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# LEVEL CROSSINGS OF A STOCHASTIC PROCESS WITH ABSOLUTELY CONTINUOUS SAMPLE PATHS

BY MICHAEL B. MARCUS

*Nothwestern University*

Let $X(t)$, $t \in [0, 1]$ be a real valued stochastic process with absolutely continuous sample paths. Let $M(a, X(t))$ denote the number of times $X(t) = a$ for $t \in (0, 1]$ and $N(a, X(t))$ the number of times $X(t)$ crosses the level $a$ for $t \in (0, 1]$. Under certain conditions on the joint density function of $X(t)$ and its derivative $\dot{X}(t)$, integral expressions are obtained for $E[\prod_{i=1}^{k} N(a_i, X(t))^{j_i}]$ for $j_i$ positive integers (similarly with $M$ replacing $N$).

Examples of Gaussian processes $X(t)$ are found for which $X(0) \equiv 0$, $EN(a, X(t)) < \infty$, $a \neq 0$ but $EN(0, X(t)) = \infty$. Also examples of stationary Gaussian processes are given for which $EN(a, X(t)) < \infty$ for all $a$, $EN^2(0, X(t)) = \infty$ but $E[N(0, X(t))N(a, X(t))] < \infty$ for $a \neq 0$. These examples are used to describe the clustering of the zeros of a certain class of Gaussian processes.

**1. Introduction.** Let $X(t)$ be a stochastic process with absolutely continuous sample paths. A great amount of attention has been given to studying the random variables $M(a, X(t))$, the number of times $X(t) = a$ (say for $t \in (0, 1]$), and $N(a, X(t))$ the number of times the paths of $X(t)$ "cross" the level $a$. Rice (1945) and Kac (1943), (1943a), obtained expressions for the expected value, $EN(a, X(t))$, for a certain class of Gaussian processes and polynomials with random coefficients. Itô (1964) found necessary and sufficient conditions for $EN(a, X(t)) < \infty$ when $X(t)$ is a stationary Gaussian process. Higher moments for $N(a, X(t))$ were considered in a series of papers by Cramér and Leadbetter and Ylvisaker. An explicit expression for the $k$th factorial moment $E[N(a, X(t))(N(a, X(t)) - 1) \cdots (N(a, X(t)) - k + 1)]$ was obtained for stationary Gaussian processes under mild regularity conditions in Cramér and Leadbetter (1965) and Ylvisaker (1966). Other references can be found in Cramér and Leadbetter (1967).

The results of Cramér and Leadbetter and Ylvisaker are generalized in two directions. We do not restrict ourselves to stationary Gaussian processes (see also Leadbetter (1966)) and we obtain integral expressions for quantities such as

$$E[N(a_1, X(t))^{j_1} \cdots N(a_k, X(t))^{j_k}] \tag{1.1}$$

for levels $a_1, \cdots, a_k$ and integers $j_1, \cdots, j_k$. The integral will be called a generalized Rice's function. It is usually not a continuous function of $\mathbf{a} = (a_1, \cdots, a_k)$. This observations bears upon the results of Brillinger (1972). It is this lack

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of continuity that accounts for the analytical complications that arise in the proofs of many results on level crossings.

The integral expression for (1.1) is obtained under certain conditions on the joint density function of $X(t)$ and its derivative $\dot{X}(t)$ (in the sense of absolute continuity). Under rather weak conditions, upper bounds are obtained for (1.1) but with $M(a_i, X(t))$ replacing $N(a_i, X(t))$, $i = 1, \cdots, k$.

A precise statement of our results will be given in Section 2. Proofs of the main theorems are given in Sections 3 and 4. In the earlier papers on this subject Kac (1943, 1943a), Ivanov (1960) and Itô (1964) obtain $EN(a, X(t))$ by first finding a function that counts the level crossings of a real valued function. Then they substitute $X(t)$ for the function and take the expectation. This method was not used for finding the $k$th factorial moment of the number of level crossings for stationary Gaussian processes but we return to it in this paper.

In Section 5 sufficient conditions are presented, simpler than the ones given in Section 2, which show that our results hold for stationary and nonstationary Gaussian processes subject to certain obvious minimal conditions. In Section 6 we exhibit Gaussian processes $X(t)$, with $X(0) \equiv 0$, for which $EN(a, X(t)) < \infty$ for $a \neq 0$ but $EN(0, X(t)) = \infty$. In Section 7 examples are given of stationary Gaussian processes for which $EN(a, X(t)) < \infty$ for all $a$, $EN^2(0, X(t)) = \infty$ but $E[N(0, X(t))N(a, X(t))] < \infty$ whenever $a \neq 0$. These examples and some other considerations are used to give a picture of what the zeros are like for certain Gaussian processes for which $EN(0, X(t)) < \infty$ but $EN^2(0, X(t)) = \infty$.

**2. Results and discussion.** Let $X(t)$, $t \in [0, 1]$ be a stochastic process with absolutely continuous sample paths. Let

$$p_{t_1,\cdots,t_k}(x_1, \cdots, x_k; y_1, \cdots, y_k) \tag{2.1}$$

be the joint density function of $X(t_1), \cdots, X(t_k); \dot{X}(t_1), \cdots, \dot{X}(t_k)$ where $\dot{X}$ denotes the derivative of $X$ in the sense of absolute continuity. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{t}$ denote vectors in $R^k$. The function in (2.1) will also be written as $p_{\mathbf{t}}(\mathbf{x}; \mathbf{y})$. Define (with $d\mathbf{y} = dy_1 \cdots, dy_n$)

$$g_{\mathbf{t}}(\mathbf{x}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\mathbf{t}}(\mathbf{x}; \mathbf{y})|y_1| \cdots |y_k|\, d\mathbf{y}\,; \tag{2.2}$$

$$A_\delta = \{\mathbf{t} \mid t_i > \delta;\ |t_i - t_j| > \delta;\ i, j = 1, \cdots, k;\ i = j\}\,. \tag{2.3}$$

Let $\mathbf{a} = (a_1, \cdots, a_k)$. $X(t)$ will be said to satisfy condition I$k$ at $\mathbf{a}$, for a fixed positive integer $k$, if the following hold:

(I$k$, 1) $p_{\mathbf{t}}(\mathbf{x}; \mathbf{y})$ exists for all $k$-tuples $(t_1, \cdots, t_k)$ of distinct values of $t \in (0, 1]$ ($t = 0$ is omitted to allow $X(0) \equiv 0$);

(I$k$, 2) given $\delta > 0$ there exists an $\eta > 0$ so that for $\mathbf{x} \in \bigotimes_{i=1}^{k} (a_i - \eta, a_i + \eta)$ and $\mathbf{t} \in A_\delta$

$$g_{\mathbf{t}}(\mathbf{x}) \leqq M_\delta \quad \text{for some constant} \quad M_\delta\,; \qquad \text{and} \tag{2.4}$$

(I$k$, 3) for each $\mathbf{t} \in A_\delta$, $g_{\mathbf{t}}(\mathbf{x})$ is continuous at $\mathbf{a}$.

Let $X_n(t)$ be the $n$th polygonal approximation of $X(t)$ formed by taking $X_n(t) = X(t)$ for $t = k/2^n$, $k = 0, 1, \cdots, 2^n$. If the joint density function of $X_n(t_1), \cdots, X_n(t_k); \dot{X}_n(t_1), \cdots, \dot{X}_n(t_k)$ exists we denote it by $p_{\mathbf{t},n}(\mathbf{x}; \mathbf{y})$. Define

$$g_{\mathbf{t},n}(\mathbf{x}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\mathbf{t},n}(\mathbf{x}; \mathbf{y})|y_1| \cdots |y_k| \, d\mathbf{y} . \tag{2.5}$$

The family $\{X_n(t)\}$ of polygonal approximations of $X(t)$ will be said to satisfy condition II$k$ at $\mathbf{a}$ if the following hold:

(II$k$, 1) For each $\delta > 0$ there is an $N_\delta$ so that for $n \geqq N_\delta$ $p_{\mathbf{t},n}(\mathbf{x}; \mathbf{y})$ exists for all $\mathbf{t} \in A_\delta$.

(II$k$, 2) For each $n \geqq N_\delta$ there exists an $\eta > 0$ so that for $\mathbf{x} \in \bigotimes_{i=1}^k (a_i - \eta, a_i + \eta)$ and $\mathbf{t} \in A_\delta$

$$g_{\mathbf{t},n}(\mathbf{x}) \leqq M_{\delta,n} \tag{2.6}$$

for some constant $M_{\delta,n}$. (Note $M_{\delta,n}$ and $\eta$ can both depend on $n$.)

(II$k$, 3) For each $n \geqq N_\delta$ and $\mathbf{t} \in A_\delta$, $g_{\mathbf{t},n}(\mathbf{x})$ is continuous at $\mathbf{a}$.

(II$k$, 4) For each $\mathbf{t} \in A_\delta$, $g_{\mathbf{t},n}(\mathbf{a}) \leqq T_\delta$ uniformly in $n \geqq N_\delta$, for some constant $T_\delta$.

(II$k$, 5) For each $\mathbf{t} \in A_\delta$, $\lim_{n\to\infty} g_{\mathbf{t},n}(\mathbf{a}) = g_{\mathbf{t}}(\mathbf{a})$.

