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WIENER FUNCTIONALS AS ITÔ INTEGRALS[1]

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Every measurable real-valued function f on the space of Wiener process paths $\{W(t): 0 \leqq t \leqq 1\}$ can be represented as an Itô stochastic integral $\int_0^1 \varphi(t, \omega)\, dW(t, \omega)$ where φ is a nonanticipating functional with $\int_0^1 \varphi(t, \omega)^2\, dt < \infty$ for almost all ω.

Let $W(t, \omega)$ be a standard Wiener process, $W_t \equiv W(t) \equiv W(t, \cdot)$. A function $\varphi(t, \omega)$ is called nonanticipating iff for all $t \geqq 0$, $\varphi(t, \cdot)$ is measurable with respect to $\{W_s: 0 \leqq s \leqq t\}$. The Itô stochastic integral

$$f(\omega) \equiv \int_0^1 \varphi(t, \omega)\, d_t W(t, \omega)$$

is defined for any jointly measurable, nonanticipating φ such that for almost all ω, $\int_0^1 \varphi^2(t, \omega)\, dt < \infty$ (Gikhman and Skorokhod (1968), Chapter 1, Section 2). It is known that if $E \int_0^1 \varphi^2(t, \omega)\, dt < \infty$, then $Ef = 0$ and $Ef^2 < \infty$. Representation of an arbitrary measurable f as a stochastic integral was stated, but later retracted, by J.M.C. Clark (1970, 1971).

To illustrate our method, we will first show that for an arbitrary probability law P on R, there is a stochastic integral f with law P. Indeed there is a measurable g such that $g(W_{\frac{1}{2}})$ has law P. Let $\varphi(t, \omega) = 0$, $0 \leqq t \leqq \frac{1}{2}$. Let $\varphi(t, \omega) = 1/(1-t)$ for $\frac{1}{2} < t < \tau(\omega)$, the least time such that

$$\int_{\frac{1}{2}}^{\tau} 1/(1-t)\, dW(t, \omega) = g(W_{\frac{1}{2}})\,.$$

Then $\tau < 1$ a.s. since $\int_{\frac{1}{2}}^1 (1-t)^{-2}\, dt = +\infty$. Let $\varphi(t, \omega) = 0$ for $t \geqq \tau(\omega)$. This yields the desired result.

To prove the theorem stated in the abstract, let $g = \arctan f$. Then $|g| < \pi/2$ everywhere. For a sequence $t(n) \uparrow 1$ to be specified later, let B_n be the smallest σ-algebra with respect to which $W(t, \cdot)$ are measurable for all $t \in [0, t(n)]$. Sample continuity of $W(t)$ at 1 implies that g is measurable with respect to the σ-algebra generated by the union of the B_n. Thus, by martingale convergence, $g_n \equiv E(g \mid B_n) \to g$ almost surely. So $f_n \equiv \tan g_n \to f$ a.s., with f_n measurable (B_n).

Now, beginning with any sequence $s(n) \uparrow 1$ such as $s(n) = 1 - 1/n$, we choose $t(n)$ as a subsequence with $\Pr\{|f_n - f| > 1/n^3\} < 1/n^2$, so that if $x_n \equiv x_n(\omega) \equiv (f_{n+1} - f_n)(\omega)$, then

$$\Pr\{(n+1)|x_n(\omega)| > 4n^{-2}\} < 2n^{-2}\,. \tag{1}$$

Now we consider integrals of the form $X_t \equiv \int_a^t v(s)\, dW(s, \omega)$ for $0 \leqq a \leqq t$ and nonrandom v. Then X_t is a Gaussian process with mean 0 and covariance

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$EX_tX_u = \min(h(t), h(u))$ where $h(t) = \int_a^t v^2(s)\,ds$. Thus, X_t has the same law as $W_{h(t)}$. If $\int_a^b v^2(s)\,ds = +\infty$, then for any x, a.s. there is some $t < b$ with $X_t = x$, $t > a$. Taking τ as the least such t, let G be the random variable $\int_a^{\tau(\omega)} v^2(s)\,ds = h\{\tau(\omega)\}$. Then G has the distribution of the least time $T = T(x, \omega)$ such that $W_T = x$ (starting at $W_0 = 0$, as usual). This law has density, for any $u \geqq 0$ (e.g., Itô–McKean (1965), page 25),

$$\Pr\{u \leqq T \leqq u + du\} = (2/\pi)^{\frac{1}{2}} \exp(-x^2/2u)\{|x|/2u^{\frac{3}{2}}\}\,du < (|x|/2u^{\frac{3}{2}})\,du$$

so $\Pr(T \geqq u) \leqq |x|/u^{\frac{1}{2}}$. Hence

$$\Pr(G \geqq u) \leqq |x|/u^{\frac{1}{2}} . \tag{2}$$

Now we define φ. Let $\varphi(t, \omega) = 0$ for $0 \leqq t \leqq t(1)$. For $n = 1, 2, \cdots$, let $v_n(s) = 1/\{t(n+1) - s\}$. Let $\tau_n(\omega)$ be the least $t > t(n)$ such that $\int_{t(n)}^t v_n(s)\,dW(s, \omega) = f_n(\omega) - f_{n-1}(\omega)$ (letting $f_0(\omega) \equiv 0$). Define

$$\begin{aligned} \varphi(s, \omega) &= v_n(s)\,, \quad t(n) < s \leqq \tau_n(\omega)\,; \\ &= 0\,, \qquad \tau_n(\omega) < s \leqq t(n+1)\,. \end{aligned}$$

This defines a nonanticipating function φ such that for each n,

$$\int_0^{t(n+1)} \varphi(s, \omega)\,dW(s, \omega) = f_n(\omega)\,. \tag{3}$$

We have

$$\int_0^1 \varphi^2(s, \omega)\,ds = \sum_{n=1}^\infty \int_{t(n)}^{\tau_n(\omega)} v_n(s)^2\,ds = \sum_{n=1}^\infty G_n$$

where by (2), $\Pr(G_n \geqq n^{-2} \mid B_n) \leqq n|x_{n-1}(\omega)|$. Thus by (1),

$$\Pr(G_n \geqq n^{-2}) \leqq 2(n-1)^{-2} + 4(n-1)^{-2} = 6(n-1)^{-2}\,,$$

so $\sum G_n < \infty$ a.s. and $\varphi(\cdot, \omega) \in \mathscr{L}^2[0, 1]$ a.s.

Thus $\int_0^t \varphi(s, \omega)\,dW(s, \omega)$ is a.s. continuous in t (Gikhman and Skorokhod (1968), Chapter 1, Section 3, Theorem 2). This and (3) give the desired result since $f_n \to f$. □

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