergodic case are parallel to the results of Liggett (1973, 1974).

In order to specify the modified simple exclusion process let $\beta(x, \eta) \geqq 0$ and

$\delta(x, \eta) \geqq 0$ be respectively the exponential creation and destruction rates at site x whenever the process is in configuration η. This means that when the process is in configuration η particles will be created at vacant sites x during $[0, t]$ with probability $t\beta(x, \eta) + o(t)$ and will be destroyed at filled sites u during $[0, t]$ with probability $t\delta(u, \eta) + o(t)$. For $f \in \mathscr{F}$ the infinitesimal generator of the process is given by

$$(1.1) \qquad \begin{aligned} \Omega f(\eta) = {} & \textstyle\sum_{\eta(x)=1, \eta(y)=0} p(x, y)[f(\eta_{xy}) - f(\eta)] \\ & + \textstyle\sum_{\eta(x)=0} \beta(x, \eta)[f(\eta_x) - f(\eta)] \\ & + \textstyle\sum_{\eta(x)=1} \delta(x, \eta)[f(\eta_x) - f(\eta)] , \end{aligned}$$

where $\eta_x(u) = \eta(u)$ if $u \neq x$ and $\eta_x(u) = 1 - \eta(u)$ if $u = x$. Then η_x represents a birth at site x if $\eta(x) = 0$ and a death at site x if $\eta(x) = 1$. The proof that the closure of Ω in $C(X)$ is the infinitesimal generator of a unique strong Markov process η_t on X is an application of Liggett's existence criteria in [8] under the following assumptions:

(i) $\sup_u \sup_\eta (\beta(u, \eta) + \delta(u, \eta)) < \infty$,

(ii) $\sup_u \sum_x \sup_\eta |\beta(u, \eta_x) - \beta(u, \eta)| < \infty$, $\sup_u \sum_x \sup_\eta |\delta(u, \eta_x) - \delta(u, \eta)| < \infty$.

We take η_t to be the desired modification of the simple exclusion process and will refer to it as a $\beta - \delta$ process. We note that the $\beta - \delta$ process exists whenever (i) holds and $\beta(u, \eta)$ and $\delta(u, \eta)$ depend on at most $M < \infty$ coordinates of η where M can be chosen independently of u. If $M = 0$ we write $\beta(x, \eta) = \beta(x)$ and $\delta(x, \eta) = \delta(x)$.

REMARK. A basic assumption of this paper is the symmetry of $p(x, y)$ in x and y. The existence of the $\beta - \delta$ process can be proved under weaker assumptions (i.e., $p(x, y)$ is doubly stochastic) but the main results require the symmetry of $p(x, y)$ as in Spitzer (1974) and Liggett (1973, 1974).

Denote the semigroup on $C(X)$ associated with η_t by $S(t)$ and as usual let $S(t)$ determine a semigroup on $\mathscr{M}$ by

$$S(t)\mu(A) = \int_X P^\eta[\eta_t \in A]\, d\mu(\eta) .$$

A probability measure ν is invariant for η_t if $S(t)\nu = \nu$ for all t. The process η_t will be called ergodic if there exists a unique invariant measure ν and $\lim_{t\to\infty} S(t)\mu = \nu$ for all $\mu \in \mathscr{M}$. Let X_t be a continuous time Markov chain on S with transition function $p(x, y)$ and exponential waiting times of mean one.

(1.2) THEOREM. *Suppose* $\beta(x, \eta) = \beta(x)$ *and* $\delta(x, \eta) = \delta(x)$ *are independent of* η *and satisfy* $\sup_x [\beta(x) + \delta(x)] < \infty$. *Then* η_t *is ergodic if and only if*

(iii) $P^x[\int_0^\infty \beta(X_t) + \delta(X_t)\, dt = \infty] = 1$ *for all* $x \in S$.

REMARK. Condition (iii) is satisfied whenever X_t is recurrent and $\sup_x [\beta(x) + \delta(x)] > 0$. If $\beta(x) \equiv 0$ (no births) then point mass on $\eta \equiv 0$ is invariant and hence, under the hypothesis of Theorem (1.2), must be the unique invariant measure.

Similarly if $\delta(x) \equiv 0$ (no deaths) then pointmass on $\eta \equiv 1$ is the unique invariant measure. The proof of Theorem (1.2) is in Section 4.

In order to state Theorems (1.3) and (1.4) we need the following notation. Define

$$\mathscr{H} = \{\alpha(\cdot) \text{ on } S \mid 0 \leqq \alpha(x) \leqq 1 \text{ and } \textstyle\sum_y p(x, y)\alpha(y) = \alpha(x) \text{ for each } x \in S\}.$$

For each $\alpha \in \mathscr{H}$ let ν_α be the product measure on X whose values on cylinder sets is given by

$$\nu_\alpha\{\eta \mid \eta(x_i) = 1; i = 1, \cdots, n\} = \textstyle\prod_{i=1}^n \alpha(x_i).$$

Let $\Lambda = \{\int_0^\infty \beta(X_t) + \delta(X_t)\, dt < \infty\}$ and define an equivalence relation R on $\mathscr{H}$ by

$$\alpha_1 R \alpha_2 \quad \text{if} \quad \lim_{t\to\infty} [\alpha_1(X_t) - \alpha_2(X_t)] = 0 \quad \text{a.s. on} \quad \Lambda.$$

Let I be the set of invariant probability measures for η_t. Since I is a convex, compact set in the topology of weak convergence the Krein–Millman theorem says I is the closed convex hull of I_e, the extreme points of I.

(1.3) THEOREM. *Assume that $p(x, y)$ is a transient Markov chain on S and that $\beta(x, \eta)$ and $\delta(x, \eta)$ are independent of η. Then*

(i) *For all α_1 and $\alpha_2 \in \mathscr{H}$, $\lim_{t\to\infty} S(t)\nu_{\alpha_1} = \lim_{t\to\infty} S(t)\nu_{\alpha_2}$ if and only if $\alpha_1 R \alpha_2$.*

(ii) *$I_e = \{\lim_{t\to\infty} S(t)\nu_\alpha\}_{\alpha \in \mathscr{H}_R}$ where $\mathscr{H}_R$ is any set of representatives of the equivalence classes determined by R.*

(iii) *Let $\theta_\alpha = \lim_{t\to\infty} S(t)\nu_\alpha$. Then*

$$\lim_{t\to\infty} [\theta_\alpha\{\eta \mid \eta(X_t) = 1\} - \alpha(X_t)] = 0 \quad \text{a.s. on} \quad \Lambda.$$

There is some overlap between Theorem (1.2) and (1.3) because it is possible that $p(x, y)$ is transient and $P^x[\int_0^\infty \beta(X_t) + \delta(X_t)\, dt = \infty] \equiv 1$. But in this situation there is only one equivalence class and hence a unique invariant probability measure. It is not possible that $p(x, y)$ is recurrent and

$$P^x[\textstyle\int_0^\infty \beta(X_t) + \delta(X_t)\, dt = \infty] < 1, \qquad \text{unless} \quad \beta(x) \equiv \delta(x) \equiv 0.$$

NOTATION. For convenience of notation let

$$\lim_{s\to\infty} \overline{\underline{\lim}}_{t\to\infty} f(t, s) = 0 \quad \text{mean}$$
$$\lim_{s\to\infty} \limsup_{t\to\infty} f(t, s) = \lim_{s\to\infty} \liminf_{t\to\infty} f(t, s) = 0.$$

(1.4) THEOREM. *Assume that $p(x, y)$ is transient and that $\beta(x, \eta)$ and $\delta(x, \eta)$ are independent of η. Then for $\mu \in \mathscr{M}$, $\lim_t S(t)\mu = \lim_t S(t)\nu_\alpha$ if and only if*

$$\lim_{s\to\infty} \overline{\underline{\lim}}_{t\to\infty} \int_X \{\textstyle\sum_x p_s(w, x)P^x[\Lambda] \sum_y p_t(x, y)[\eta(y) - \alpha(y)]\}^2\, d\mu(\eta) = 0. \tag{1.5}$$

REMARK. If $P^x[\Lambda] = 1$ for each $x \in S$ then $\lim_{t\to\infty} S(t)\mu = \lim_{t\to\infty} S(t)\nu_\alpha$ if and only if

$$\lim_{t\to\infty} \int \{\textstyle\sum_y p_t(x, y)[\eta(y) - \alpha(y)]\}^2\, d\mu(\eta) = 0 \tag{1.6}$$

which is equivalent to Liggett's result (1974, Theorem 1.5) for the symmetric simple exclusion process. In fact it follows easily from Theorem (1.4) that (1.6) is always sufficient for convergence to $\lim_{t\to\infty} S(t)\nu_\alpha$. If $P^x[\Lambda] = 0$ for each $x \in S$ then η_t is ergodic. Sections 5–8 are devoted to proving Theorems (1.3) and (1.5).