For fixed $k$ we define the integral on $R^k$

$$I(\mathbf{a}, X(t)) = \int_1^0 \cdots \int_0^1 g_{\mathbf{t}}(\mathbf{a}) \, d\mathbf{t} . \tag{2.7}$$

$I(\mathbf{a}, X(t))$ is the familiar integral that appears in Cramér and Leadbetter (1965) and Ylvisaker (1966) except that we do not require $a_1 = a_2 = \cdots = a_k$; we call $I(\mathbf{a}, X(t))$ a generalized Rice's function. Written out,

$$I(\mathbf{a}, X(t)) = \int_0^1 \cdots \int_0^1 p_{t_1,\cdots,t_k}(a_1, \cdots, a_k; y_1, \cdots, y_k)|y_1| \cdots |y_k| \, dy_1 \cdots dy_k \, dt_1 \cdots dt_k .$$

Let $G_a$ be the set of continuous functions $f(t)$ on $[0, 1]$ which have the property that $f(t)$ is not identically equal to $\mathbf{a}$ on any interval of $[0, 1]$. Following Cramér and Leadbetter (1967, page 192) a function $f(t) \in G_a$ is said to cross the level $a$ at $t = t_0$ if in each neighborhood of $t_0$ there are points $t_1$ and $t_2$ such that $[f(t_1) - a][f(t_2) - a] < 0$. We define $N(a, f(t))$ as the number of times that $f(t)$ crosses the level $a$ for $t \in (0, 1]$ and $M(a, f(t))$ is defined as the number of times $f(t) = a$ for $t \in (0, 1]$. Clearly $M(a, f(t)) \geqq N(a, f(t))$. If $N(a, f(t)) < \infty$ then each crossing of the level $a$ is either an upcrossing or a downcrossing.

If a process $X(t)$ satisfies (I$k$, 1) then $X(t) \in G_a$ a.s. Therefore we can consider the random variables $N(a, X(t))$ and $M(a, X(t))$.

If $\mathbf{a} = (a_1, \cdots, a_k)$ the integral $I(\mathbf{a}, X(t))$ is related to the expectation of a function of the $k$ random variables $M(a_i, X(t))$, $i = 1, \cdots, k$. We now define that function. Consider the numbers $a_1, \cdots, a_k$. Suppose that $n_1$ of them are equal to $b_1, \cdots, n_j$ of them are equal to $b_j$; $\sum_{i=1}^j n_i = k$. Define

$$\begin{aligned} P_k[M(a_1, X(t)), \cdots, M(a_k, X(t))] \\ = \prod_{i=1}^j M(b_i, X(t))[M(b_i, X(t)) - 1] \cdots [M(b_i, X(t)) - n_i + 1] . \end{aligned} \tag{2.8}$$

Note that if $a_1 = a_2 = \cdots = a_k = a$, $P_k[M(a_1, X(t)), \cdots, M(a_k, X(t))] = M(a, X(t))[M(a, X(t)) - 1] \cdots [M(a, X(t)) - k + 1]$ the $k$th factorial moment of $M(a, X(t))$ and if all the $a_i$ are different

$$P[M(a_1, X(t)), \cdots, M(a_k, X(t))] = \prod_{i=1}^{k} M(a_i, X(t)) .$$

We can now state our main results:

THEOREM 2.1. *Let $X(t)$, $t \in [0, 1]$ be a stochastic process with absolutely continuous sample paths. Let $X_n(t)$ be the* $n$th *polygonal approximation of $X(t)$ as defined above. For a fixed integer $k$ and $\mathbf{a} = (a_1, \cdots, a_k) \in R^k$ assume that $X(t)$ satisfies condition* I$k$ *at* $\mathbf{a}$ *then*

$$E[P_k(M(a_1, X(t)), \cdots, M(a_k, X(t))] \leqq I(\mathbf{a}, X(t)) . \tag{2.9}$$

*If in addition $\{X_n(t)\}$ satisfies condition* II$k$ *at* $\mathbf{a}$ *then*

$$\begin{aligned} E[P_k(M(a_1, X(t)), \cdots, M(a_k, X(t)))] \\ = E[P_k(N(a_1, X(t)), \cdots, N(a_k, X(t)))] = I(\mathbf{a}, X(t)) \end{aligned} \tag{2.10}$$

*and if $I(\mathbf{a}, X(t)) < \infty$, then*

$$N(a_i, X(t)) = M(a_i, X(t)) \quad \text{a.s.}, \qquad i = 1, \cdots, k , \tag{2.11}$$

i.e., *tangencies to the levels $a_i$, $i = 1, \cdots, k$ occur on a set of measure zero of the process.*

Let $\{Y_n(t)\}$ be a refining sequence of polygonal processes in the following sense:

$$Y_n(t) \quad \text{is linear} \quad t \in \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right); \qquad k = 1, \cdots, 2^n; n = 1, \cdots \tag{2.12}$$

$$Y_{n+1}(t) = Y_n(t) \qquad t = k/2^n ; \quad k = 0, 1, \cdots, 2^n ; \quad n = 1, \cdots . \tag{2.13}$$

Assume that for each $n$, $Y_n(t) \in G_{a_i}$ a.s., $i = 1, \cdots, k$ and consider $N(a_i, Y_n(t))$. $N(a_i, Y_n(t))$ increases: denote its limit by $\tilde{N}(a_i)$.

THEOREM 2.2. *Let $\{Y_n(t)\}$ be a refining sequence of polygonal stochastic processes as defined above, assume that $\{Y_n(t)\}$ satisfies condition* II$k$ *at* $\mathbf{a}$, *then*

$$\begin{aligned} E[P_k(\tilde{N}(a_1), \cdots, \tilde{N}(a_k))] &= \lim_{n\to\infty} E[P_k(N(a_1, Y_n(t)), \cdots, N(a_k, Y_n(t)))] \\ &= \lim_{n\to\infty} I(\mathbf{a}, Y_n(t)) . \end{aligned}$$

In Theorem 3.1 of Brillinger (1972) a result like our Theorem 2.1 is shown to hold for almost all $\mathbf{a} \in R_k$. Corollary 3.2 of that paper gives conditions under which our Theorem 2.1 holds for a fixed $\mathbf{a} \in R^k$ with $a_1 = a_2 = \cdots = a_k$. One of the conditions of the corollary is that $I(\mathbf{a}, X(t))$ is continuous at $\mathbf{a}$, but one can see from Theorem 2.1 that for $k > 1$, $I(\mathbf{a}, X(t))$ is not continuous at $\mathbf{a}$ (when $a_1 = a_2 = \cdots = a_k$). Brillinger's methods for obtaining results that are true for almost all $\mathbf{a}$, and those of Geman and Horowitz (1973), who obtain a similar result for the first moment of stationary processes, are different from the ones that we use.

In dealing with the polygonal families $\{X_n(t)\}$ approximating $X(t)$, care must be taken. Condition I$k$, 1 is not the same as condition II$k$, 1. The density function $p_{\mathbf{t},n}(\mathbf{x}; \mathbf{y})$ can not exist for all $k$-tuples $(t_1, \cdots, t_k)$ of distinct values of $\mathbf{t} \in (0, 1]$ because for $t \in [(k-1)/2^n, k/2^n]$ both $X_n(t)$ and $\dot{X}_n(t)$ are determined by $X((k-1)/2^n)$ and $X(k/2^n)$. In II$k$, 1 we require only that $p_{\mathbf{t},n}(\mathbf{x}; \mathbf{y})$ exist for $\mathbf{t} \in A_\delta$ for sufficiently large $n$. This means that the $t_i$; $i = 1, \cdots, k$ in $(t_1, \cdots, t_k)$ are all separated by at least $\delta$. Therefore, if $n$ is taken large enough, each $t_i$ lies in a different interval of length $2^{-n}$. Let $\bar{p}_t(x_1, \cdots, x_{2k})$ be the joint density of $X(t_1), \cdots, X(t_{2k})$. Fix a value of $k$. If $\bar{p}_t(x_1, \cdots, x_{2k})$ exists for all $2k$-tuples $(t_1, \cdots, t_{2k})$ of distinct values of $t \in (0, 1]$ then, for this value of $k$, $p_{\mathbf{t},n}(\mathbf{x}, \mathbf{y})$ will exist for $t \in A_\delta$, $n \geqq N_\delta$.

**3. Proofs for the case $k = 1$.** The method of proof is the following: a functional is obtained that counts the level crossings of a real valued function. Replacing the real valued function by a stochastic process we obtain an integer valued random variable and take its expectation. Define

$$\varphi_{a,\Delta}(x) = 1 \qquad |x - a| \leqq \Delta \tag{3.1}$$
$$= 0 \qquad \text{otherwise.}$$

LEMMA 3.1. *Let $f(t) \in G_a$, $f(1) \neq a$ and assume that $f(t)$ is absolutely continuous with derivative $\dot{f}(t)$, then*

$$M(a, f(t)) \leqq \lim_{\delta\to 0} \liminf_{\Delta\to 0} \frac{1}{2\Delta} \int_\delta^1 \varphi_{a,\Delta}(f(t))|\dot{f}(t)|\, dt\,. \tag{3.2}$$

*Define $M_\delta(a, f(t))$ as the number of times $f(t) = a$ for $t \in (\delta, 1]$.*

PROOF. This lemma is given in Ivanov (1960) and also in Itô (1964). Kac (1943) first used the expression on the right in (3.2) (with $\dot{f}(t)$ continuous) to count the zeros of a function. We will give a proof both for completeness and because we shall refer to it in the proof of Lemma 3.2.

Suppose $f(t) = a$ for a finite number of times $\delta < t_1 < \cdots < t_n < 1$. For each $\Delta \leqq \Delta_0$ for some sufficiently small $\Delta_0$ we can find $n$ disjoint sets $(s_k, s'_{k'})$ such that $t_k \in (s_k, s_k')$, $s_1 > \delta$, $s_k' < 1$ and

$$\left|\int_{s_k}^{t_k} \dot{f}(s)\, ds\right| = \Delta \qquad \text{and} \qquad \left|\int_{t_k}^{s_k'} \dot{f}(s)\, ds\right| = \Delta\,. \tag{3.3}$$

(Note that $s_k = \max\{s : s < t_k, |f(s) - a| = \Delta\}$ and $s_k' = \min\{s : s > t_k, |f(s) - a| = \Delta\}$. The absolute continuity of $f$ is also used in (3.3).) Since $|f(s) - a| \leqq \Delta$ for $s \in (s_k, s_k')$,

$$\frac{1}{2\Delta} \int_\delta^1 \varphi_{a,\Delta}(f(t))|\dot{f}(t)|\, dt \geqq \sum_{k=1}^n \left(\int_{s_k}^{s_k'} |\dot{f}(s)|\, ds\right)/2\Delta \geqq n \tag{3.4}$$

(for the last inequality use (3.3)).

