The Annals of Probability
1977, Vol. 5, No. 1, 42-51

RANDOM STOPPING PRESERVES REGULAR VARIATION OF PROCESS DISTRIBUTIONS

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Let S_n be a stochastic process with either discrete or continuous time parameter and stationary independent increments. Let N be a stopping time for the process such that $EN < \infty$. If the upper tail of the process distribution, F, is regularly varying, certain conditions on the lower tail of F and on the tail of the distribution of N imply that $\lim_{y\to\infty} P(S_N > y)/(1 - F(y)) = EN$. A similar asymptotic relation is obtained for $\sup_n S_{n\wedge N}$, if n is discrete. These asymptotic results are related to the Wald moment identities and to moment inequalities of Burkholder. Applications are given for exit times at fixed and square-root boundaries.

1. Introduction. If a random walk $S_n = \sum_{i=1}^n X_i$, where the X_i are independent, identically distributed random variables, is evaluated at a stopping time N, the resulting random variable S_N satisfies a family of identities called Wald's equations. For instance $ES_N = EX_1EN$ and, if $EX_1 = 0$, $ES_N^2 = EX_1^2EN$ when appropriate moments are finite. The identities arise from a family of polynomial expressions in n and S_n which form martingales in n and when stopped at N have zero expected value. They have been obtained for random walks by Chow, Robbins and Teicher (1965), (1966), for martingales by Brown (1969), and for processes with stationary independent increments by Hall (1970) and recently by Athreya and Kurtz (1973) using Dynkin's formula. From these identities and Hölder's inequality one can derive certain of the stopping time moment inequalities of Burkholder and Gundy (1970). An example is computed by Athreya and Kurtz. On the other hand, such a moment inequality may give improved moment conditions under which Wald's equations hold. Hall gave an example.

This paper concerns the relation of the asymptotic properties of the distributions of S_N, X_1, and N when $1 - F(y) = P(X_1 > y)$ is regularly varying. Greenwood (1973) found asymptotic conditions on the joint distribution of S_n and N equivalent to

$$\lim_{y\to\infty} P(S_N > y)/(1 - F(y)) = EN . \tag{1}$$

Here we give conditions on F and the distribution of N viewed separately which imply (1). Under these conditions, if S_N is replaced by $\sup_{n\leq N} S_n$, (1) remains true. If $S_N^* = \sup_{n\leq N} |S_n|$, Burkholder and Gundy's (1970) relation $c_pEN^{p/2} \leqq ES_N^{*p} \leqq C_pEN^{p/2}$, holds for random walks S_n if $EX_1 = 0$, $EX_1^2 = 1$, with $0 < p \leqq 2$, and with any positive p if S is Brownian motion. By analogy, we expect

Received February 26, 1975; revised April 29, 1976.

AMS 1970 *subject classifications.* 60G40, 60J15, 60J30.

Key words and phrases. Stopping times, random walk, independent increments, regular variation, asymptotics, boundary crossing, maximum process.

that when relation (1) holds, the condition that $P(N > y)$ decreases like $y^{-p/2}$ should correspond to the condition that $P(X_1 > y)$ decreases like y^{-p}. This expectation would be supported, alternatively, by the observation that Wald's equations for $ES_N{}^p$ contain powers of N up to $N^{p/2}$ for integral p.

We find, in fact (Theorems 1 and 2), that if $P(X_1 > y) \sim y^{-p}$, $P(N > y) = o(y^{-r})$, $r > p/2$, $EX_1 = 0$, and $F(-y) = O(y^{-p})$ then (1) holds. If instead $F(-y) = O(y^{-q})$ where $q < p$, relation (1) still holds if $P(N > y)$ decreases rapidly enough. Corollary 1 says that S_N may be replaced by $\sup_{n \leqq N} S_n$ in the foregoing.

For processes with $EX_1 \neq 0$ we expect a different answer. If $F(1) = 0$, for instance, then $S_N > N$ and $P(S_N > y) \geqq P(N > y)$. If $P(S_N > y)$ is $O(y^{-p})$ then so is $P(N > y)$. We find that (1) holds under the above hypotheses if the condition on N is strengthened to $P(N > y) = o(y^{-r})$, $r > p > 1$.