2. Finite particle systems. Spitzer (1970, page 280) observed that whenever $p(x, y)$ is symmetric certain probabilities concerning the simple exclusion process with infinitely many particles could be expressed in terms of a finite particle simple exclusion process. This is the basis of the proofs in Liggett's (1973, 1974) and Spitzer's (1974) work characterizing the invariant measures of γ_t. In this section we state Spitzer's observation in a form suitable for our purposes and describe a new finite particle system which relates similarly to the $\beta - \delta$ process when $\beta(x, \eta) = \beta(x)$ and $\delta(x, \eta) = \delta(x)$ are independent of η.

For each $n \geqq 1$ let X_t^n be a continuous time Markov chain on $T_n = \{(x_1, \cdots, x_n) \in S^n \mid x_i \neq x_j \text{ for } i \neq j\}$ with infinitesimal generator given by

$$\begin{aligned}\Gamma_n f(x_1, \cdots, x_n) \\ = \textstyle\sum_{i=1}^n \sum_{u \neq x_j \text{ for } i \neq j} p(x_i, u)[f(x_1, \cdots, x_{i-1}, u, x_{i+1}, \cdots, x_n) \\ - f(x_1, \cdots, x_n)]\,. \end{aligned} \tag{2.1}$$

where Γ_n is defined for bounded functions on T_n. Intuitively $X_t^n = (X_t^n(1), \cdots, X_t^n(n))$ describes the motion of n interacting particles on S with simple exclusion interaction and one particle motion determined by $p(x, y)$.

(2.2) LEMMA (Spitzer). *Suppose $p(x, y)$ is a symmetric function of x and y. Then for $x \in T_n$*

$$P^\gamma[\gamma_t(x_i) = 1; i = 1, \cdots, n] = P^x[\gamma(X_t^n(i)) = 1; i = 1, \cdots, n]\,.$$

For each $\mu \in \mathscr{M}$ and $x \in T_n$ define $f_n(x; \mu) = \mu\{\eta \mid \eta(x_i) = 1; i = 1, \cdots, n\}$.

(2.3) THEOREM. *Suppose $\mu, \nu \in \mathscr{M}$. Then $\lim_{t\to\infty} U(t)\mu = \nu$ if and only if $\lim_{t\to\infty} E^x[f_n(X_t^n; \mu)] = f_n(x; \nu)$ for each $x \in T_n$, for all $n \geqq 1$.*

PROOF. Theorem (2.3) is a consequence of Lemma (2.2) and the following equalities:

$$\begin{aligned} U(t)\mu\{\eta \mid \eta(x_i) = 1; i = 1, \cdots, n\} &= \textstyle\int_X P^\gamma[\gamma_t(x_i) = 1; i = 1, \cdots, n]\, d\mu(\gamma) \\ &= \textstyle\int P^x[\gamma(X_t^n(i)) = 1; i = 1, \cdots, n]\, d\mu(\gamma) \\ &= E^x[f_n(X_t^n; \mu)]\,. \end{aligned}$$

We turn now to describing a finite particle system which relates to the $\beta - \delta$ process so as to yield a version of Theorem (2.3) when $\beta(x, \eta)$ and $\delta(x, \eta)$ are independent of η. The following notation is needed. First adjoin to S two distinguished sites Δ_B and Δ_D and let $S' = S \cup \{\Delta_B, \Delta_D\}$. Next extend the domain of $p(x, y)$ to $S' \times S'$ by setting $p(x, y) = 0$ whenever x or y belongs to $\{\Delta_B, \Delta_D\}$ and $x \neq y$. Set $p(\Delta_B, \Delta_B) = p(\Delta_D, \Delta_D) = 1$ and $\beta(u) = \delta(u) = 0$ whenever $u \in \{\Delta_B, \Delta_D\}$. Then for each n let Y_t^n be a continuous time Markov chain with state space

$$S_n = \{(x_1, \cdots, x_n) \mid x_i \in S' \text{ for each } i \text{ and } x_i \neq x_j \text{ whenever } x_i \in S \text{ and } i \neq j\}$$

and with infinitesimal generator given by

$$\begin{aligned}\Omega_n f(x_1, \cdots, x_n) \\ &= \textstyle\sum_{i=1}^n \sum_{y \neq x_j \text{ for } j \neq i} p(x_i, y)[f(x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n) \\ &\quad - f(x_1, \cdots, x_n)] \\ &\quad + \textstyle\sum_{i=1}^n \beta(x_i)[f(x_1, \cdots, x_{i-1}, \Delta_B, y_{i+1}, \cdots, x_n) - f(x_1, \cdots, x_n)] \\ &\quad + \textstyle\sum_{i=1}^n \delta(x_i)[f(x_1, \cdots, x_{i-1}, \Delta_D, x_{i+1}, \cdots, x_n) - f(x_1, \cdots, x_n)]\end{aligned} \tag{2.4}$$

where Ω_n is defined for bounded functions on S_n.

The intuitive description of the $Y_t^n = (Y_t^n(1), \cdots, Y_t^n(n))$ process is that of an n-particle simple exclusion process on S using transition function $p(x, y)$ modified to allow a particle at x to jump to death point Δ_B at an exponential rate $\beta(x)$ and to death point Δ_D at an exponential rate $\delta(x)$. The death points Δ_B and Δ_D will be absorbing, hence multiple occupancy at Δ_B and Δ_D is permitted. The basic relation between Y_t^n and η_t is given in the following theorem.

(2.5) THEOREM. *Extend the domain of $\eta \in X$ by defining $\eta(\Delta_B)=1$ and $\eta(\Delta_D)=0$. Then for $x \in S_n$*

$$P^\eta[\eta_t(x_i) = 1; i = 1, \cdots, n] = P^x[\eta(Y_t^n(i)) = 1; i = 1, \cdots, n] .$$

PROOF. For $u \in S_n$ define $F_u[\eta] = \prod_{i=1}^n \eta(u_i)$. Then

$$\begin{aligned}\Omega F_u[\eta] &= \textstyle\sum_{x,y} p(x, y)\eta(x)[1 - \eta(y)][F_u[\eta_{xy}] - F_u[\eta]] \\ &\quad + \textstyle\sum_x (\beta(x)(1 - \eta(x)) + \delta(x)\eta(x))[F_u[\eta_x] - F_u[\eta]] \\ &= -\textstyle\sum_{i=1}^n \sum_y p(u_i, y)[1 - \eta(y)] - \sum_{i=1}^n \delta(u_i) \qquad \text{if} \quad F_u[\eta] = 1 , \\ &= \textstyle\sum_{x, x \neq u_j \text{ if } i \neq j} p(x, u_i)\eta(x) + \beta(u_i) \qquad \text{if} \quad (1 - \eta(u_i)) \prod_{j \neq i} \eta(u_j) = 1 , \\ &= 0 \qquad \text{elsewhere.}\end{aligned}$$

Thus,

$$\begin{aligned}\Omega F_u[\eta] &= -\textstyle\sum_{i=1}^n \sum_y p(u_i, y)(1 - \eta(y))F_u[\eta] - \sum_{i=1}^n \delta(u_i)F_u[\eta] \\ &\quad + \textstyle\sum_{i=1}^n \sum_{x \neq u_j, j \neq i} p(x, u_i)\eta(x)[\prod_{j \neq i} \eta(u_j) - F_u[\eta]] \\ &\quad + \textstyle\sum_{i=1}^n \beta(u_i)[\prod_{j \neq i} \eta(u_j) - F_u[\eta]] \\ &= \textstyle\sum_{i=1}^n \sum_{y \neq u_j, j \neq i} p(u_i, y)[F_{(u_1, \cdots, u_{i-1}, y, u_{i+1}, \cdots, u_n)}[\eta] - F_u[\eta]] \\ &\quad + \textstyle\sum_{i=1}^n \delta(u_i)[F_{(u_i, \cdots, u_{i-1}, \Delta_D, u_{i+1}, \cdots, u_n)}[\eta] - F_u[\eta]] \\ &\quad + \textstyle\sum_{i=1}^n \beta(u_i)[F_{(u_i, \cdots, u_{i-1}, \Delta_B, u_{i+1}, \cdots, u_n)}[\eta] - F_u[\eta]] \\ &= \Omega_n F_u[\eta] .\end{aligned}$$

Letting $w(t, u, \eta) = E^u[F_{Y_t^n}[\eta]]$ it follows that $\Omega w(t, u, \eta) = \Omega_n w(t, u, \eta)$ and that

$$\lim_{h \to 0} \sup_\eta \left| \frac{w(t + h, \eta, u) - w(t, \eta, u)}{h} - \Omega w(t, u, \eta) \right| = 0 .$$

Hence by the uniqueness of solutions to $w'(t) = \Omega w(t)$ (e.g., Theorem 1.3 of Dynkin (1965)) it follows that $S(t)F_u[\eta] = E^u[F_{Y_t^n}[\eta]]$ or, equivalently,

$$P^\eta[\eta_t(u_i) = 1; i = 1, \cdots, n] = P^u[\eta(Y_t^n(i)) = 1; i = 1, \cdots, n] .$$

Extend the definition of $f_n(x;\mu)$ for $x \in S'$ by defining $\eta(\Delta_D) = 0$ and $\eta(\Delta_B) = 1$ for all $\eta \in X$.