Therefore, in this case

$$M_\delta(a, f(t)) \leqq \liminf_{\Delta\to 0} \frac{1}{2\Delta} \int_\delta^1 \varphi_{a,\Delta}(f(t))|\dot{f}(t)|\, dt$$

and if $f(t) = a$ a finite number of times in $[0, 1]$, (3.2) follows. If $f(t) = a$ for an infinite number of points $t$, then for each $m$ we can choose a $\delta$ and $t_1, \cdots, t_m \in (\delta, 1]$ and show that the right side of (3.4) is greater than or equal to $m$. Therefore both sides in (3.2) will be infinite.

Define $G_{a,n}$ as the set of continuous functions $f(t) \in G_a$ which have the further property that $f(t) \neq a$ for $t = k/2^n$, $k = 1, \cdots, 2^n$.

LEMMA 3.2. *Let $f_n(t) \in G_{a,n}$ be linear for $t \in ((k-1)/2^n, k/2^n)$, $k = 1, \cdots, 2^n$. Let $N_\delta(a, f_n(t))$ be the number of times $f_n(t)$ crosses $a$ for $t \in (\delta, 1]$. Then*

$$N_\delta(a, f_n(t)) = \lim_{\Delta \to 0} \frac{1}{2\Delta} \int_\delta^1 \varphi_{a,\Delta}(f_n(t)) |\dot{f}_n(t)| \, dt \tag{3.5}$$

*and*

$$\frac{1}{2\Delta} \int_0^1 \varphi_{a,\Delta}(f_n(t)) |\dot{f}_n(t)| \, dt \leqq 2^n . \tag{3.6}$$

PROOF. Suppose $f_n(t) = a$ for a finite number of times $\delta < t_1 < \cdots < t_n < 1$. For all $\Delta \leqq \Delta_0$ for some sufficiently small $\Delta_0$ we can find $n$ disjoint sets $(s_k, s_k')$ with $t_k \in (s_k, s_k')$, $(s_k, s_k') \in (j_k/2^n, (j_k+1)/2^n)$ for some $0 \leqq j_k < 2^n$, $s_1 > \delta$, $s_k' < 1$ and such that (3.3) holds with $f_n(t)$ replacing $f(t)$. Furthermore, since $f_n(t)$ only has a finite number of maxima and minima the intervals $(s_k, s_k')$ can also be chosen so that $\varphi_{a,\Delta}(f_n(t)) = 0$ if $t \notin \bigcup_{k=1}^n (s_k, s_k')$ and since $f_n(t)$ does not change sign in $(s_k, s_k')$ there is equality throughout (3.4) when $f_n(t)$ replaces $f(t)$. Clearly $M_\delta(a, f_n(t)) = N_\delta(a, f_n(t))$.

To establish (3.6),

$$\frac{1}{2\Delta} \int_0^1 \varphi_{a,\Delta}(f_n(t)) |\dot{f}_n(t)| \, dt = \sum_{k=1}^{2^n} \frac{1}{2\Delta} \int_{(k-1)/2^n}^{k/2^n} \varphi_{a,\Delta}(f_n(t)) |\dot{f}_n(t)| \, dt \leqq 2^n , \tag{3.7}$$

since $\dot{f}_n(t)$ does not change sign in $((k-1)/2^n, k/2^n)$.

PROOF OF THEOREM 2.1, (2.9). In this case $P_1(M(a, X(t))) = M(a, X(t))$. Because the joint density function $p_t(x, y)$ exists for all $t \in (0, 1]$ the paths of $X(t) \in G_a$ and $X(1) \neq a$ except possibly on a set of measure zero. Therefore, almost surely, the paths of $X(t)$ satisfy the hypothesis of Lemma 3.1. Using Lemma 3.1, the monotone convergence theorem and Fubini's theorem we get

$$\begin{aligned} E[M(a, X(t))] &\leqq \lim_{\delta \to 0} \liminf_{\Delta \to 0} \frac{1}{2\Delta} \int_\delta^1 E\varphi_{a,\Delta}(X(t)) |\dot{X}(t)| \, dt \\ &= \lim_{\delta \to 0} \liminf_{\Delta \to 0} \frac{1}{2\Delta} \int_\delta^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_{a,\Delta}(x) |y| p_t(x; y) \, dx \, dy \, dt \\ &= \lim_{\delta \to 0} \liminf_{\Delta \to 0} \int_\delta^1 \frac{1}{2\Delta} \int_{a-\Delta}^{a+\Delta} \int_{-\infty}^{\infty} |y| p_t(x; y) \, dy \, dx \, dt . \end{aligned} \tag{3.8}$$

Using (2.4), (3.8) is

$$\leqq \lim_{\delta \to 0} \int_\delta^1 \limsup_{\Delta \to 0} \frac{1}{2\Delta} \int_{a-\Delta}^{a+\Delta} \int_{-\infty}^{\infty} |y| p_t(x; y) \, dy \, dx \, dt .$$

The proof is completed by using condition (I$k$, 3).

PROOF OF THEOREM 2.2, (2.10). In this case $P_1(N(a, X(t))) = N(a, X(t))$. Since the joint density $p_t(x; y)$ exists for $t \in (0, 1]$, $X(t) \in G_{a,n}$ for all $n$ a.s. Therefore, Lemma 3.2 can be applied to the paths of $X_n(t)$ for all $n$.

$$E[N_\delta(a, X_n(t))] = E\left[\lim_{\Delta\to 0} \frac{1}{2\Delta} \int_\delta^1 \varphi_{a,\Delta}(X_n(t))|\dot{X}_n(t)|\, dt\right].$$

By (3.6) and the dominated convergence theorem this is

$$= \lim_{\Delta\to 0} E\left[\frac{1}{2\Delta} \int_\delta^1 \varphi_{a,\Delta}(X_n(t))|\dot{X}_n(t)|\, dt\right]$$

$$= \lim_{\Delta\to 0} \int_\delta^1 \frac{1}{2\Delta} \int_{a-\Delta}^{a+\Delta} \int_{-\infty}^{\infty} p_{t,n}(x, y)|y|\, dy\, dx\, dt \tag{3.9}$$

$$= \int_\delta^1 \int_{-\infty}^{\infty} p_{t,n}(a, y)|y|\, dy\, dt\,. \tag{3.10}$$

(3.9) uses Fubini's theorem; (3.10) follows by dominated convergence using conditions (II$k$, 2) and (II$k$, 3). (Condition (II$k$, 1) is not needed when $k = 1$.) Let $\delta_m = 1/2^m$. Observe that as $n \to \infty$ $N_{\delta_m}(a, X_n(t))$ increases to $N_{\delta_m}(a, X(t))$. Therefore

$$E[N_{\delta_m}(a, X(t))] = \lim_{n\to\infty} \int_{\delta_m}^1 \int_{-\infty}^{\infty} p_{t,n}(a, y)|y|\, dy\, dt\,. \tag{3.11}$$

Condition (II$k$, 4) enables us to use the dominated convergence theorem in (3.11) and applying (II$k$, 5) we get

$$E[N_{\delta_m}(a, X(t))] = \int_{\delta_m}^1 \int_{-\infty}^{\infty} p_t(a, y)|y|\, dy\, dt\,.$$

The proof now follows by the monotone convergence theorem since $N_{\delta_m}(a, X(t))$ increases, as $m \to \infty$, to $N(a, X(t))$. Since $M(a, X(t)) \geqq N(a, X(t))$ we get (2.11).

PROOF OF THEOREM 2.2, $k = 1$. It follows from the proof of Theorem 2.1 that

$$E[N_\delta(a, Y_n(t))] = \int_\delta^1 \int_{-\infty}^{\infty} p_{t,n}(a, y)|y|\, dy\, dt\,.$$

Theorem 2.2 now follows from the monotone convergence theorem with respect to $\delta$ and $n$.

**4. Proofs for $k \geqq 1$.** Given $A_\delta$ as defined in Section 2, let $P_{k,\delta}(M(a_1, f(t)), \cdots, M(a_k, f(t)))$ be the number of different $k$-tuples $(t_1, \cdots, t_k) \in A_\delta$ where $t_i \in f^{-1}(a_i)$ (i.e., each $t_i$ is a solution of $f(t) = a_i$). Clearly

$$P_{k,\delta}(M(a_1, f(t)), \cdots, M(a_k, f(t))) \uparrow P_k(M(a_1, f(t)), \cdots, M(a_k, f(t))) \tag{4.1}$$

as $\delta \downarrow 0$. Let $I_\delta(t_1, \cdots, t_k) = \chi(A_\delta)$ where $\chi(A)$ denotes the characteristic function of the set $A$.