Limit theorems for ratios of tails of measures have been obtained by Chover, Ney and Wainger (1973) and by Rudin (1973). Their results are related to ours for stopping times N which are independent of the process S_n. They are stronger in some respects and weaker in others.

A result of Monroe (1972) implies that for a strictly stable process S_t, $\lim_{y\to\infty} P(S_T > y)/P(S_1 > y) = ET$ if T is any stopping time such that ET is finite. Corollary 2 extends this property to other processes with stationary independent increments whose Lévy measures ν satisfy $\nu(y, \infty) \sim y^{-p}$.

A straightforward argument extends the results from $P(X_1 > y) \sim y^{-p}$ as written to $P(X_1 > y)$ a regularly varying function. Possibly results similar to these can be proved for processes with distribution functions in the subexponential class discussed by J. Teugels (1975).

Sections 4 and 5 contain applications to boundary-crossing times and a discussion of some remaining problems.

2. Dominant terms and sums of truncated terms. The asymptotic property of the distribution of S_N arises from just one randomly determined summand which dominates the sum. This becomes apparent already in the following lemma.

The notations $f(y) \sim g(y)$, $f(y) \lesssim g(y)$ mean that $\lim_{y\to\infty} f(y)/g(y) = 1$, $\limsup_{y\to\infty} f(y)/g(y) \leqq 1$, respectively. By A' we mean the complement of A.

LEMMA 1. *Let $A_1(b)$, $A_2(b)$ be two parameterized families of Borel sets in R^1 such that $A_i(b) \to R^1$ as $b \to \infty$, $i = 1, 2$ and $A_1' \cap A_2' = \emptyset$ for each $b \in (0, \infty)$. Let $T_i(b) = \min\{n : X_n \notin A_i(b)\}$, $i = 1, 2$, and $T_3(b) = \min\{n > T_1(b) : X_n \notin A_1(b)\}$. Suppose $P(X_1 \notin A_1(b)) \sim b^{-p}$, $P(X_1 \notin A_2(b)) = O(b^{-q})$ as $b \to \infty$. Let N be a stopping time such that $EN < \infty$ and $P(N > n) = o(n^{-r})$ as $n \to \infty$ for some $r \geqq 1$. Then as $b \to \infty$,*

i. $P(T_1(b) \leqq N) \sim b^{-p}EN$,
ii. $P(T_1(b) \vee T_2(b) \leqq N) = o(b^{-p-q+(p\vee q)/r})$,
iii. $P(T_3(b) \leqq N) = o(b^{-2p+p/r})$.

PROOF. i. Each X_n is independent of X_j, $j = 1, \cdots, n-1$ and the event

$(N \geqq n)$ so that

$$\begin{aligned} P(T_1(b) \leqq N) &= \textstyle\sum_{n=1}^{\infty} P(n = \min j\colon X_j \notin A_1(b),\, N \geqq n) \\ &= \textstyle\sum_{n=1}^{\infty} P(X_n \notin A_1(b)) P(X_j \in A_1(b),\, j < n,\, N \geqq n) \\ &\leqq P(X_1 \notin A_1(b)) \textstyle\sum_{n=1}^{\infty} P(N \geqq n) \sim b^{-p} EN \,. \end{aligned}$$

Also,

$$P(T_1(b) \leqq N) \gtrsim b^{-p} \textstyle\sum_{n=1}^{\infty} (P(N \geqq n) - P(T_1(b) < n)) \vee 0 \,.$$

The sum converges to EN by dominated convergence since each term increases to $P(N \geqq n)$ as $b \to \infty$.

ii. We suppress the b and write

$$\begin{aligned} P(T_1 \vee T_2 \leqq N) &= P(T_1 < T_2 \leqq N) + P(T_2 < T_1 \leqq N) \\ &= I + II \,. \end{aligned}$$