(2.6) THEOREM. *Suppose $\mu \in \mathscr{M}$. Then $\lim_{t\to\infty} S(t)\mu = \nu$ if and only if $\lim_{t\to\infty} E^x[f(Y_t^n;\mu)] = f_n(x,\nu)$ for each $x \in S_n$, $n \geqq 1$.*

PROOF. Theorem (2.6) follows from (2.5) in the same way that (2.3) follows from (2.2).

In the following chapters several theorems are proved by comparing the Y_t^n process with the n-particle simple exclusion model X_t^n. First extend the definition of X_t^n so that it has state space S_n by using our extension of $p(x,y)$ in the generator Γ_n of X_t^n. Then couple the Y_t^n and X_t^n processes in such a way that for $x \in S_n$

$$P^x[Y_t^n = X_t^n \text{ for all } t < T] = 1$$

where

$$T = \{\text{first time a particle in the } Y_t^n \text{ process jumps from } S \text{ to } \{\Delta_B, \Delta_D\}\}.$$

We also assume that

$$\{Y_t^n(i);\, i = 1, \cdots, n\} \cap S \subset \{X_t^n(i);\, i = 1, \cdots, n\}$$

for all $t \geqq 0$. The coupling is such that the processes Y_t^n and X_t^n have the same exponential waiting times governing the $p(x,y)$ motion and if possible make the same transition. The transition intensities of the coupling are easily constructed and so are omitted. In later sections all references to X_t^n and Y_t^n will assume these processes have been coupled.

(2.7) REMARK. Suppose $x \in S_n$ and $x_i \in \{\Delta_B, \Delta_D\}$ for some i. Then $P^x[X_t^n \in A_1 \times \cdots \times A_n] = P^{{}^i x}[x_t^{n-1} \in A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n]$ where ${}^i x = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$. The same statement also holds when X_t^n is replaced by Y_t^n. This means that whenever the n particle processes X_t^n and Y_t^n are started with at least one particle in $\{\Delta_B, \Delta_D\}$ they behave like $n-1$ particle processes.

It is convenient to think of $X_t = X_t^1$ as a one particle process on S with transition function $p(x,y)$. In the next section a special case of Theorem (1.2) is proved by coupling the X_t and Y_t^n processes so that $X_t \in \{Y_t^n(1), \cdots, Y_t^n(n)\}$ until a particle in the Y_t^n process which coincides with X_t jumps to $\{\Delta_B, \Delta_D\}$, in particular

(2.8) $$X_t \in \{Y_t^n(1), \cdots, Y_t^n(n)\} \quad \text{for all} \quad t \leqq T.$$

This coupling is accomplished in the following manner. If $X_t = Y_t^n(i)$ then with regard to the $p(x,y)$ motion each particle has the same waiting time and attempts to make the same jump. It is possible that X_t jumps to another site occupied by the Y_t^n process while $Y_t^n(i)$ remains where it is, in which case X_t is still coupled to Y_t^n. If $X_t = Y_t^n(i)$ and $Y_t^n(i)$ jumps to $\{\Delta_B, \Delta_D\}$ then the X_t and Y_t^n processes have uncoupled since X_t does not jump to $\{\Delta_B, \Delta_D\}$. Note that the infinitesimal rate at which X_t and Y_t^n will uncouple given $X_t = Y_t^n(i) = x$

is $\beta(x) + \delta(x)$. As before the transition intensities for this coupling are easily constructed, hence are omitted.

3. Preliminary lemma for Theorem (1.2). This lemma is a result of a general nature. Suppose $\mathscr{E}_t$ is a continuous time nonexplosive jump process on a countable set N. Let the infinitesimal parameters of $\mathscr{E}_t$ be given by q_{xy}. For $A \subset N$ define

$$Q_A(x) = \sum_{y \in A, y \neq x} q_{xy} .$$

(3.1) LEMMA. *Suppose there exist constants $0 < \alpha_1 < \alpha_2 < \infty$ such that for each $x \in N$ either $Q_N(x) = 0$ or $\alpha_1 \leqq Q_N(x) \leqq \alpha_2$. Then almost surely*

$$\{\omega \mid \int_0^\infty Q_A(\mathscr{E}_t)\, dt = \infty\} \subset \{\omega \mid \mathscr{E}_t \in A \text{ for some } t\} .$$

Furthermore if $\alpha_1 \leqq Q_N(x) \leqq \alpha_2$ and $P^x[\mathscr{E}_t \in A$ for some $t] = 1$ for all $x \in N$ then $P^x[\int_0^\infty Q_A(\mathscr{E}_t)\, dt = \infty] = 1$ for all $x \in N$.

PROOF. Let $\mathscr{E}_k$ be the embedded Markov chain in $\mathscr{E}_t$ and let τ_k be the time of the kth jump of $\mathscr{E}_t$. If $\mathscr{E}_t$ is absorbed create fictitious jumps to itself at rate α_1. Then

$$\{\omega \mid \int_0^\infty Q_A(\mathscr{E}_t)\, dt = \infty\} = \{\omega \mid \sum_{k=0}^\infty Q_A(\mathscr{E}_k) = \infty\} \quad \text{a.s.} \tag{3.2}$$

To see this note that

$$\int_0^\infty Q_A(\mathscr{E}_t)\, dt = \sum_{k=0}^\infty Q_A(\mathscr{E}_k)[\tau_{k+1} - \tau_k] .$$

By assumption $\alpha_2^{-2} \leqq E[\tau_{k+1} - \tau_k] \leqq \alpha_1^{-1}$ and $\alpha_2^{-2} \leqq \operatorname{Var}[\tau_{k+1} - \tau_k] \leqq \alpha_1^{-2}$. Hence it follows that $[\tau_{k+1} - \tau_k \mid \mathscr{E}_0, \mathscr{E}_1, \cdots]$ are independent random variables with bounded means and variances. Therefore (3.2) is a consequence of Kolmogorov's three series theorem.

The above remarks have reduced the problem to proving

$$\{\omega \mid \sum_{k=0}^\infty Q_A(\mathscr{E}_k) = \infty\} \subseteqq \{\omega \mid \mathscr{E}_k \in A \text{ i.o.}\} \quad \text{a.s.} \tag{3.3}$$

with equality holding if $\alpha_1 \leqq Q_N(x) \leqq \alpha_2$ for all x. By the extended Borel–Cantelli lemma [e.g., Breiman (1968); Corollary 5.29] we have

$$\begin{aligned}\{\omega \mid \mathscr{E}_k \in A \text{ i.o.}\} &= \{\omega \mid \sum_{k=0}^\infty P^{\mathscr{E}_k}[\mathscr{E}_{k+1} \in A] = \infty\} \\ &\supseteqq \{\omega \mid \sum_{k=0}^\infty Q_A(\mathscr{E}_k) = \infty\} \quad \text{a.s.}\end{aligned}$$

with equality holding if $\alpha_1 \leqq Q_N(x) \leqq \alpha_2$ for all x, hence (3.3) is proved.

4. Proof of Theorem (1.2). In Lemma (4.1) it is assumed that X_t and Y_t^n are coupled as in Section 2.

(4.1) LEMMA. *Assume that* $0 < \sup_x (\beta(x) + \delta(x)) < \infty$. *Then*

$$P^x[\int_0^\infty \beta(X_t) + \delta(X_t)\, dt = \infty] = 1 \qquad \text{for all} \quad x \in S \tag{4.2}$$

if and only if

$$P^y[Y_t^n \in \{\Delta_B, \Delta_D\}^n \text{ eventually}] = 1 \tag{4.3}$$

for all $y \in S_n$, for any n.