LEMMA 4.1. *Given* $\mathbf{a} = (a_1, \cdots, a_k)$ *assume* $f(t) \in G_{a_i}$, $f(1) \neq a_i$ $i = 1, \cdots, k$. *Assume that* $f(t)$ *is absolutely continuous with derivative* $\dot{f}(t)$. *Then*

$$\begin{aligned} &P_{k,\delta}(M(a_1, f(t)), \cdots, M(a_k, f(t))) \\ &\leqq \liminf_{\Delta\to 0} \left(\frac{1}{2\Delta}\right)^k \int_0^1 \cdots \int_0^1 I_\delta(t_1, \cdots, t_k)\{\textstyle\prod_{i=1}^k \varphi_{a_i,\Delta}(f(t_i))|\dot{f}(t_i)|\}\, d\mathbf{t}\,. \end{aligned} \tag{4.2}$$

PROOF. Assume that $f(t) = a_i$ for $n(i)$ values of $t$ denoted by $t_{j(i)}$; $j(i) = 1, \cdots, n(i)$; $i = 1, \cdots, k$. Consider the finite number of $k$-tuples $(t_{j(1)}, \cdots, t_{j(k)}) \in A_\delta$. For each $(t_{j(1)}, \cdots, t_{j(k)}) \in A_\delta$ we can find a $\Delta_0$ such that for each $\Delta \leqq \Delta_0$ there are intervals (of $R^1$), $(s_{j(1)}, s'_{j(1)}) \ni t_{j(1)}, \cdots, (s_{j(k)}, s'_{j(k)}) \ni t_{j(k)}$ for which

$$\bigcup \otimes_{i=1}^k (s_{j(i)}, s'_{j(i)}) \subset A_\delta \, , \tag{4.3}$$

where the union is taken over each of the intervals (of $R^k$) $\otimes_{i=1}^k (s_{j(i)}, s'_{j(i)})$ containing one of the $k$-tuples $(t_{j(1)}, \cdots, t_{j(k)}) \in A_\delta$, $j(i) = 1, \cdots, n(i)$; $i = 1, \cdots, k$, the terms in the union in (4.3) are disjoint and

$$|\textstyle\int_{s_{j(i)}}^{t_{j(i)}} \dot{f}(t)\, dt| = \Delta \, ; \qquad |\textstyle\int_{t_{j(i)}}^{s'_{j(i)}} \dot{f}(t)\, dt| = \Delta \, .$$

Therefore

$$\int_{s_{j(1)}}^{s'_{j(1)}} \cdots \int_{s_{j(k)}}^{s'_{j(k)}} \left(\frac{1}{2\Delta}\right)^k \{\textstyle\prod_{i=1}^k \varphi_{a_i,\Delta}(f(t_i))|\dot{f}(t_i)|\}\, d\mathbf{t} \tag{4.4}$$

$$\geqq \prod_{i=1}^k \frac{\int_{s_{j(i)}}^{s'_{j(i)}} |\dot{f}(t_i)|\, dt_i}{|\int_{s_{j(i)}}^{t_{j(i)}} \dot{f}(t_i)\, dt_i| + |\int_{t_{j(i)}}^{s'_{j(i)}} \dot{f}(t_i)\, dt_i|} \geqq 1 \, . \tag{4.5}$$

(See the proof of Lemma 3.1 for more details on how the intervals $(s_{j(i)}, s'_{j(i)})$ are chosen.)

By (4.3) and using the fact that the terms in the union are disjoint, for $\Delta \leqq \Delta_0$

$$P_{k,\delta}(M(a_1, f(t)), \cdots, M(a_k, f(t))) \leqq \left(\frac{1}{2\Delta}\right)^k \int_0^1 \cdots \int_0^1 I_\delta(t_1, \cdots, t_k)\{\textstyle\prod_{i=1}^k \varphi_{a_i,\Delta}(f(t_i))|\dot{f}(t_i)|\}\, d\mathbf{t} \, .$$

Therefore the lemma holds whenever the left side in (4.2) is finite. It follows as in the proof of Lemma 3.1 that if the left side of (4.2) is infinite so is the right side. This completes the proof of the lemma.

LEMMA 4.2. *Let $f_n(t) \in G_{a_i,n}$; $i = 1, \cdots, k$ and be linear for $t \in ((k-1)/2^n, k/2^n)$, $k = 1, \cdots, 2^n$. Let $\mathbf{a} = (a_1, \cdots, a_k)$, then*

$$P_{k,\delta}(N(a_1, f_n(t)), \cdots, N(a_k, f_n(t))) = \lim_{\Delta\to 0} \left(\frac{1}{2\Delta}\right)^k \int_0^1 \cdots \int_0^1 I_\delta(t_1, \cdots, t_k)\{\textstyle\prod_{i=1}^k \varphi_{a_i,\Delta}(f_n(t_i))|\dot{f}_n(t_i)|\}\, d\mathbf{t} \, , \tag{4.6}$$

*and*

$$\left(\frac{1}{2\Delta}\right)^k \int_0^1 \cdots \int_0^1 I_\delta(t_1, \cdots, t_k)\{\textstyle\prod_{i=1}^k \varphi_{a_i,\Delta}(f_n(t_i))|\dot{f}_n(t_i)|\}\, d\mathbf{t} \leqq 2^{kn} \, . \tag{4.7}$$

PROOF. Following the proof of Lemma 3.1 we can obtain (4.4) with $f_n(t)$ replacing $f(t)$ and with the additional condition that each $(s_{j(i)}, s'_{j(i)}) \subset ((k-1)/2^n, k/2^n)$ for some $k = 1, \cdots, 2^n$ since $f_n(t) \in G_{a_i,n}$, $i = 1, \cdots, k$. Therefore $\dot{f}_n(t_{j(i)})$ does not change sign in $(s_{j(i)}, s'_{j(i)})$. Using this in (4.4) and

(4.5) (with $f_n(t)$ replacing $f(t)$) we get that for $\Delta \leqq \Delta_0$

$$\int_{s_{j(1)}}^{s'_{j(1)}} \cdots \int_{s_{j(k)}}^{s'_{j(k)}} \left(\frac{1}{2\Delta}\right)^k \{\prod_{i=1}^k \varphi_{a_i,\Delta}(f_n(t_i))|\dot{f}_n(t_i)|\}\, d\mathbf{t} = 1 \tag{4.8}$$

for all $\otimes_{i=1}^k (s_{j(i)}, s'_{j(i)}) \subset A_\delta$. Also since $f_n(t)$ has only a finite number of maxima and minima, if $\Delta_0$ is small enough then for $\Delta \leqq \Delta_0$ the integral in (4.8) is zero outside of $\bigcup \otimes_{i=1}^k (s_{j(i)}, s'_{j(i)})$. Therefore (4.6) holds. Inequality (4.7) follows immediately from (3.6) since (4.7)

$$\leqq \prod_{i=1}^k \left\{\frac{1}{2\Delta} \int_\delta^1 \varphi_{a_i,\Delta} (f_n(t)|\dot{f}_n(t)|\, dt\right\} \leqq 2^{kn}$$

by (3.6).

PROOF OF THEOREM 2.1. The joint density $p_{\mathbf{t}}(\mathbf{x}; \mathbf{y})$ exists for all distinct values of $(t_1, \cdots, t_k) \in (0, 1]^k$, $X(t) \in G_{a_i}$; $i = 1, \cdots, k$ a.s. and $X(1) \neq a_i$, $i = 1, \cdots, k$ a.s. Therefore, almost surely, the paths of $X(t)$ satisfy the hypothesis of Lemma 3.1 so

$$P_{k,\delta}(M(a_1, X(t)), \cdots, M(a_k, X(t)))$$
$$\leqq \liminf_{\Delta\to 0} \left(\frac{1}{2\Delta}\right)^k \int_0^1 \cdots \int_0^1 I_\delta(t_1, \cdots, t_k)\{\prod_{i=1}^k \varphi_{a_i,\Delta}(X(t_i))|\dot{X}(t_i)|\}\, d\mathbf{t}$$

almost surely. Using Fatou's lemma and Fubini's theorem we get

$$E\{P_{k,\delta}(M(a_1, X(t)), \cdots, M(a_k, X(t)))]$$
$$\leqq \liminf_{\Delta\to 0} \int_0^1 \cdots \int_0^1 I_\delta(t_1, \cdots, t_k) \left(\frac{1}{2\Delta}\right)^k$$
$$\times \int_{a_1-\Delta}^{a_1+\Delta} \cdots \int_{a_k-\Delta}^{a_k+\Delta} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\mathbf{t}}(\mathbf{x}; \mathbf{y})|y_1| \cdots |y_k|\, d\mathbf{y}\, d\mathbf{x}\, d\mathbf{t}\, .$$

Condition (I$k$, 2) allows us to use the dominated convergence theorem to bring the limit on $\Delta$ inside the integral with respect to $\mathbf{t}$ and the continuity condition (I$k$, 3) enables us to take the limit. Therefore

$$E[P_{k,\delta}(M(a_1, X(t)), \cdots, M(a_k, X(t)))] \leqq \int_0^1 \cdots \int_0^1 I_\delta(t_1, \cdots, t_k) g_{\mathbf{t}}(\mathbf{a})\, d\mathbf{t}\, . \tag{4.9}$$

(2.9) now follows by (4.1) and the monotone convergence theorem. To prove (2.10) we note that since $X(t)$ satisfies condition (I$k$, 1), almost surely $X(t) \in G_{a_i,n}$, $i = 1, \cdots, k$; $n = 1, 2, \cdots$. Consequently $X_n(t) \in G_{a_i,n}$, $i = 1, \cdots, k$; $n = 1, 2, \cdots$, almost surely and Lemma 3.2 can be applied to the paths of $X_n(t)$. Given $\delta$ take $N_\delta$ large enough so that $p_{\mathbf{t},n}(\mathbf{x}, \mathbf{y})$ exists for all $\mathbf{t} \in A_\delta$, $n \geqq N_\delta$. For $n \geqq N_\delta$, by Lemma 3.2

$$E[P_{k,\delta}(N(a_1, X_n(t)), \cdots, N(a_k, X_n(t)))] \tag{4.10}$$

$$= \lim_{\Delta\to 0} E\left[\left(\frac{1}{2\Delta}\right)^k \int_0^1 \cdots \int_0^1 I_\delta(t_1, \cdots, t_k) \times \{\prod_{i=1}^k \varphi_{a_i,\Delta}(X_n(t_i))|\dot{X}_n(t_i)|\}\right] d\mathbf{t}\, . \tag{4.11}$$