It is sufficient to study

$$\begin{aligned} I &= \textstyle\sum_{m=1}^{\infty} P(T_1 < T_2 \leqq N \mid T_1 = m) P(T_1 = m) \\ &= \textstyle\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P(T_2 = m + n \leqq N \mid T_1 = m) P(T_1 = m) \\ &\leqq \textstyle\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P(X_{m+n} \notin A_2(b),\, X_j \in A_2(b), \\ &\qquad m < j < m + n \leqq N \mid T_1 = m) P(T_1 = m) \,. \end{aligned}$$

As in i, the event determined by X_{m+n} is independent of the others. We have

$$I \lesssim b^{-q} \textstyle\sum_{n=1}^{\infty} P(N \geqq T_1 + n) \leqq b^{-q} E(N - T_1,\, N > T_1) \,.$$

If $r = 1$, since $E(N, N > T_1) \to 0$ we have $I = o(b^{-q})$. Similarly $II = o(b^{-p})$. Finally

$$I + II = o(b^{-p \wedge q}) = o(b^{-p-q+p \vee q}) \,.$$

In the case $r > 1$, for any $\varepsilon > 0$ one has

$$\begin{aligned} E(N - T_1, N > T_1) &\leqq E(N, N > \varepsilon b^{p/r}) + E(N, T_1 < N < \varepsilon b^{p/r}) \\ &\leqq [\varepsilon b^{p/r}] P(N > \varepsilon b^{p/r}) + \textstyle\sum_{n=[\varepsilon b^{p/r}]}^{\infty} P(N > n) \\ &\quad + \varepsilon b^{p/r} P(T_1 < N) \,. \end{aligned}$$

Since $P(N > n) = o(n^{-r})$, the first two terms are $o(b^{-p+p/r})$, while by i,

$$\varepsilon b^{p/r} P(T_1 < N) \lesssim \varepsilon b^{-p+p/r} EN \,.$$

But ε is arbitrary, so $I = o(b^{-q-p+p/r})$. For T_2 the proof of i gives $P(T_2 \leqq N) \lesssim b^{-q} EN$. The above computation results in an asymptotic bound for II. Finally

$$\begin{aligned} P(T_1 \vee T_2 \leqq N) &= o(b^{-p-q+p/r}) + o(b^{-q-p+q/r}) \\ &= o(b^{-p-q+(p \vee q)/r}) \,. \end{aligned}$$

If T_2 is replaced by T_3 in the proof of ii, II is zero and iii follows.

In proofs using truncation the outer part pruned off is commonly discarded while the remaining truncated portion is essential. Here just the opposite happens.

The previous lemma will give the limiting value in (1) from outer portions of the X_i, while the following lemma will show that the inner parts can be discarded.

We use a distribution inequality of Burkholder (1973, 21.2),

$$(2) \qquad P(f^* > \beta\lambda, s(f) \vee d^* \leqq \delta\lambda) \leqq \frac{\delta^2}{(\beta - \delta - 1)^2} P(f^* > \lambda), \qquad \lambda > 0,$$

where f is a martingale, $f^* = \sup_n |f_n|$ its maximal function, $s(f) = (\sum_{k=1}^{\infty} E(d_k^2 \mid \mathscr{F}_{k-1}))^{\frac{1}{2}}$, $d_k = f_k - f_{k-1}$. Burkholder obtains a moment inequality from (2) by integrating. We iterate the distribution inequality to obtain a bound on the tail of the distribution of f^* in terms of that of $s(f)$ with the d_k truncated. The lemma is stated in the particular form needed later.

LEMMA 2. *Let X_i, $i = 1, 2, \cdots$ be a sequence of independent, identically distributed random variables. Let $\mu_b = E(X_i, |X_i| \leqq b)$. Let $Y_i(b) = X_i - \mu_b$ if $|X_i| \leqq b$, otherwise $Y_i(b) = -\mu_b$. Let σ_b^2 denote the variance of $Y_i(b)$. Let N be a stopping time for the sequence $\{X_i\}$ such that $P(N > n) = o(n^{-r})$ for some $r > 0$. Let $Q_n = \sum_{i=1}^{n \wedge N} Y_i$, and $Q^* = \sup_n |Q_n|$. If $s > 0$, $\varepsilon > 0$, then*

$$P(Q^* > \varepsilon b^{1+s}) = o(b^{-(1+s)2r} \sigma_b^{2r}) \qquad \text{as} \quad b \to \infty .$$