PROOF. (4.2) $\Rightarrow$ (4.3). The proof is by induction on n. Let $n = 1$. If $Y_t^1 \notin \{\Delta_B, \Delta_D\}$ eventually it follows from (2.8) that $X_t = Y_t^1$ for all t. Therefore almost surely

$$\int_0^\infty \beta(Y_t^1) + \delta(Y_t^1)\, dt = \int_0^\infty \beta(X_t) + \delta(X_t)\, dt = \infty$$

and hence by Lemma (3.1) $Y_t^1 \in \{\Delta_B, \Delta_D\}$ eventually which is a contradiction. Suppose $n > 1$. If $Y_t^n(i) \in \{\Delta_B, \Delta_D\}$ eventually for some i then Remark (2.7) and the induction hypothesis yield $Y_t^n \in \{\Delta_B, \Delta_D\}^n$ eventually. If $Y_t^n(i) \notin \{\Delta_B, \Delta_D\}$ eventually for any i then it follows from (2.8) that $X_t \in \{Y_t^n(1), \cdots, Y_t^n(n)\}$ for all t. Therefore almost surely

$$\int_0^\infty \sum_{i=1}^n \beta(Y_t^n(i)) + \delta(Y_t^n(i))\, dt \geqq \int_0^\infty \beta(X_t) + \delta(X_t)\, dt = \infty$$

and hence by Lemma (3.1) $Y_t^n(i) \in \{\Delta_B, \Delta_D\}$ eventually for at least one i which is a contradiction implying the desired result.

(4.3) $\Rightarrow$ (4.2). Let $\mathscr{E}_t = (X_t, \zeta_t)$ be a Markov jump process on $N = Sx\{0, 1, 2, \cdots\}$ with jump rates given by $q_{(x,n),(y,0)} = p(x, y)$ and $q_{(x,n),(x,n+1)} = \beta(x) + \delta(x)$. Let $A = Sx\{1, 2, 3, \cdots\}$ then using the notation of Lemma (3.1) $Q_A((x, n)) = \beta(x) + \delta(x)$ and $Q_N((x, n)) \geqq 1$ for all $(x, n) \in N$. Hypothesis (4.3) implies that $P^{(x,n)}[\mathscr{E}_t \in A$ for some $t] = 1$ for all $(x, n) \in N$ and therefore, by Lemma (3.1), $P^{(x,n)}[\int_0^\infty Q_A(\mathscr{E}_t)\, dt = \infty] = 1$ for all $(x, n) \in N$. Consequently $P^x[\int_0^\infty \beta(x_t) + \delta(x_t)\, dt = \infty] = 1$ for all $x \in S$.

(4.4) THEOREM. *Assume that $\beta(x, \eta) = \beta(x)$ and $\delta(x, \eta) = \delta(x)$ are independent of η and satisfy* $\sup_x \{\beta(x) + \delta(x)\} < \infty$. *Then the following conditions are equivalent*:

(a) $P^x[\int_0^\infty \beta(X_t) + \delta(X_t)\, dt = \infty] = 1$ *for all* $x \in S$.

(b) *There exists a unique invariant probability measure ν for process η_t such that $S(t)\mu$ converges weakly to ν for any initial distribution μ.*

Furthermore, if (a) *or* (b) *holds the finite distributions of ν are given by*

(4.5) $\nu\{\eta \mid \eta(y_i) = 1; i = 1, \cdots, n\} = P^y[Y_t^n = (\Delta_B, \cdots, \Delta_B)$ *eventually*$]$

whenever $y = (y_1, \cdots, y_n)$ and the y_i are distinct.

PROOF. (a) $\Rightarrow$ (b). From Theorem (2.6) it is sufficient to prove that

$$\lim_{t\to\infty} E^y[f_n(Y_t^n; \mu)] = P^y[Y_t^n = (\Delta_B, \cdots, \Delta_B) \text{ eventually}]$$

for all $y \in S_n$, n, μ. Since $f_n(x; \cdot) \equiv 0$ whenever x has at least one coordinate equal to Δ_D it is sufficient to prove $P^x[Y_t^n \in \{\Delta_B, \Delta_D\}^n] \equiv 1$. But this is a consequence of Lemma (4.1).

(b) $\Rightarrow$ (a). Assume that (b) is true and that $P^x[\int_0^\infty \beta(x_t) + \delta(x_t)\, dt = \infty] < 1$. Let ζ and η be those elements of X satisfying $\zeta \equiv 0$ and $\eta \equiv 1$. Then from Theorem (2.5)

$$P^\zeta[\eta_t(x) = 1] = P^x[\zeta(Y_t) = 1] = P^x[Y_t = \Delta_B]$$

and

$$P^{\eta}[\eta_t(x) = 1] = P^x[\eta(Y_t) = 1]$$
$$= P^x[Y_t \in S] + P^x[Y_t = \Delta_B] .$$

Lemma (4.1) implies that $P^x[Y_t \in S$ for all $t] > 0$. Consequently $\lim_{t\to\infty} P^{\gamma}[\eta_t(x) = 1] \neq \lim_{t\to\infty} P^{\eta}[\eta_t(x) = 1]$ which contradicts assumption (b).

(4.6) COROLLARY. *The unique invariant measure in Theorem* (4.4) *is pointmass on* $\eta \equiv 1$ *if* $\delta(x) \equiv 0$ *and is pointmass on* $\eta \equiv 0$ *if* $\beta(x) \equiv 0$.

5. Preliminaries for Theorem (1.3). Suppose $\beta(x, \eta)$ and $\delta(x, \eta)$ are independent of η. In Section 4 we proved there is a unique invariant probability measure for the process η_t whenever $P^x[\int_0^\infty \beta(X_t) + \delta(X_t)\, dt = \infty] = 1$. The characterization of I is more complicated in the case $P^x[\int_0^\infty \beta(X_t) + \delta(X_t)\, dt = \infty]\, dt = \infty] < 1$ since the associated finite processes Y_t^n no longer necessarily die out (Lemma (4.1)). However, in this latter case, $p(x, y)$ is transient and we are able to characterize I by comparing for large t the coupled processes X_t^n and Y_t^n with the motion of n noninteracting particles. This section is devoted to the proofs of several lemmas concerning these n-particle processes which are needed in the proof of Theorem (1.3). Under the assumption that $p(x, y)$ is transient Lemma (5.1) makes precise the notion that the particles in the X_t^n process eventually behave like n independent copies of X_t. Lemma (5.3) says that the Y_t^n process eventually behaves like a finite particle simple exclusion process with possibly some particles started in $\{\Delta_B, \Delta_D\}$.

Let $W_t^n = (W_t(1), \cdots, W_t(n))$ where $W_t(i)$ are independent continuous time Markov chains with transition function $p(x, y)$ and waiting time of mean one. Define the event

$$\mathscr{A} = \{W_t(i) = W_t(j) \text{ for some } t \geqq 0 \text{ and } i \neq j\} .$$

Assume that X_t^n and W_t^n are coupled in such a way that $X_t^n = W_t^n$ until a particle in the W_t^n process jumps to an occupied site, hence

$$\mathscr{A} = \{W_t^n \neq X_t^n \text{ for some } t \geqq 0\} .$$

(5.1) LEMMA. *Suppose* $p(x, y)$ *is transient. Let* X_k^n *be the embedded Markov chain in* X_t^n. *Then* $\lim_{k\to\infty} P^{X_k^n}[\mathscr{A}] = 0$ a.s. *and, consequently*, $\lim_{t\to\infty} P^{X_t^n}[\mathscr{A}] = 0$ a.s.

PROOF. Liggett (1974, page 204) has shown that $\lim_{k\to\infty} E^x[P^{X_k^n}[\mathscr{A}]] = 0$. Since $P^{X_k^n}[\mathscr{A}]$ is a bounded supermartingale this implies that $\lim_{k\to\infty} P^{X_k^n}[\mathscr{A}]$ exists a.s. and hence must be 0.

In order to state Lemma (5.2) let

$$\Lambda = \{\textstyle\int_0^\infty \beta(X_t) + \delta(X_t)\, dt < \infty\}$$

and

$$\Lambda_n(i) = \{\textstyle\int_0^\infty \beta(X_t^n(i)) + \delta(X_t^n(i))\, dt < \infty\} .$$

(5.2) LEMMA. *Suppose that* $p(x, y)$ *is transient and that* $g: S \to [0, 1]$. *Then*

(i) *If* $\lim_{t\to\infty} g(X_t) = 0$ a.s. *on* Λ *then* $\lim_{t\to\infty} g(X_t^n(i)) = 0$ a.s. *on* $\Lambda_n(i)$.
(ii) *If* $\lim_{t\to\infty} g(X_t) = 0$ a.s. *on* Λ^c *then* $\lim_{t\to\infty} g(X_t^n(i)) = 0$ a.s. *on* $\Lambda_n^c(i)$.