In (4.11) the limit and integral can be interchanged because of (4.7). Following the proof of (2.9) and using conditions (II$k$, 1), (II$k$, 2) and (II$k$, 3) for $\{X_n(t)\}$ we get that (4.11) is equal to

$$\int_0^1 \cdots \int_0^1 I_\delta(t_1, \cdots, t_k) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{t,n}(\mathbf{x}; \mathbf{y})|y_1| \cdots |y_k| \, d\mathbf{y} \, d\mathbf{t} . \tag{4.12}$$

Using conditions (II$k$, 4) and (II$k$, 5) we can take the limit in (4.12) as $n \to \infty$; it is equal to

$$\int_0^1 \cdots \int_0^1 I_\delta(t_1, \cdots, t_k) g_t(\mathbf{a}) \, d\mathbf{t} . \tag{4.13}$$

Clearly

$$\begin{aligned} \lim_{n\to\infty} P_{k,\delta}(N(a_1, X_n(t)), \cdots, N(a_k, X_n(t))) \\ = P_{k,\delta}(N(a_1, X(t)), \cdots, N(a_k, X(t))) . \end{aligned} \tag{4.14}$$

If we show that for a fixed $\delta$ and $N_\delta$ sufficiently large

$$\begin{aligned} P_{k,\delta}(N(a_1, X_n(t)), \cdots, N(a_k, X_n(t))) \\ \leqq P_{k,\delta/8}(N(a_1, X(t)), \cdots, N(a_k, X(t))) \end{aligned} \tag{4.15}$$

for $n \geqq N_\delta$ the proof of (2.10) can be completed. Assume (4.15) by (4.9) and condition (I$k$, 2) for $X(t)$

$$E[P_{k,\delta/8}(N(a_1, X(t)), \cdots, N(a_k, X(t)))] < \infty . \tag{4.16}$$

Therefore, taking the limit as $n \to \infty$ in (4.10) and using the dominated convergence theorem by virtue of (4.15) and (4.16), we can interchange the limit and expectation and using (4.14) we get that (4.10) is equal to

$$E[P_{k,\delta}(N(a_1, X(t)), \cdots, N(a_k, X(t)))] .$$

Finally we use the monotone convergence theorem to take the limit with respect to $\delta$ and obtain the second equality in (2.10). The first equality comes from (2.9) and the fact that $M(a_i, X(t)) \geqq N(a_i, X(t))$.

We now obtain (4.15). Let $\mathscr{B}_m$ be the set of all subsets formed by taking unions of the sets $\otimes_{i=1}^k [(j_i - 1)/2^m, j_i/2^m]$, $j_i = 1, \cdots, 2^m$. For a given set $A_\delta$ let $\mathscr{B}_{m,\delta}$ be the smallest set contained in $\mathscr{B}_m$ such that $A_\delta \subset \mathscr{B}_{m,\delta}$. Analogous to the definition of $P_{k,\delta}$ let $Q_{k,\delta,m}(N(a_1, X(t)), \cdots, N(a_k, X(t)))$ be a random variable equal to the number of different $k$-tuples $(t_1, \cdots, t_k) \in \mathscr{B}_{m,\delta}$ where $t_i = X^{-1}(a_i)$. Then

$$\begin{aligned} P_{k,\delta}(N(a_1, X_n(t)), \cdots, N(a_k, X_n(t))) &\leqq Q_{k,\delta,m}(N(a_1, X_n(t)), \cdots, N(a_k, X_n(t))) \\ &\leqq Q_{k,\delta,m}(N(a_1, X(t)), \cdots, N(a_k, X(t))) . \end{aligned}$$

For any $\delta < 2^{-k}$ we can find an $M_\delta$ so that for $m \geqq M_\delta$, $\mathscr{B}_{m,\delta} \subset A_{\delta/8}$, consequently

$$Q_{k,\delta,m}(N(a_1, X(t)), \cdots, N(a_k, X(t))) \leqq P_{k,\delta/8}(N(a_1, X(t)), \cdots, N(a_k, X(t)))$$

and we have (4.15). (2.11) follows as in Section 3.

The proof of Theorem 2.2 also follows as in Section 3.

**5. Sufficient conditions for I$k$ and II$k$ to be satisfied at a and applications to Gaussian processes.** Let $X(t)$, $t \in [0, 1]$ be a stochastic process with absolutely continuous sample paths that satisfies (I$k$, 1). If the joint density function satisfies the following conditions then $X(t)$ satisfies condition I$k$ at $\mathbf{a}$:

(5.1) Given $\delta > 0$ there exists an $\eta > 0$ so that for $t \in A_\delta$ and $\mathbf{x} \in \bigotimes_{i=1}^{k} (a_i - \eta, a_i + \eta)$, $p_t(\mathbf{x}, \mathbf{y}) \leqq h_\delta(y)$ where $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |y_1| \cdots |y_k| h_\delta(\mathbf{y})\, d\mathbf{y} < \infty$

and

(5.2) for each $\mathbf{t} \in A_\delta$, $p_t(\mathbf{x}, \mathbf{y})$ is continuous at $\mathbf{a}$.

Let $\{X_n(t)\}$ be the family of polygonal process associated with $X(t)$. As we remarked at the end of Section 2, under mild conditions on $X(t)$ given a $\delta > 0$ the density function $p_{\mathbf{t},n}(\mathbf{x}; \mathbf{y})$ will exist for $n \geqq N_\delta$ for all $\mathbf{t} \in A_\delta$. If the density function satisfies the following conditions then $\{X_n(t)\}$ satisfies condition II$k$ at $\mathbf{a}$:

(5.3) Given $\delta > 0$ there exists an $\eta > 0$ so that for $t \in A_\delta$, $n \geqq N_\delta$ and $\mathbf{x} \in \bigotimes_{i=1}^{k} (a_i - \eta, a_i + \eta)$, $p_{\mathbf{t},n}(\mathbf{x}, \mathbf{y}) \leqq h_\delta(\mathbf{y})$ where $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |y_1| \cdots |y_k| h_\delta(\mathbf{y})\, d\mathbf{y} < \infty$,

(5.4) for each $n \geqq N_\delta$ and $\mathbf{t} \in A_\delta$, $p_{\mathbf{t},n}(\mathbf{x}; \mathbf{y})$ is continuous at $\mathbf{a}$

and

$$\lim_{n\to\infty} p_{\mathbf{t},n}(\mathbf{a}, \mathbf{y}) = p_{\mathbf{t}}(\mathbf{a}, \mathbf{y})\,. \tag{5.5}$$

These assertions are easily proved. Given (5.1) and (5.2), (I$k$, 2) and (I$k$, 3) follow from the dominated convergence theorem. Similarly (5.3), (5.4) and (5.5) imply (II$k$, 2), (II$k$, 3), (II$k$, 4) and (II$k$, 5).

Let $X(t)$ be a separable Gaussian process with covariance $R(t, s) = EX(t)X(s)$. Assume that $R(t, s)$ has continuous first and second partial derivatives in a closed set in $R^2$ containing $[0, 1] \times [0, 1]$. Then $X(t)$, $t \in [0, 1]$ has a version with absolutely continuous sample paths $\dot{X}(t)$. The process $\dot{X}(t)$ is the Gaussian process with covariance function $r(s, t) = \partial^2 R(s, t)/\partial t\, \partial s$. For a process of this type if the joint density function exists then I$k$ and II$k$ are satisfied for any $\mathbf{a} \in R^k$ and consequently (2.10) holds.

THEOREM 5.1. *Let $X(t)$ be a separable Gaussian process with covariance function $R(s, t)$ satisfying the conditions given above. Assume that $X(t)$ satisfies condition* (I$k$, 1) *and* (II$k$, 1); *then $X(t)$ satisfies* I$k$ *at* $\mathbf{a}$ *and* II$k$ *at* $\mathbf{a}$ *for every* $\mathbf{a} \in R^k$ *and consequently Theorem* 2.1 *holds for every* $\mathbf{a} \in R^k$.

PROOF. Recall $\mathbf{t} = (t_1, \cdots, t_k)$. Let $|R(\mathbf{t})|$ denote the determinant of the $2k \times 2k$ matrix $R = \{EZ_i(s_i)Z_j(s_j)\}$ $i, j = 1, \cdots, 2k$ where $s_i = s_{i+k} = t_i$, $Z_i = X$ and $Z_{i+k} = \dot{X}$, $i = 1, \cdots, k$. (This is the covariance matrix of the random variables $X(t_1), \cdots, X(t_k), \dot{X}(t_1), \cdots, \dot{X}(t_k)$.) The hypothesis that $p_{\mathbf{t}}(\mathbf{x}; \mathbf{y})$ exists for all $k$-tuples $(t_1, \cdots, t_k)$ of distinct values of $t \in (0, 1]$ implies that $|R(\mathbf{t})| > 0$ for these values. For any $\delta$ we can find a compact set $G_\delta$ such that $A_\delta \subset G_\delta \subset A_{\delta/2}$. Since $|R(\mathbf{t})|$ is a continuous function of $\mathbf{t}$, $|R(\mathbf{t})| \geqq \alpha(\delta) > 0$ for $\mathbf{t} \in A_\delta$, where $\alpha(\delta)$ is a constant that depends on the set $A_\delta$. Furthermore $|R(\mathbf{t})| \leqq C$ for $\mathbf{t} \in [0, 1]^k$ for some constant $C$. Let $P(\mathbf{t}) = R^{-1}(\mathbf{t})$; $P(\mathbf{t})$ exists for $\mathbf{t} \in A_\delta$ for all $\delta > 0$ and $\lambda(\mathbf{t})$, the smallest eigenvalue of $P(\mathbf{t})$, is greater than zero, $\mathbf{t} \in A_\delta$, $\delta > 0$. Let $P = \{P_{ij}\}$, $i, j = 1, \cdots, 2k$, we have

$$\sum_{i,j=1}^{2k} P_{ij}(\mathbf{t}) z_i z_j \geqq \lambda(\mathbf{t}) \sum_{i=1}^{2k} z_i^2 .$$

