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NONINVARIANCE OF $\bar{d}$-CONVERGENCE OF k-STEP MARKOV APPROXIMATIONS

BY GIDEON SCHWARZ

Tel Aviv University and Hebrew University of Jerusalem

Among the class of totally ergodic stationary discrete stochastic processes, the Bernoulli processes are characterized by $\bar{d}$-convergence of their canonical k-step Markov approximations. Here this property is shown to be no longer invariant under isomorphism if we leave the totally ergodic class. On the other hand, the isomorphism class of all Markov chains is shown to be not closed under $\bar{d}$-convergence.

1. Introduction. Among the various characterizations of the class of Bernoulli processes that arise out of the recent work on measure-preserving transformations (Friedman and Ornstein (1970), Ornstein (1970), Ornstein (1973)), one of the most appealing is the following: for a stationary discrete stochastic process X, let $M^k(X)$ be the (unique) k-step Markov process that shares with X the (joint) distribution of $k+1$ consecutive variables. Then X is a Bernoulli process if and only if the $M^k(X)$ have no cycles, and X is their $\bar{d}$-limit.

By definition, the class of Bernoulli processes is closed under isomorphism. This leads to the question whether when cycles are not ruled out $\bar{d}$-convergence of the k-step Markov approximations is still an isomorphism invariant. We show here that this is not the case. For every $\eta > 0$ and integer $n > 1$, we construct two processes isomorphic to the direct product of an n-rotation and a Bernoulli shift of entropy η, such that one of the processes is the $\bar{d}$-limit of its M^k, and the other one is not.

Furthermore, we construct a process whose M^k do converge $\bar{d}$, yet the process is not isomorphic to any Markov chain.

2. The example for noninvariance. For the case where $\bar{d}$-convergence of the M^k is to be proved, we chose the process itself to be Markov, so that for $k > 0$, the M^k are all the same as the process they approximate, and convergence holds trivially. For the other process, the M^k will be shown to have no cycles, which for a non-Bernoulli process implies nonconvergence.

Heuristically, the construction of the processes can be described as follows. In a (doubly infinite) sequence of independent, identically distributed random variables, replace each entry by a block of n equal entries. If the starting point of the sequence is then placed uniformly in a block, the result is a stationary process Y. If we pair the variables of this process with a cyclic counting process,

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that keeps track of the location in the block, it turns into a Markov process, while not leaving its isomorphism class.

In the formal definition, we find it convenient to define the second process—the Markov one—first. Let $(p) = (p_1, \cdots, p_h)$ be a probability vector of entropy $n\eta$. Let X be the stationary Markov chain with states (i, m), $i = 1, \cdots, h$, $m = 1, \cdots, n$, and the following transitions: if $m < n$, (i, m) leads with probability 1 to $(i, m + 1)$, and states of the form (i, n) are followed with probability p_j by $(j, 1)$, independently of i. The entropy of this process is easily found to be $H((p))/n = \eta$, and since it is a Markov chain of period n, it is isomorphic to the direct product of an n-rotation with a Bernoulli shift of that entropy.

Now, let Y be the process that consists just of the first elements of the pairs that are the values of X. In other words, we erase the counting of repetitions. Outside the null-event of constant Y-sequences, the X-sequences can be reconstructed from the Y-sequences. We simply look for two consecutive entries in the Y-sequence that are different. The second one we pair with $m = 1$, and this determines all the m-values that were erased. The mapping that created Y out of X is therefore almost surely invertible, and X and Y are isomorphic.

Now the probability of $k + 1$ consecutive entries in Y all being equal to j, is positive. Therefore, in $M^k(Y)$ a block of k j's has positive conditional probability of being followed by another j. Equivalently, a block of k j's leads with positive probability to a similar block in the (1-step) Markov chain we obtain from $M^k(Y)$ by passing to overlapping blocks of length k. Hence this Markov chain is not periodic, and neither is $M^k(Y)$ itself. Consequently, if the $M^k(Y)$ were to converge in the $\bar{d}$-topology to Y, it would be a Bernoulli process.

3. A non-Markov limit of Markov chains. Construct a sequence G_n of processes with states ± 1 as follows:

Let S_0 be the sequence of length 1 whose only entry is 1. For $n > 0$, let S_n be the sequence of length 2^n obtained by writing down S_{n-1} twice consecutively, and then changing the sign of the last entry. Denote by G_n the stationary process whose typical strings are infinite repetitions of S_n. Each G_n is $(2^n - 1)$-step Markov of period 2^n. The sequence S_∞, whose first 2^n-block is S_n for every n, is the Toeplitz sequence, whose nth entry is -1 raised to the power of 2 that divides n. It is generic for a process G_∞, that has a 2^n-rotation factor for every n, and is therefore not isomorphic to any Markov chain. It is, in fact, isomorphic to the von Neumann transformation. However, since the easily established relation $\bar{d}(G_{n-1}, G_n) = 2^{-n}$ implies $\bar{d}(G_n, G_\infty) \leqq 2^{-n}$ (in fact, $\bar{d}(G_n, G_\infty) = \frac{2}{3}2^{-n}$), the process G_∞ is the $\bar{d}$-limit of the G_n.

4. Remark. A characterization of the class of processes whose M^k converge $\bar{d}$, and of the $\bar{d}$-closure of all k-step Markov chains will appear in a joint paper with Daniel Rudolph [5].

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