PROOF. Let $E_1 = \{\lim_{t\to\infty} g(X_t^n(i)) = 0\}$ and $E_2 = \Lambda_n(i)$. Suppose $\varepsilon > 0$. By applying Egoroff's theorem to Lemma (5.1) there exists a set A and a number T such that $P^x[A] < \varepsilon$ and $P^{X_t^n}[\mathscr{A}] < \varepsilon$ for all $t > T$ on A^c. Therefore

$$\begin{aligned} P^x[E_1 \cap E_2] &\geqq E^x[P^{X_T^n}[E_1 \cap E_2 \cap \mathscr{A}^c]; A^c] \\ &\geqq E^x[P^{X_T^n}[E_2 \cap \mathscr{A}^c]; A^c] \\ &\geqq E^x[(P^{X_T^n}[E_2] - \varepsilon); A^c] \\ &\geqq P^x[E_2] - 2\varepsilon . \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary $P^x[E_1 \cap E_2] = P^x[E_2]$ and part (i) is proved. Part (ii) follows in the same way if $E_2 = \Lambda_n^c(i)$.

Next assume that X_t^n and Y_t^n are coupled as in Section 2 and define the event

$$\mathscr{B} = \{X_t^n \neq Y_t^n \quad \text{for some} \quad t \geqq 0\} .$$

(5.3) LEMMA. $\lim_t P^{Y_t^n}[\mathscr{B}] = 0$ a.s.

PROOF. Since $P^{Y_t^n}[\mathscr{B}]$ is a bounded supermartingale it is sufficient to prove that $\lim_{t\to\infty} E^x[P^{Y_t^n}[\mathscr{B}]] = 0$ where x has no coordinates in $\{\Delta_B, \Delta_D\}$. The proof will be by induction on n. Let

$$\mathscr{E} = \{Y_t^n(i) \in \{\Delta_B, \Delta_D\} \quad \text{for some} \quad i, \quad \text{some} \quad t \geqq 0\} .$$

Suppose $n = 1$. Then $\mathscr{B} \subset \mathscr{E}$ implies that

$$\lim_t E^x[P^{Y_t}[\mathscr{B}]] \leqq \lim_t E^x[P^{Y_t}[\mathscr{E}]; \mathscr{E}^c] + \lim_t E^x[P^{Y_t}[\mathscr{B}]; \mathscr{E}] .$$

The first term after the inequality is 0 since $\mathscr{E}$ is a tail event whereas the second term is 0 because $P^{\Delta_B}[\mathscr{B}] = P^{\Delta_D}[\mathscr{B}] = 0$. Suppose $n > 1$ and for convenience drop the superscript n on Y_t^n. For $\varepsilon > 0$ choose t^* such that

$$P^x[Y_{t^*}(i) \in \{\Delta_B, \Delta_D\} \quad \text{for some} \quad i; \mathscr{E}] \geqq P^x[\mathscr{E}] - \varepsilon .$$

Then

$$\lim_t E^x[P^{Y_t}[\mathscr{B}]] \leqq \lim_t E^x[E^{Y_{t^*}}[P^{Y_t}[\mathscr{B}]]; \mathscr{E}] + E^x[P^{Y_{t^*}}[\mathscr{E}]; \mathscr{E}^c] .$$

Since $\mathscr{E}$ is a tail event the second term after the inequality can be made arbitrarily small by choosing t^* large enough. Using the induction hypothesis and Remark (2.7) the first term is bounded by ε. Since $\varepsilon > 0$ is arbitrary Lemma (5.3) is proved.

6. Sketch of the proof of Theorem (1.3). The proof of Theorem (1.3) requires the following results proved by Liggett (1973, 1974) and Spitzer (1974). Let J be the set of invariant probability measures for the simple exclusion process γ_t and let J_e be the set of extreme points of J. Recall the definitions of $\mathscr{H}$ and ν_α from Section 1.

(6.1) THEOREM (Liggett, Spitzer). *Assume that* $p(x, y)$ *is the transition function* *for a symmetric irreducible Markov chain on S. Then for each* $\alpha \in \mathscr{H}$ *there exists a unique* $\mu_\alpha \in J_e$ *such that*

$$\mu_\alpha\{\eta \mid \eta(x) = 1\} = \alpha(x)$$

for $x \in S$. *Furthermore,* $J_e = \{\mu_\alpha \mid \alpha \in \mathscr{H}\}$, $\mu_\alpha = \lim_{t\to\infty} U(t)\nu_\alpha$ *and* $\mu_\alpha = \nu_\alpha$ *if and only if* α *is a constant.*

SKETCH OF THE PROOF OF THEOREM (1.3). Define

$$\mathscr{H}' = \{\alpha \in \mathscr{H} \mid \lim_{t\to\infty} \alpha(X_t) = 1 \text{ a.s. on } \{\textstyle\int_0^\infty \beta(X_t) + \delta(X_t)\,dt = \infty\}\}$$
$$J' = \{\mu \in J \mid \lim_{t\to\infty} \mu\{\eta \mid \eta(X_t) = 1\} = 1 \text{ a.s. on } \{\textstyle\int \beta(X_t) + \delta(X_t)\,dt = \infty\}\}\,.$$

Then $\tau : J' \to I$ defined by $\tau(\mu) = \lim_{t\to\infty} S(t)\mu$ is a 1–1 onto affine map between J' and I. The proof that τ is 1–1 and onto is first proved in the case where $\delta(x) \equiv 0$ by comparing the finite processes Y_t^n and X_t^n. In the general case ($\beta(x) \geqq 0$, $\delta(x) \geqq 0$) let $\tilde{\eta}_t$ be a new $\beta - \delta$ process with birth rates $\beta(x) + \delta(x)$ and death rates identically 0. Let $\tilde{S}(t)$ and $\tilde{I}$ be the corresponding semigroup and set of invariant measures. Define $\tilde{\tau} : J' \to \tilde{I}$ by $\tilde{\tau}(\mu) = \lim_{t\to\infty} \tilde{S}(t)\mu$ and $\sigma : \tilde{I} \to I$ by $\sigma(\mu) = \lim_{t\to\infty} S(t)\mu$. From the first step $\tilde{\tau}$ is a 1–1 onto map. We prove that σ is also a 1–1 onto map again by comparing the finite processes and then complete the proof that τ is 1–1 onto by proving that $\tau = \sigma \circ \tilde{\tau}$.

Since the extreme points of J' are mapped by τ onto the extreme points of I,

$$\begin{aligned} I_e &= \{\lim_{t\to\infty} S(t)\mu \mid \mu \text{ is an extreme point of } J'\} \\ &= \{\lim_{t\to\infty} S(t)\mu_\alpha \mid \mu_\alpha \in J \text{ and } \alpha \in \mathscr{H}'\} \\ &= \{\lim_{t\to\infty} S(t)\nu_\alpha \mid \alpha \in \mathscr{H}'\} \end{aligned}$$

where the third equality follows from a proof that $\lim_{t\to\infty} S(t)\mu_\alpha = \lim_{t\to\infty} S(t)\nu_\alpha$. The proof of Theorem (1.3) is concluded by proving that $\mathscr{H}'$ is a set of class representatives for R and then proving parts (i) and (iii) of Theorem (1.3).

7. **Proof of Theorem** (1.3). It is assumed that $p(x, y)$ is transient throughout this section. In addition to the definition of $\mathscr{H}'$, J', J_e and μ_α given in Section 6 we need to define

$$\mathscr{M}' = \{\mu \in \mathscr{M} \mid \lim_{t\to\infty} \mu\{\eta \mid \eta(X_t) = 1\} = 1 \text{ a.s. on } \{\textstyle\int_0^\infty \beta(X_t) + \delta(X_t)\,dt = \infty\}\}\,.$$

(7.1) LEMMA. *Assume that* $\delta(x) \equiv 0$. *Suppose* $\mu \in I$ *or* $\mu = \lim_{t\to\infty} U(t)\nu$ *for some* $\nu \in I$. *Then* $\mu \in \mathscr{M}'$.

PROOF. Suppose $\mu \in I$. From Theorem (2.6) and Lemma (3.1)

$$\begin{aligned} \mu\{\eta(x) = 1\} &= \lim_{t\to\infty} E^x[\mu\{\eta(Y_t^1) = 1\}] \\ &\geqq P^x[Y_t^1 = \Delta_B \text{ eventually}] \\ &\geqq P^x[\textstyle\int_0^\infty \beta(X_t)\,dt = \infty]\,. \end{aligned}$$

Hence $\mu \in \mathscr{M}'$ since the last term goes to 1 a.s. on $\{\int_0^\infty \beta(X_t)\,dt = \infty\}$. Suppose $\mu = \lim_{t\to\infty} U(t)\nu$ for $\nu \in I$. From Theorem (2.3) and above argument it follows that

$$\begin{aligned}\mu\{\eta(x) = 1\} &= \lim_t E^x[\nu\{\eta(X_t) = 1\}] \\ &\geqq \lim_t E^x[P^{X_t}[Y_s^1 = \Delta_B \text{ eventually}]] \\ &\geqq \lim_t E^x[P^{X_t}[\textstyle\int_0^\infty \beta(X_s)\, ds = \infty]] \\ &= P^x[\textstyle\int_0^\infty \beta(X_s)\, ds = \infty],\end{aligned}$$

and, consequently, $\mu \in \mathscr{M}'$.