Since $\lambda(\mathbf{t})$ is continuous in $\mathbf{t}$, $\lambda(\mathbf{t}) \geqq \beta(\delta) > 0$ for $\mathbf{t} \in A_\delta$ where $\beta(\delta)$ is a constant depending on $A_\delta$. Therefore

$$p_{\mathbf{t}}(\mathbf{x}; \mathbf{y}) \leqq (2\pi)^{-k} |\alpha(\delta)|^{-\frac{1}{2}} \exp\left\{-\frac{\beta(\delta)}{2}(x_1^2 + \cdots + x_k^2 + y_1^2 + \cdots + y_k^2)\right\}$$

$$\leqq (2\pi)^{-k} |\alpha(\delta)|^{-\frac{1}{2}} \exp\left\{-\frac{\beta(\delta)}{2}(y_1^2 + \cdots + y_k^2)\right\} . \tag{5.6}$$

The function in (5.6) serves as the function $h_\delta(\mathbf{y})$ in (5.1). Condition (5.2) is also satisfied. Therefore $X(t)$ satisfies I$k$ at $\mathbf{a}$ for every $\mathbf{a} \in R^k$.

The conditions on $R(t, s)$ imply that the elements of the covariance matrix of $X_n(t_1), \cdots, X_n(t_k)$; $\dot{X}_n(t_1), \cdots, \dot{X}_n(t_k)$, ($\mathbf{t} \in A_\delta$, $n \geqq N_\delta$) converge uniformly to the elements of the covariance matrix of $X(t_1), \cdots, X(t_k)$; $\dot{X}(t_1), \cdots, \dot{X}(t_k)$. If we denote by $R_n$, $P_n$, $\alpha_n(\delta)$ and $\beta_n(\delta)$ the matrices and constants relating to $X_n(t_1), \cdots, X_n(t_k)$; $\dot{X}_n(t_1), \cdots, \dot{X}_n(t_k)$, in analogy to $R$, $P$, $\alpha(\delta)$ and $\beta(\delta)$ defined above, we get that $\alpha_n(\delta)$ and $\beta_n(\delta)$ converge uniformly to $\alpha(\delta)$ and $\beta(\delta)$. Therefore for $n > N_\delta'$, for some integer $N_\delta'$ depending on $\delta$

$$p_{\mathbf{t},n}(\mathbf{x}; \mathbf{y}) \leqq \frac{2}{|\alpha(\delta)|^{\frac{1}{2}}} \exp\left\{-\frac{\beta(\delta)}{4}(y_1^2 + \cdots + y_k^2)\right\} . \tag{5.7}$$

The function on the right in (5.7) serves as $h_\delta(\mathbf{y})$ in (5.3); (5.4) is clearly satisfied. Condition (5.5) is also satisfied because of the convergence of the entries in $R^n$ to those in $R$. Therefore $X(t)$ satisfies II$k$ at $\mathbf{a}$ for every $\mathbf{a}$. (Note that inequality (5.6) is similar to the inequality on page 208 in Cramér and Leadbetter (1967).)

REMARK 5.2. The existence of $p_{\mathbf{t}}(\mathbf{x}, \mathbf{y})$ for distinct values $(t_1, \cdots, t_k)$ of $t[0, 1]$ is assured if $X(t)$ is stationary, mean square differentiable and has a continuous component in its spectrum (Cramér and Leadbetter (1967), page 203). However, sometimes $X(t)$ satisfies (I$k$, 1) for some but not all values of $k$. Examples of this are the Gaussian polynomials considered by Kac (1943).

**6. Expected number of zeros of a special class of Gaussian processes.** Let $\xi(t)$ be a stationary Gaussian process, $E\xi^2(t) = 1$, $E\xi(t) = 0$. Define $R(t) = E\xi(s+t)\xi(s)$ and $\sigma^2(t) = E(\xi(t+s) - \xi(s))^2 = 2(1 - R(t))$. Consider the process

$$X(t) = \int_0^t \xi(u)\, du \tag{6.1}$$

$t \in [0, 1]$. $X(t)$ is a zero mean Gaussian process with absolutely continuous sample paths.

THEOREM 6.1. *For $X(t)$ as defined above*

$$\text{If} \quad a \neq 0 , \qquad EN(a, X(t)) < \infty , \tag{6.2}$$

*otherwise for some* $\delta > 0$

$$EN(0, X(t)) < \infty \Leftrightarrow \int_0^\delta \frac{1}{s^2} \left|\int_0^s \tau\sigma^2(\tau)\, d\tau\right|^{\frac{1}{2}} ds < \infty . \tag{6.3}$$

PROOF. Let $Q = \{q_{ij}\}_{i,j=1,2}$ be the covariance matrix for $X(t)$ and its derivative $\xi(t)$, i.e.,

$$\begin{aligned} q_{11} = EX^2(t) &= \int_0^t \int_0^t R(u - v)\, dv\, du = 2 \int_0^t (t - \tau)R(\tau)\, d\tau \\ &= t^2 - \int_0^t (t - \tau)\sigma^2(\tau)\, d\tau \end{aligned} \tag{6.4}$$

$$\begin{aligned} q_{12} = q_{21} &= EX(t)\xi(t) = \int_0^t R(t - u)\, du = \int_0^t R(u)\, du = t - \tfrac{1}{2} \int_0^t \sigma^2(u)\, du \\ q_{22} &= E\xi^2(t) = 1 . \end{aligned} \tag{6.5}$$

Let $|Q|$ denote the determinant of $Q$,

$$|Q| = \int_0^t \tau\sigma^2(\tau)\, d\tau + \tfrac{1}{4}\left(\int_0^t \sigma^2(u)\, du\right)^2 . \tag{6.6}$$

Since $|Q| > 0$, $t \in (0, 1]$, the joint density function $p_t(x, y)$ of $X(t)$ and $\xi(t)$ exists for $t \in (0, 1]$. One can also check that the joint density of $X(t)$ and $X(s)$ exists for $t \neq s$, $t$, $s \in (0, 1]$. Therefore, by Theorem 5.1 and Remark 5.2, $EN(a, X(t)) = I(a, X(t))$.

Let $P = \{p_{ij}\}_{i,j=1,2} = Q^{-1}$, then

$$\begin{aligned} I(a, X(t)) &= \int_0^1 \int_{-\infty}^{\infty} p_t(a, y)|y|\, dy\, dt \\ &= \frac{1}{2\pi} \int_0^1 \int_{-\infty}^{\infty} \frac{1}{|Q|^{\frac{1}{2}}} e^{-\frac{1}{2}(p_{11}a^2 + 2p_{12}ay + p_{22}y^2)} |y|\, dy\, dt , \\ &= \frac{1}{2\pi} \int_0^1 \int_{-\infty}^{\infty} \frac{1}{|Q|^{\frac{1}{2}} p_{22}} e^{-a^2/2p_{22}|Q|} \left| s - \frac{ap_{12}}{(p_{22})^{\frac{1}{2}}} \right| e^{-s^2/2}\, ds\, dt . \end{aligned} \tag{6.7}$$

Since $p_{22} = q_{11}/|Q|$ and $p_{12} = -q_{12}/|Q|$, (6.7) is equal to

$$\frac{1}{2\pi} \int_0^1 \int_{-\infty}^{\infty} \frac{|Q|^{\frac{1}{2}}}{q_{11}} e^{-a^2/2q_{11}} \left| s + \frac{aq_{12}}{|Q|^{\frac{1}{2}}(q_{11})^{\frac{1}{2}}} \right| e^{-s^2/2}\, ds\, dt . \tag{6.8}$$

Suppose $a \neq 0$. If $t \in (\delta, 1]$ for some $\delta > 0$ all the functions of $t$ in (6.8) are finite. Therefore we need only consider this integral for $t \in (0, \delta]$. From (6.4) and (6.5) we see that $q_{11} \sim t^2$ and $q_{12} \sim t$ as $t \downarrow 0$. Therefore $I(a, X(t)) < \infty$ if

$$\int_0^\delta \frac{1}{q_{11}} e^{-a^2/2q_{11}}\, dt < \infty . \tag{6.9}$$

This integral is finite so (6.2) is established.

When $a = 0$

$$I(0, X(t)) = \frac{1}{\pi} \int_0^1 \frac{|Q|^{\frac{1}{2}}}{q_{11}}\, dt .$$

Again we need only consider the integral for $t \in (0, \delta]$. Substituting for $|Q|$, and noting that $q_{11} \sim t^2 \downarrow 0$, (6.3) follows since

$$\left(\int_0^s \sigma^2(u)\, du\right)^2 \leqq 4\left(\int_0^s \tau\sigma^2(\tau)\, d\tau\right) \tag{6.10}$$

for $s$ sufficiently small. To see (6.10) observe that the left side in (6.10) is equal to

$$(6.11)\qquad \int_0^s \int_0^v \sigma^2(u)\, du\sigma^2(v)\, dv + \int_0^s \int_v^s \sigma^2(u)\, du\sigma^2(v)\, dv\,.$$

For $s$ sufficiently small the first term in (6.11) is less than

$$2 \int_0^s v\sigma^2(v)\, dv\,.$$

The same is true for the second term in (6.11) as can be seen by interchanging the order of integration.