PROOF. We have arranged that $EY_i = 0$ and $|Y_i| \leqq 2b$. In (2) let f be the martingale Q_n. Then $s(f) = N^{\frac{1}{2}}\sigma_b$, $d_k = Y_k I_{(N \geqq k)}$ and (2) implies

$$P(Q^* > \beta\lambda, N^{\frac{1}{2}}\sigma_b \vee Y^* \leqq \delta\lambda) \leqq \frac{\delta^2}{(\beta - 1 - \delta)^2} P(Q^* > \lambda), \qquad \lambda > 0 .$$

Let $\lambda = b$, $\beta = 3$. Then

$$P(Q^* > 3b) \leqq \frac{\delta^2}{(2 - \delta)^2} P(Q^* > b) + P(N^{\frac{1}{2}}\sigma_b \vee Y^* > \delta b) .$$

Iteration produces for any integer i,

$$\begin{aligned} P(Q^* > 3^i b) &\leqq \frac{\delta^2}{(2 - \delta)^2} P(Q^* > 3^{i-1}b) + P(N^{\frac{1}{2}}\sigma_b \vee Y^* > \delta 3^{i-1} b) \\ &\leqq \cdots \\ &\leqq \left(\frac{\delta}{2 - \delta}\right)^{2i} P(Q^* > b) \\ &\quad + \sum_{k=1}^{i} \left(\frac{\delta}{2 - \delta}\right)^{2(k-1)} (P(N > (\delta b 3^{i-k} \sigma_b^{-1})^2) \\ &\qquad\qquad + P(Y^* > \delta b 3^{i-k})) . \end{aligned}$$

Let i be the largest integer $\leqq (\log \varepsilon + s \log b)/\log 3$. Then $3^i \leqq \varepsilon b^s < 3^{i+1}$ and

$$\begin{aligned} \delta^{2i} &= \exp(2i \log \delta) \\ &\leqq \exp((2 \log \varepsilon + 2s \log b) \log \delta / \log 3) \\ &= b^{2s \log \delta / \log 3} e^{2 \log \varepsilon \log \delta / \log 3} . \end{aligned}$$

Choose $\delta > 0$ so that $2s \log \delta / \log 3 < -(1 + s)2r$, and so that $\delta^2 < (2 \cdot 3^{2r})^{-1}$.

Then $\delta^{2i} = o(b^{-2r(1+s)})$ as $b \to \infty$, since $e^{2 \log \varepsilon \log \delta / \log 3}$ is constant. We have

$$\begin{aligned} P(Q^* > \varepsilon b^{1+s}) &\leqq P(Q^* > 3^i b) \\ &\leqq o(b^{-2r(1+s)}) + \sum_{k=1}^{i} \delta^{2(k-1)}[o((\delta b 3^{i-k} \sigma_b^{-1})^{-2r}) + P(Y^* > \delta b 3^{i-k})] \\ &\leqq o(b^{-2r(1+s)}) + o((\delta b 3^i \sigma_b^{-1})^{-2r}) \sum_{k=1}^{i} \delta^{2(k-1)} 3^{2rk} \\ &\quad + \sum_{k=i-l}^{i} \delta^{2(k-1)} P(Y^* > \delta b 3^{i-k}) . \end{aligned}$$

The sum in the second term is bounded as $i \to \infty$, by choice of δ. Because $|Y| < 2b$, the last sum has a finite number l of terms, l depending only on δ. The last term, then, is bounded by a multiple of δ^{2i}. Since $\delta^{2i} = o(b^{-2r(1+s)})$, σ_b is bounded away from zero as $b \to \infty$, and $3^i \sim \varepsilon b^s$, all three terms are of the desired order.

3. Stopped random walks and Lévy processes. Lemma 1 could have been written for X_i having distribution F with arbitrary asymptotic properties. Lemma 2 is distribution-free with respect to the X_i if the second moment is finite. To obtain (1) we assume that $1 - F(y) \sim y^{-p}$. The reader familiar with the theory of regularly varying functions will see that $1 - F(y)$ could be regularly varying as $y \to \infty$. Whether the following theorems can be extended to F with other asymptotic properties we do not know.