(7.2) LEMMA. *Assume that* $\delta(x) \equiv 0$. *If* $\mu \in \mathscr{M}'$ *then* $\lim_{t\to\infty} \mu\{\eta(X_t^n(i)) = 1\} = 1$ a.s. *on* $\{\int_0^\infty \beta(X_t^n(i))\, dt = \infty\}$.

PROOF. Apply Lemma (5.2).

Recall from Section 2 the definition

$$f_n(x; \mu) = \mu\{\eta \mid \eta(x_i) = 1;\ i = 1, \cdots, n\}$$

where it is always assumed that $\eta(\Delta_B) = 1$ and $\eta(\Delta_D) = 0$ for all $\eta \in X$. Define for $x \in S_n$

$$\begin{aligned}G_n(x) = \min\{&P^x[X_t^n \neq X_t^n \text{ for some } t], \\ &\textstyle\sum_{i=1}^n 1_{\{x_i \in S\}} P^x[\int_0^\infty \beta(X_t^n(i))\, dt < \infty,\ Y_t^n(i) = \Delta_B \text{ eventually}]\}.\end{aligned}$$

(7.3) LEMMA. *Assume that* $\delta(x) \equiv 0$. *If* $\mu \in \mathscr{M}'$ *then*

$$\limsup_{t\to\infty} E^x[|f_n(Y_t^n; \mu) - f_n(X_t^n; \mu)|] \leqq G_n(x).$$

If $\nu \in I$ *and* $\lim_t U(t)\nu = \mu$ *or if* $\mu \in J'$ *and* $\nu = \lim_{t\to\infty} S(t)\mu$ *then*

$$|f_n(x; \mu) - f_n(x; \nu)| \leqq G_n(x).$$

PROOF. Assume that X_t^n and Y_t^n are coupled as in Section 2, then the first statement of Lemma (7.3) is a direct consequence of Lemmas (7.1), (7.2) and the definition of $\mathscr{M}'$. To prove the second statement note that $\mu = \lim_{t\to\infty} U(t)\nu$ for $\nu \in I$ implies

$$|f_n(x; \mu) - f_n(x; \nu)| = \lim_{t\to\infty} |E^x[f_n(X_t^n; \nu) - f_n(Y_t^n; \nu)]| \leqq G_n(x).$$

Similarly for the remaining case.

(7.4) THEOREM. *Assume that* $\delta(x) \equiv 0$. *Then*

(i) $\lim_{t\to\infty} E^x[G_n(X_t^n)] = 0$,
(ii) $\lim_{t\to\infty} E^x[G_n(Y_t^n)] = 0$.

PROOF. (i) It is sufficient to prove that

$$\lim_{t\to\infty} E^x[P^{X_t^n}[\textstyle\int_0^\infty \beta(X_s^n(i))\, ds < \infty,\ Y_s^n(i) = \Delta_B \text{ eventually}]] = 0,$$

hence it is sufficient to prove that

(7.5) $$\lim_{t\to\infty} E^x[P^{X_t^n}[Y_s^n(i) = \Delta_B \text{ eventually}];\ \textstyle\int_0^\infty \beta(X_s^n(i))\, ds < \infty] = 0.$$

For $n = 1$ we construct a proof similar to the proof of (4.3) $\Rightarrow$ (4.2) in Lemma

(4.1). Using the notation of Lemma (4.1)

$$\lim_{t\to\infty} E^x[P^{X_t}[Y_s \in \Delta_B \text{ eventually}]; \textstyle\int_0^\infty \beta(x_r)\,dr < \infty]$$
$$= \lim_{t\to\infty} P^x[\mathscr{E}_s \text{ jumps to } A \text{ after time } t; \textstyle\int_0^\infty \beta(X_r)\,dr < \infty]$$
$$= P^x[\mathscr{E}_s \in A \text{ i.o.}, \textstyle\int_0^\infty \beta(X_r)\,dr < \infty] = 0$$

where the last equality follows from (3.3). For $n > 1$ use Lemma (5.1) and Egoroff's theorem to show that (7.5) follows from

$$\lim_{t\to\infty} E^x[P^{X_t^n(i)}[Y_s^n(i) = \Delta_B \text{ eventually}]; \textstyle\int_0^\infty \beta(X_s^n(i))\,ds < \infty] = 0\,,$$

then apply Lemma (5.2). The proof of (ii) follows immediately from Lemma (5.3).

(7.6) THEOREM. *Assume that $p(x, y)$ is transient and that $\delta(x) \equiv 0$. Let J' be as defined in Section 6. Then $\tau(\mu) = \lim_{t\to\infty} S(t)\mu$ defines a 1–1 onto affine map between J' and I. Furthermore, $\tau^{-1}(\nu) = \lim_{t\to\infty} U(t)\nu$.*

PROOF. Assume that X_t^n and Y_t^n are coupled as in Section 2, then $f_n(X_t^n; \mu) \leqq f_n(Y_t^n; \mu)$ a.s. for $\mu \in \mathscr{M}$. For $\mu \in J$ this implies that

$$f_n(x; \mu) = E^x[f_n(X_t^n; \mu)] \leqq E^x[f_n(Y_t^n; \mu)]$$

and hence $E^x[f_n(Y_s^n; \mu)] \leqq E^x[f_n(Y_{s+t}^n; \mu)]$. Therefore $\lim_{t\to\infty} E^x[f_n(Y_t^n; \mu)]$ exists for each $x \in S_n$ and by Theorem (2.6) $\lim_t S(t)\mu$ exists. A similar argument shows that $\lim_t U(t)\nu$ exists whenever $\nu \in I$ and by Lemma (7.1) $\lim_{t\to\infty} U(t)\nu \in J'$. To prove that τ is 1–1 suppose $\tau(\mu) = \nu$. Then by Lemma (7.3)

$$|f_n(x; \nu) - f_n(x; \mu)| \leqq G_n(x)$$

and by Theorem (7.4) $\lim_t E^x[f_n(X_t^n; \nu)] = f_n(x; \mu)$. Hence $\lim_t S(t)\nu = \mu$ which proves τ is 1–1. A similar argument proves that τ is onto.

In order to prove that $\tau(\mu) = \lim_{t\to\infty} S(t)\mu$ defines a 1–1 onto map between J' and I for the general case ($\beta(x) \geqq 0$, $\delta(x) \geqq 0$), we compare the $\beta - \delta$ process η_t which has birth rates, $\beta(x)$, and death rates $\delta(x)$, with the $\beta - \delta$ process $\tilde{\eta}_t$ which has birth rates $\tilde{\beta}(x) = \beta(x) + \delta(x)$ and death rates identically 0. Hence $\tilde{\eta}_t$ fits into the case covered by Theorem (7.6). Let the associated finite processes of η_t and $\tilde{\eta}_t$ be Y_t^n and $\tilde{Y}_t^n$ respectively. Couple these processes so that Y_t^n and $\tilde{Y}_t^n$ make the same transitions except whenever $Y_t^n(i)$ jumps to Δ_D, $\tilde{Y}_t^n(i)$ jumps to Δ_B.

(7.7) LEMMA. *Let $h(x) = P^x[Y_t^n \neq \tilde{Y}_t^n$ for some $t]$. Then*

$$\lim_{t\to\infty} E^x[h(Y_t^n)] = \lim_{t\to\infty} E^x[h(\tilde{Y}_t^n)] = 0\,.$$

PROOF. Since $h(x) \leqq P^x[Y_t^n \neq X_t^n$ for some $t] = P^x[\tilde{Y}_t^n \neq X_t^n$ for some $t]$, Lemma (7.7) follows from Lemma (5.3).

(7.8) LEMMA. *If $\sigma : \tilde{I} \to I$ is given by $\sigma(\nu) = \lim_{t\to\infty} S(t)\nu$ then σ is a 1–1 onto affine map between $\tilde{I}$ and I. Also $\sigma^{-1}(\mu) = \lim_{t\to\infty} \tilde{S}(t)\mu$.*

PROOF. The proof of Lemma (7.8) proceeds exactly as the proof of Lemma (7.6) where we now use Lemma (7.7) instead of Lemma (7.4).

(7.9) THEOREM. *The map* $\tau(\mu) = \lim_{t\to\infty} S(t)\mu$ *is a* 1–1 *onto affine map between* J' *and* I.