Suppose $\sigma^2(\tau)$ is a slowly varying function, in this case the condition in (6.3) simplifies to

$$(6.12)\qquad \int_0^\delta \frac{\sigma(\tau)}{\tau}\, d\tau < \infty\,.$$

It is well known (Marcus and Shepp (1970)) that there are continuous stationary Gaussian processes $\xi(t)$ for which the integral in (6.12) is not finite. For these processes $I(a, X(t))$ is not continuous at $a = 0$. Also for such a process $EN(0, X(t)) = \infty$ but $N(0, X(t)) < \infty$ a.s. This answers a question of Dudley (1973), page 95, at least if $X(t)$ is not required to be stationary.

**7. Mixed second moments of certain stationary Gaussian processes.** In this section we find examples of certain stationary Gaussian processes for which $EN^2(0, X(t)) = \infty$ but $E[N(a, X(t))N(0, X(t))] < \infty$ for any $a \neq 0$. This provides concrete examples of processes for which $I(\mathbf{a}, X(t))$ is discontinuous when $\mathbf{a} \in R^2$. The major application of these examples will be in Section 8 where we speculate on the nature of the zeros of certain stationary Gaussian processes.

It is easily checked that (given $0 < d < 1$) we may define a covariance function $r(t)$, $t \in [0, 1]$ such that

$$(7.1)\qquad r(t) = 1 - t^2/2(1 - (\log 1/t)^{-d}) + O(t^4)$$

for $t \in [0, \delta]$ for some $\delta > 0$. Let $\theta(t) = (\log 1/t)^{-d}$ and $\rho(t) = 1 - \theta(t)$. Note that $\theta'(t) = d/t(\log 1/t)^{d+1}$ and $\theta''(t) = -d/t^2(\log 1/t)^{d+1} + d(d+1)/t^2(\log 1/t)^{d+2}$. Let $\varphi(t) = t\theta'(t) = d\theta(t)/\log 1/t$ so

$$(7.2)\qquad \varphi(t) = o(\theta^2(t))\,.$$

In this notation

$$(7.3)\qquad r'(t) = -t\rho(t) + \frac{t}{2}\varphi(t) + O(t^3)$$

and

$$(7.4)\qquad r''(t) = -\rho(t) + \tfrac{3}{2}\varphi(t) + \frac{(d+1)\varphi(t)}{2(\log 1/t)} + O(t^2)\,.$$

Let $X(t)$, $t \in [0, 1]$ be a stationary Gaussian process with covariance $r(t)$. Such a process satisfies the hypothesis of Theorem 5.1 so for these examples $E[N^2(0, X(t))] = I((0, 0)) = \infty$ by (8.3). Also $E[N(a, X(t)) \cdot N(0, X(t))] = I((a, 0))$. We will show that $I((a, 0))$ is finite for these examples. Since $X(t)$ is stationary we have

$$(7.5)\qquad I((a,0)) = \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1-t)p_{0,t}(a,0;y_1,y_2)|y_1||y_2|\,dy_1\,dy_2\,dt\,.$$

Let $\Lambda(t)$ denote the covariance matrix of $X(t)$. As in (10.7.3) of Cramér and Leadbetter (1967) we have

$$(7.6)\qquad \Lambda(t) = \begin{bmatrix} 1 & r(t) & 0 & r'(t) \\ r(t) & 1 & -r'(t) & 0 \\ 0 & -r'(t) & 1 & -r''(t) \\ r'(t) & 0 & -r''(t) & 1 \end{bmatrix}.$$

Let $M_{ij}$ be the cofactor of the $(ij)$th element of $\Lambda$. The values of $M_{ij}$ that we will need are listed below (here $r = r(t)$, $\theta = \theta(t)$, etc.).

$$\begin{aligned} M_{11} &= 1 - (r'')^2 - (r')^2 = 2\theta - \theta^2 + 3\varphi + o(\varphi) \\ M_{12} &= -r(1-(r'')^2) - r''(r')^2 \\ M_{13} &= -r'(r+r'') = t\rho\theta + \tfrac{3}{2}t\varphi + o(t\varphi) \\ M_{14} &= -r'[1+rr''] + (r')^3 = t\rho\theta + \tfrac{3}{2}t\varphi + o(t\varphi) \\ M_{34} &= r''(1-r^2) + r(r')^2 = \tfrac{1}{2}t^2\rho\varphi + o(t^2\rho\varphi) \\ M_{33} &= M_{44} = (1-r^2) - (r')^2 = t^2\rho\theta + t^2\varphi + o(t^2\varphi) \end{aligned}$$

$$M_{11} + M_{12} = (1-r)(1-(r'')^2) - (r')^2(1+r'') = \frac{t^2\rho\theta^2}{2} + o(t^2\rho\theta^2)\,.$$

Let $|\Lambda|$ denote the determinant of $\Lambda$; then

$$\begin{aligned} |\Lambda| &= M_{11} + rM_{12} + r'M_{14} = M_{11} + M_{12} - (1-r)M_{12} + r'M_{14} \\ &= t^2\rho\theta^2 + o(t^2\rho\theta^2)\,. \end{aligned}$$

In this calculation we use $M_{12} = (M_{11} + M_{12}) - M_{11}$.

We proceed to estimate the integral in (7.5).

$$p_{0,t}(a,0;y_1,y_2) = \frac{1}{(2\pi)^2|\Lambda|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2|\Lambda|}(M_{11}a^2 + M_{33}y_1^2 + M_{44}y_2^2 + 2M_{13}ay_1 + 2M_{14}ay_2 + 2M_{34}y_1y_2)\right\}.$$

To simplify the notation let

$\alpha = M_{11}$, $\beta = M_{33} = M_{44}$, $\gamma = M_{34}$, $\delta = M_{13}$ and $\eta = M_{14}$;

let $y_1 = s_1 + s_2$, $y_2 = s_1 - s_2$, then

$$\begin{aligned} &\beta y_1^2 + \beta y_2^2 + 2\gamma y_1 y_2 + 2\delta a y_1 + 2\eta a y_2 \\ &\quad = 2(\beta+\gamma)s_1^2 + 2(\beta-\gamma)s_2^2 + 2a(\delta+\eta)s_1 + 2a(\delta-\eta)s_2 \\ &\quad = 2(\beta+\gamma)\left(s_1 + \frac{a(\delta+\eta)}{2(\beta+\gamma)}\right)^2 + 2(\beta-\gamma)\left(s_2 + \frac{a(\delta-\eta)}{2(\beta-\gamma)}\right)^2 \\ &\qquad - \frac{a^2(\delta+\eta)^2}{2(\beta+\gamma)} - \frac{a^2(\delta-\eta)^2}{2(\beta-\gamma)}\,. \end{aligned}$$

Therefore

$$(7.7)\qquad (2\pi)^2|\Lambda|^{\frac{1}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{0,t}(a, 0; y_1, y_2)|y_1||y_2|\, dy_1 dy_2$$

$$(7.8)\qquad = \Big[2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\Big\{-\frac{1}{2|\Lambda|}\Big((\beta+\gamma)\Big(s_1 + \frac{a(\delta+\eta)}{2(\beta+\gamma)}\Big)^2 + (\beta-\gamma)\Big(s_2 + \frac{a(\delta-\eta)}{2(\beta-\gamma)}\Big)^2\Big)\Big\} \cdot |s_1^2 - s_2^2|\, ds_1\, ds_2\Big] Q(a; t).$$

where

$$(7.9)\qquad Q(a; t) = \exp\Big\{-\frac{a^2}{2|\Lambda|}\Big(\alpha - \frac{(\delta+\eta)^2}{2(\beta+\gamma)} - \frac{(\delta-\eta)^2}{2(\beta-\gamma)}\Big)\Big\}.$$

Let us first estimate $Q(a; t)$ for $t$ small.

$$\begin{aligned}
\alpha &= 2\theta - \theta^2 + 3\varphi + o(\varphi)\\
(\delta+\eta) &= 2t\rho\theta + 6t\varphi + o(t\varphi)\\
(\delta-\eta) &= o(t\varphi)\\
(\beta+\gamma) &= t^2\rho\theta + \tfrac{3}{2}t^2\varphi + o(t^2\varphi)\\
(\beta-\gamma) &= t^2\rho\theta - \tfrac{3}{2}t^2\varphi + o(t^2\varphi)\\
\frac{(\delta+\eta)^2}{2(\beta+\gamma)} &= 2\rho\theta + 9\varphi + o(\varphi)\\
\frac{(\delta-\eta)^2}{2(\beta-\gamma)} &= o(\theta^3)\,.
\end{aligned}$$

Therefore

$$\Big\{\alpha - \frac{(\delta+\eta)^2}{2(\beta+\gamma)} - \frac{(\delta-\eta)^2}{2(\beta-\gamma)}\Big\} = \theta^2 + o(\theta^2)$$

and

$$(7.10)\qquad Q(a; t) = \exp\Big\{-\frac{a^2}{2t^2} + o\Big(\frac{1}{t^2}\Big)\Big\}.$$

Notice that $Q(a; t)$ is very rapidly decreasing as $t \downarrow 0$ unless $a = 0$. This will account for the fact that $I((a, 0)) < \infty$ but $I((0, 0)) = \infty$. In order to complete this demonstration we will estimate the integral in (7.8).