PROOF. Using the notation Theorem (7.6) and Lemma (7.8) consider the diagram

$$\begin{array}{ccc} J' & \xrightarrow{\tilde\tau} & \tilde I \\ & & \Big\downarrow \sigma \\ & & I \end{array}$$

where $\tilde\tau(\mu) = \lim_t \tilde S(t)\mu$, $\sigma(\tilde\nu) = \lim_t S(t)\tilde\nu$ and $\sigma \circ \tilde\tau$ is a 1–1 onto affine map. To prove Theorem (7.9) we need only show that $\lim_t S(t)\mu = \sigma \circ \tilde\tau$. Let $\mu \in J'$, $\tilde\nu = \lim_t \tilde S(t)\mu$ and $\nu = \lim_t S(t)\tilde\nu$. Then

$$\begin{aligned}
&\lim_t |E^x[f_n(Y_t^n;\mu) - f_n(Y_t^n;\tilde\nu)]| \\
&\qquad \leqq \lim_t |E^x[f_n(Y_t^n;\mu) - f_n(\tilde Y_t^n;\mu)]| + \lim_t |E^x[f_n(\tilde Y_t^n;\mu) - f_n(x;\tilde\nu)]| \\
&\qquad\qquad + \lim |E^x[f_n(\tilde Y_t^n;\tilde\nu) - f_n(Y_t^n;\tilde\nu)]| \\
&\qquad \leqq 2P^x[Y_t^n \neq \tilde Y_t^n \text{ for some } t]\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_t |E^x[f_n(Y_t^n;\mu) - f_n(Y_t^n;\tilde\nu)]| &\leqq \lim_t E^x[|E^{Y_s^n}[f_n(Y_t^n;\mu) - f_n(Y_t^n;\tilde\nu)]|] \\
&\leqq 2E^x[P^{Y_s^n}[Y_t^n \neq \tilde Y_t^n \text{ for some } t]] \to 0
\end{aligned}$$

as $s \to \infty$ by Lemma 7.7. Hence $\lim_t S(t)\mu = \lim_t S(t)\tilde\nu = \sigma \circ \tilde\tau(\mu)$.

(7.10) LEMMA. *The extreme points of* J' *are* $\{\mu_\alpha \in J_e \mid \alpha \in \mathscr{H}'\}$.

PROOF. Let μ be an extreme point of J'. If $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ for $\mu_1, \mu_2 \in J$ then $\mu_1, \mu_2 \in J'$ and hence $\mu = \mu_1 = \mu_2$. Therefore by Theorem (6.1) $\mu \in J_e \cap J' = \{\mu_\alpha : \alpha \in \mathscr{H}'\}$. On the other hand any $\mu_\alpha \in J'$ must be an extreme point of J'.

(7.11) LEMMA. *Suppose* $\mu, \nu \in \mathscr{M}$. *If* $\lim_{t\to\infty} U(t)\nu = \mu$ *then* $\lim_{t\to\infty} S(t)\mu = \lim_{t\to\infty} S(t)\nu$. *In particular*, $\lim_{t\to\infty} S(t)\mu_\alpha = \lim_{t\to\infty} S(t)\nu_\alpha$.

PROOF.

$$\begin{aligned}
&\lim_{t\to\infty} |E^x[f_n(Y_t^n;\nu) - f_n(Y_t^n;\mu)]| \\
&\qquad \leqq \lim_{t\to\infty} |E^x[f_n(Y_t^n;\nu) - f_n(X_t^n;\nu)]| \\
&\qquad\qquad + \lim_{t\to\infty} |E^x[f_n(X_t^n;\mu) - f_n(Y_t^n;\mu)]| \\
&\qquad \leqq 2P^x[X_t^n \neq Y_t^n \text{ for some } t]\,.
\end{aligned} \tag{7.12}$$

Hence conditioning (7.12) on Y_s^n and using Lemma (5.3) yields desired result.

(7.13) LEMMA. $\mathscr{H}'$ *is a set of class representatives for the equivalence relation* R.

PROOF. Recall that $\Lambda = \{\int_0^\infty \beta(X_t) + \delta(X_t)\,dt < \infty\}$. For each $\alpha \in \mathscr{H}$ let

$Z_\alpha = \lim_t \alpha(X_t)$ and define

$$r_\alpha(x) = P^x[\Lambda^c] + E^x[Z; \Lambda] .$$

Then $\alpha R r_\alpha$ and $r_\alpha \in \mathscr{H}'$. To prove $\alpha R r_\alpha$ note that on Λ, $\lim_s [r_\alpha(X_s) - \alpha(X_s)] = \lim_s [E^{X_s}[Z] - \alpha(X_s)] = 0$. To prove $r_\alpha \in \mathscr{H}$ let X_k be the embedded Markov chain in X_t, then

$$\begin{aligned} r_\alpha(x) &= P^x[\textstyle\sum_{k=1}^\infty \beta(X_t) + \delta(X_k) = \infty] \\ &\quad + E^x[\lim_{k\to\infty} \alpha(X_k); \textstyle\sum_{k=1}^\infty \beta(X_t) + \delta(X_k) < \infty] \\ &= \textstyle\sum_y p(x, y) r_\alpha(y) . \end{aligned}$$

It now follows immediately that $r_\alpha \in \mathscr{H}'$.

Proof of Theorem (1.3)(i). First consider that

$$\begin{aligned} \lim_t S(t)\nu_\alpha\{\eta \mid \eta(x) = 1\} &= \lim_t E^x[\nu_\alpha\{\eta \mid \eta(Y_t^1) = 1\} \\ &= P^x[Y_t^1 = \Delta_B \ \text{ eventually}] + E^x[Z_\alpha; X_t = Y_t^1 \ \text{ for all } \ t] \end{aligned}$$

where $Z_\alpha = \lim_t \alpha(X_t)$. Therefore $\lim_{t\to\infty} S(t)\nu_{\alpha_1} = \lim_{t\to\infty} S(t)\nu_{\alpha_2}$ implies that

$$E^x[Z_{\alpha_1}; X_t = Y_t^1 \ \text{ for all } \ t] = E^x[Z_{\alpha_2}; X_t = Y_t^1 \ \text{ for all } \ t] .$$

Since $P^{X_s}[X_t = Y_t^1 \text{ for all } t] \to 1$ on $\{\int_0^\infty \beta(X_s) + \delta(X_s)\, ds < \infty\}$ (see (7.5)) and since $E^{X_s}[Z_{\alpha_i}] = \alpha_i(X_s)$ it follows that $\alpha_1 R \alpha_2$. Conversely suppose $\alpha_1 R \alpha_2$. Then

$$\begin{aligned} &\lim_t E^x[\textstyle\prod_{i=1}^n \alpha_1(Y_t^n(i)) - \prod_{i=1}^n \alpha_2(Y_t^n(i))] \\ &\quad \leqq \textstyle\sum_{j=1}^2 |\lim_t E^x[\prod_{i=1}^n \alpha_j(Y_t^n(i)) - \prod_{i=1}^n \alpha_j(X_t^n(i))]| \\ &\qquad + \lim_t |E^x[\textstyle\prod_{i=1}^n \alpha_1(X_t^n(i)) - \prod_{i=1}^n \alpha_2(X_t^n(i))]| \\ &\quad \leqq 2P^x[Y_t^n \neq X_t^n \ \text{ for some } \ t] \\ &\qquad + P^x[\textstyle\sum_{i=1}^n \int_0^\infty \beta(X_t^n(i)) + \delta(X_t^n(i))\, dt = \infty] \\ &\quad \leqq 3P^x[Y_t^n \neq X_t^n \ \text{ for some } \ t] , \end{aligned} \tag{7.14}$$

where the last inequality follows from Lemma 3.1. Therefore conditioning (7.14) on Y_s^n, taking limits as $s \to \infty$ and applying Lemma (5.3) yields (7.14) $= 0$. Since $x \in S_n$ and n were arbitrary, Theorem (2.6) yields $\lim_t S(t)\nu_{\alpha_1} = \lim_t S(t)\nu_{\alpha_2}$.

Proof of Theorem (1.3)(ii). By Theorem (7.9) and Lemmas (7.10) and (7.11)

$$I_e = \{\lim_{t\to\infty} S(t)\mu_\alpha\}_{\alpha\in\mathscr{H}'} = \{\lim_t S(t)\nu_\alpha\}_{\alpha\in\mathscr{H}'} .$$

Theorem (1.3)(i) and Lemma (7.13) imply

$$I_e = \{\lim_t S(t)\nu_\alpha\}_{\alpha\in\mathscr{H}_R} ,$$

where $\mathscr{H}_R$ is any set of class representatives determined by R.

Proof of Theorem (1.3)(iii). This follows from the proof of Theorem (1.3)(i).