Consider the integral

$$(7.11)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-\alpha_1(s_1+\beta_1)^2 - \alpha_2(s_2+\beta_2)^2]|s_1^2 - s_2^2|\, ds_1\, ds_2\,.$$

Since $|s_1^2 - s_2^2| \leqq s_1^2 + s_2^2$ (7.11) is less than or equal to

$$\frac{1}{\alpha_1^{\frac{3}{2}}\alpha_2^{\frac{1}{2}}}\int_{-\infty}^{\infty} \exp[-(u_1 + (\alpha_1)^{\frac{1}{2}}\beta_1)^2]u_1^2\, du_1 \int_{-\infty}^{\infty} \exp[-(u_2 + (\alpha_2)^{\frac{1}{2}}\beta_2)^2]\, du_2$$
$$+ \frac{1}{\alpha_2^{\frac{3}{2}}\alpha_1^{\frac{1}{2}}}\int_{-\infty}^{\infty} \exp[-(u_2 + (\alpha_2)^{\frac{1}{2}}\beta_2)^2]u_2^2\, du_2 \int_{-\infty}^{\infty} \exp[-(u_1 + (\alpha_1)^{\frac{1}{2}}\beta_1)^2]\, du_1$$

$$(7.12)\qquad = \frac{2}{\alpha_1^{\frac{3}{2}}\alpha_2^{\frac{1}{2}}}(c_1 + c_2\alpha_1\beta_1^2)c_2 + \frac{2}{\alpha_2^{\frac{3}{2}}\alpha_1^{\frac{1}{2}}}(c_1 + c_2\alpha_2\beta_2^2)c_2$$

where $c_1 = \int_{-\infty}^{\infty} u^2 e^{-u^2}\,du$ and $c_2 = \int_{-\infty}^{\infty} e^{-u^2}\,du$. Referring to the integral in (7.8), $\alpha_1 = (\beta + \gamma)/2|\Lambda|$ and $\alpha_2 = (\beta - \gamma)/2|\Lambda|$, consequently $\alpha_1^{-1} = 2\theta + o(\theta)$ and also $\alpha_2^{-1} = 2\theta + o(\theta)$. Note that when $a = 0$ both $\beta_1 = 0$ and $\beta_2 = 0$ and we get

$$I((0, 0)) \leqq C \int_0^{\delta'} \frac{\theta(t)}{t}\,dt \tag{7.13}$$

for constants $C$ and $\delta' > 0$. This is what we expect from Geman (1972). (Actually, in the case we are considering the integral on the right is infinite. But this analysis will work when $d > 1$. Our purpose in making this observation is that it serves as a check on the accuracy of this lengthy computation.) For $a \neq 0$, referring to (7.12) and (7.8)

$$\beta_1^2 = (a^2/4)\frac{(\delta + \eta)^2}{(\beta + \gamma)^2} = \frac{a^2}{t^2} + o(1/t^2)$$

$$\beta_2^2 = (a^2/4)\frac{(\delta - \eta)^2}{(\beta - \gamma)^2} = o(1/t^2)\,,$$

so the term in (7.12) is less than

$$C'\,\frac{a^2\theta}{t^2} \tag{7.14}$$

for some constant $C'$. Incorporating (7.14) and (7.10)

$$I((a, 0)) \leqq \text{Const.} \int_0^{\delta'} \frac{(1 - t)}{|\Lambda|^{\frac{1}{2}}}\, Q(a; t)\, \frac{a^2\theta}{t^2}\,dt$$

$$\sim \text{Const.} \int_0^{\delta'} \frac{(1 - t)}{t^3}\, e^{-a^2/2t^2}\,dt < \infty\,,$$

where $\delta' > 0$. This completes the demonstration that $I((a, 0)) < \infty$ for $a \neq 0$.

We have actually only considered the integral carefully for $t$ near zero but it should be clear that this is all that is necessary.

(Note: We would expect that for these processes $E[N^2(a, X(t))] = \infty$ for all $a$, not just $a = 0$. Based on preliminary calculations this appears to be the case.)

In contrast to the difficulty of the above calculation, it is easy to show that the expected value of products of the number of level crossings, of a very general class of stochastic processes, is finite for almost all levels. Let $X(t)$, $t \in [0, 1]$ be a stochastic process with absolutely continuous sample paths. Then using (3.2) and Fubini's theorem we get

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} E\left[\prod_{i=1}^{n} M(a_i, X(t))\right] da_1 \cdots da_n$$

$$\leqq E\left[\int_0^1 |\dot{X}(t)|\,dt\right]^n = \int_0^1 \cdots \int_0^1 E\left[\prod_{i=1}^{n} |\dot{X}(t_i)|\right] dt_1 \cdots dt_n\,.$$

Therefore if $E[\prod_{i=1}^{n} |\dot{X}(t_i)|] < \infty$ (which is the case for $X(t)$, a stationary Gaussian process with absolutely continuous sample paths) then $E[\prod_{i=1}^{n} M(a_i, X(t))] < \infty$ a.s. with respect to Lebesgue measure on $R^n$. This observation is due to Donald Geman.

**8. A study of the zeros of a certain class of stationary Gaussian processes.** The following lemma is an immediate consequence of (2.4) and (4.9).

LEMMA 8.1. *Let $X(t)$, $t \in [0, 1]$, satisfy* I$k$ *at* **a**. *Then*

$$E[P_{k,\delta}(M(a_1, X(t)), \cdots, M(a_k, X(t)))] < \infty. \tag{8.1}$$

($P_{k,\delta}(\ )$ is defined at the beginning of Section 4.)

Let $X(t)$, $t \in [0, 1]$ be a stationary Gaussian process, with absolutely continuous sample paths, which has a spectrum with a continuous component. Such a process satisfies I$k$ at **a**. Let $k = 2$, $EX(t) = 0$, $EX^2(t) = 1$ and $E\dot{X}^2(t) = 1$. Define $r(t) = EX(s)X(t + s)$. Then

$$EN(0, X) = \frac{1}{\pi} \tag{8.2}$$

(Itô (1964), here $t \in [0, 1]$), and

$$EN^2(0, X(t)) < \infty \Leftrightarrow \int_0^\delta \frac{-r''(0) + r''(t)}{t} dt < \infty \tag{8.3}$$

for some $\delta > 0$ (Geman (1972)).

Suppose $EN^2(0, X(t)) = \infty$. By Lemma 8.1 we have $E[P_{2,\delta}(N(a, X(t)), N(a, X(t)))] < \infty$ for all $a$. We will consider what this means. Take a specific sample path $X(t, \omega)$ and let $t_1, \cdots, t_n$ denote the values of $t$ for which $X(t, \omega) = 0$. Consider the graph of the $n^2$ points $(t_i, t_j)$; $i, j = 1, \cdots, n$. If the zeros $t_1, \cdots, t_n$ were relatively uniformly distributed, the points $(t_i, t_j)$ in the square would be relatively uniformly distributed. Then, since the Lebesgue measure of $A_\delta \to 0$ as $\delta \to 0$, $E[P_{2,\delta}(N(a, X(t)), N(a, X(t)))] < \infty$ would imply that $EN^2(0, X(t)) < \infty$. However, since $EN^2(0, X(t)) = \infty$, at least for some paths the number of pairs $(t_i, t_j)$ outside $A_\delta$ is much smaller than the number of pairs in $A_\delta$. This remains true no matter how small the value of $\delta$. Considering (8.2) most paths have a small number of zeros, or none at all; however there are some paths with a large number of zeros and the zeros of most of these paths form a tight cluster in $A_\delta$.

We can carry this analysis further. The clustering of zeros must be a relatively rare event because, given a path where there is a large cluster of zeros near some point $t'$ it is unlikely that there will be a comparably large cluster near another point $t''$ if $|t' - t''| > 2\delta$. To see this suppose the path has $N$ zeros in an interval of radius $\delta/2$ about $t'$ and $M$ zeros in an interval of radius $\delta/2$ about $t''$ (assume $N \geqq M$). On the unit square such a path would place $N^2 + M^2$ points in $A_\delta$ but also $2NM$ outside $A_\delta$. However, since $E[P_{k\delta}(N(0, X(t)), N(0, X(t)))] < \infty$ and $EN^2(0, X(t)) = \infty$, $N^2 + M^2$ must generally be much larger than $2NM$; which is possible, of course, if $M$ is much smaller than $N$. Note that all the results in this paper hold equally well if $t \in [0, T]$. Consequently, these remarks that imply that the clusters of zeros are rare events hold on any finite interval.

This is how we would describe the sample paths of a stationary Gaussian process for which $EN(0, X(t)) < \infty$, $EN^2(0, X(t)) = \infty$ and $E[N(0, X(t)), N(a, X(t))] < \infty$. The large values of $N(0, X(t))$ occur on a set of paths with small measure. The paths with a large number of zeros generally have them in a single tight cluster. These paths are not likely to have many zeros away from the tight cluster. Furthermore these paths are not likely to have a great many crossings of another fixed level $a$. Since this is true for any $a \neq 0$ the paths must have very small oscillations while attaining their cluster of zeros.

The conjecture that the paths that have a large cluster of zeros must oscillate very little while attaining the zeros is supported by the results of Section 6. Let $\xi(t)$ be a stationary Gaussian process with mean square derivative $\dot{\xi}$. $EN(a, \xi(t)) < \infty$ for any level $a$ yet $EN(0, \xi(t) - \xi(0))$ can be infinite. This follows directly from Theorem 6.1 since

$$\xi(t) - \xi(0) = \int_0^t \dot{\xi}(u)\, du .$$

This suggests that the paths $\xi(t)$ do oscillate a lot but the oscillations are small and no single level is crossed too often. When the different levels are "tied together" as when we consider $\xi(t) - \xi(0)$ then the random variable $N(0, \xi(t) - \xi(0))$ can be quite large.

One can see from Section 7 that evaluating $I(\mathbf{a}, X(t))$, even in special cases, can be very difficult. A great deal of interesting work has been carried out in this direction. We refer the reader to Kac (1959), page 5 and page 259, and to Miroshin (1973). Additional references can be obtained from these sources.

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DEPARTMENT OF MATHEMATICS
NORTHWESTERN UNIVERSITY
EVANSTON, ILLINOIS 60201