8. Ergodic theorems. In this section necessary and sufficient conditions are given for the convergence of $S(t)\mu$ to an extreme point of I in the case $\beta(x, \eta) = \beta(x)$ and $\delta(x, \eta) = \delta(x)$. For a fixed $\mu \in \mathscr{M}$ and $\alpha \in \mathscr{H}$ define $Q_n(x) = f_n(x; \mu) - f_n(x; \nu_\alpha)$. Recall the definition of W_t^n from Section 5.

(8.1) LEMMA. *Assume that* $p(x, y)$ *is transient. Then the following statements are equivalent.*

$$\lim_{t\to\infty} S(t)\mu = \lim_{t\to\infty} S(t)\nu_\alpha \tag{8.2}$$

$$\lim_{t\to\infty} E^y[Q_n(Y_t^n)] = 0 \quad \textit{for all} \quad y \in S_n\,, \qquad \textit{for all} \ \ n \tag{8.3}$$

$$\lim_{s\to\infty} \overline{\lim}_{t\to\infty} E^{Y_s^n}[Q_n(X_t^n)] = 0 \ \text{ a.s. } \ P^y \ \textit{for all} \ \ y \in S_n\,, \quad \textit{for all} \ \ n\,. \tag{8.4}$$

$$\begin{aligned} &\lim_{s\to\infty} \overline{\lim}_{t\to\infty} E^{X_s^n}[Q_n(X_t^n)] = 0 \quad \text{a.s.} \quad P^x \\ &\quad \textit{on} \ \ \{\textstyle\sum_{i=1}^n \int_0^\infty \beta(X_t^n(i)) + \delta(X_t^n(i))\, dt < \infty\} \ \ \textit{for all} \quad x \in S_n\,, \\ &\quad \textit{for all} \quad n \end{aligned} \tag{8.5}$$

$$\begin{aligned} &\lim_{s\to\infty} \overline{\lim}_{t\to\infty} E^{W_s^n}[Q_n(W_t^n)] = 0 \quad \text{a.s.} \quad P^w \\ &\quad \textit{on} \ \ \{\textstyle\sum_{i=1}^n \int_0^\infty \beta(W_t(i)) + \delta(W_t(i))\, dt < \infty\} \ \ \textit{for all} \quad w \in S_n\,, \\ &\quad \textit{for all} \quad n\,. \end{aligned} \tag{8.6}$$

PROOF. (8.2) $\Leftrightarrow$ (8.3): Use Theorem (2.6).

(8.3) $\Leftrightarrow$ (8.4): Let $r_n(y) = P^y[Y_t^n \neq X_t^n$ for some $t]$ and apply Lemma (5.3) to the inequality $|E^y[Q(X_t^n)] - E^y[Q(Y_t^n)]| \leqq r_n(y)$.

(8.4) $\Rightarrow$ (8.5): Use $\lim_s P^{X_s^n}[X_l^n = Y_l^n$ for all $l] = 1$ a.s. on $\{\sum_{i=1}^n \int_0^\infty \beta(X_t^n(i)) + \delta(X_t^n(i))\, dt < \infty\}$.

(8.5) $\Rightarrow$ (8.4): Use $\lim_{s\to\infty} P^{Y_s^n}[X_t^n = Y_t^n \ \forall\, t] = 1$ and the fact that $\{X_t^n = Y_t^n \ \forall\, t\} \subset \{\sum_{i=1}^n \int_0^\infty \beta(X^n(i)) + \delta(X^n(i))\, dt < \infty\}$.

(8.5) $\Leftrightarrow$ (8.6): Use Lemma (5.1) and $\lim_{s\to\infty} P^{W_s^n}[\mathscr{A}] = 0$ a.s. (see Liggett (1974, Lemma (2.1)).

(8.7) THEOREM. *Assume that* $p(x, y)$ *is transient. Then* $\lim_{t\to\infty} S(t)\mu = \lim_{t\to\infty} S(t)\nu_\alpha$ *if and only if*

$$\begin{aligned} &\lim_{s\to\infty} \overline{\lim}_{t\to\infty} \int_X \{\textstyle\sum_x p_s(w, x) P^x[\Lambda] \sum p_t(x, y)[\eta(y) - \alpha(y)]\}^2\, d\mu(\eta) = 0 \\ &\qquad\qquad \text{a.s.} \quad P^w \quad \textit{for each} \quad w \in S \end{aligned} \tag{8.8}$$

where $\Lambda = \{\int_0^\infty \beta(X_r) + \delta(X_r)\, dr < \infty\}$.

PROOF. Using an argument similar to Liggett's proof of Theorem 4.10 in [9] we can show that (8.6) is equivalent to

$$\begin{aligned} &\lim_s \overline{\lim}_t \int_X \textstyle\prod_{i=1}^n \sum_y p_t(W_s(i), y)[\eta(y) - \alpha(y)]\, d\mu(\eta) = 0 \quad \text{a.s.} \\ &\qquad\qquad \text{on} \quad \{\textstyle\sum_{i=1}^n \int_0^\infty \beta(W_s(i)) + \delta(W_s(i))\, ds < \infty\} \end{aligned} \tag{8.9}$$

or, equivalently if $\Lambda(i) = \{\int_0^\infty \beta(W_s(i) + \delta(W_s(i))\, ds < \infty\}$

$$\lim_s \overline{\lim}_t \int_X \textstyle\prod_{i=1}^n 1_{\Lambda(i)} \sum_y p_t(W_s(i), y)[\eta(y) - \alpha(y)]\, d\mu(\eta) = 0 \quad \text{a.s.} \tag{8.10}$$

Let $n = 2$ and take expectations with respect to P^w to obtain (8.8). To show that (8.8) implies (8.10) substitute X_r for w in (8.8) and use the fact that

$$\lim_{r\to\infty} [p_s(X_r, x) P^x[\Lambda] - p_s(X_r, x)] = 0 \quad \text{a.s.} \quad \Lambda\,.$$

The result will be

$$\lim_{r\to\infty} \overline{\lim}_{t\to\infty} \int_X \{1_\Lambda \sum_y p_s(X_r, y)[\eta(y) - \alpha(y)]\}^2 \, d\mu(\eta) = 0 \quad \text{a.s.}$$

which implies (8.10).

(8.11) COROLLARY. *If* $P^x[\Lambda] = 1$ *for all* $x \in S$ *then* $\lim_{t\to\infty} S(t)\mu = \lim_{t\to\infty} S(t)\nu_\alpha$ *if and only if*

$$\lim_{t\to\infty} \int_X \{\sum_y p_t(x, y)[\eta(y) - \alpha(y)]\}^2 \, d\mu(\eta) = 0 \,. \tag{8.12}$$

If $P^x[\Lambda] = 0$ for each $x \in S$ then η_t is ergodic. If $P^x[\Lambda] \leqq 1$ then (8.12) is sufficient for $\lim_{t\to\infty} S(t)\mu = \lim_{t\to\infty} S(t)\nu_\alpha$.

REMARK. Corollary (8.11) can be proved without using Theorem (8.7). The case $P^x[\Lambda] = 0$ is done in Section 4. For the remaining cases use Lemma (7.11) and Liggett's criteria (1.6) for the convergence of $U(t)\mu$ to $\lim_{t\to\infty} U(t)\nu_\alpha$.

9. Further results. Further results are obtainable when $\beta(x, \eta)$ and $\delta(x, \eta)$ depend on η if the dependence is such that there is a nice related finite process. For example if $\beta(x, \eta) \equiv 0$ and $\delta(x, \eta) = \sum_{y\in S} q(x, y)[1 - \eta(y)]$ where $q(x, y) \geqq 0$ and $0 < \sup_x \sum_y q(x, y) < \infty$ then there is a continuous time Markov chain, A_t, on the set of finite subsets of S satisfying

$$P^\eta[\eta_t \supset A] = P^A[A_t \subset \eta]$$

where $\eta \supset A$ means $\eta(x) = 1$ for each $x \in A$. The process A_t is a finite particle simple exclusion process modified so that when the process is in state A particles are created at y at rates $\sum_{x\in A} q(x, y)$. Hence an analysis of the behavior of A_t will lead to results concerning η_t. But because the number of particles in A_t may increase, unlike X_t^n and Y_t^n, the situation is more difficult than the cases treated in this paper. Partial results, which will appear in a forthcoming paper, include $\lim_{t\to\infty} S(t)\nu_\alpha = \nu_0$ for $0 \leqq \alpha < 1$ and $I_e = \{\nu_0, \nu_1\}$ whenever $p(x, y)$ is recurrent.

REMARK. The concept of reducing the study of infinite particle systems to related finite particle systems has been further developed beyond Spitzer's original observation by Holley and Liggett (1975) and Harris (1975).

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