THEOREM 1. *Let X_i, $i = 1, 2, \cdots$ be a sequence of independent, identically distributed random variables such that $P(X_1 > y) \sim y^{-p}$, some $p > 0$, and $P(X_1 < -y)y^q$ is bounded, some $q > 0$. Let N be a stopping time for the sequence such that $P(N > n) = o(n^{-r})$, where $r > 1$, $r > p/q$, $r \geqq p/2$. If $EX_1 = \mu \neq 0$, assume $r \geqq p$. Then*

$$P\left(\sum_{i=1}^{N} X_i > y\right) \sim y^{-p} EN \qquad \text{as} \quad y \to \infty .$$

PROOF. Let $\mu_b = E(X_i, |X_i| \leqq b)$, $Y_i = X_i - \mu_b$ if $|X_i| \leqq b$, otherwise $Y_i = -\mu_b$, $Z_i = X_i - Y_i$. We replace y by $b^{1+\eta}$ where $\eta > 0$ will be chosen later. Given $1 > \varepsilon > 0$, we have the two inequalities

$$(3) \qquad P\left(\sum_{i=1}^{N} X_i > b^{1+\eta}\right) \geqq P\left(\sum_{i=1}^{N} Z_i > (1+\varepsilon) b^{1+\eta}\right) - P\left(\sum_{i=1}^{N} Y_i < -\varepsilon b^{1+\eta}\right)$$

and

$$(4) \qquad P\left(\sum_{i=1}^{N} X_i > b^{1+\eta}\right) \leqq P\left(\sum_{i=1}^{N} Z_i > (1-\varepsilon) b^{1+\eta}\right) + P\left(\sum_{i=1}^{N} Y_i > \varepsilon b^{1+\eta}\right) .$$

We have assumed $r \geqq p/2$. If $p \wedge q > 2$, σ_b in Lemma 2 is bounded and

$$(5) \qquad P\left(\left|\sum_{i=1}^{N} Y_i\right| > \varepsilon b^{1+\eta}\right) = o(b^{-(1+\eta)p}) .$$

If $p \wedge q = 2$, $\sigma_b^2 = O(\log b)$ and the left side of (5) is $o(b^{-(1+\eta)2r} \log^r b)$. Either $p = 2$ and $r > 1$ implies $2r > p$, or $q = 2$ and $r > p/q = p/2$ implies $2r > p$. In both cases $o(b^{-(1+\eta)2r} \log^r b)$ can be replaced by $o(b^{-(1+\eta)p})$. If $p \wedge q < 2$,

$$\sigma_b^2 \leqq \int_{-b}^{b} x^2 \, dF = O(b^{2-p \wedge q}) .$$

From Lemma 2 the left side of (5) is $o(b^{-(1+\eta)2r} b^{2r-(p \wedge q)r}) = o(b^{-2r\eta+(p \wedge q)r})$. The condition "$r > 1$ and $r > p/q$" is equivalent to the condition $r(p \wedge q) > p$. Since also $2r > p$, the left side of (5) is $o(b^{-(1+\eta)p})$.

Under our hypotheses on p, q and r, it will be shown that one can choose $\eta > 0$ so that

(6) $$P(\textstyle\sum_{i=1}^N Z_i > (1-\varepsilon)b^{1+\eta}) \lesssim (1-\varepsilon)^{-2p}b^{-(1+\eta)p}EN + o(b^{-(1+\eta)p}) .$$

Let $A_1(b) = (-\infty, b]$, $A_2(b) = [-b, \infty)$ in Lemma 1. Choose $0 < \eta < 1 - 1/r$. The event on the left in (6) can happen in three ways. A calculation similar to (4) gives, for $0 < \varepsilon < \frac{1}{2}$,

$$\begin{aligned} P(\textstyle\sum_{i=1}^N Z_i > (1-\varepsilon)b^{1+\eta}) &\leqq P(N\mu_b > \varepsilon(1-\varepsilon)b^{1+\eta}) \\ &\quad + P(X_{T_1} > (1-\varepsilon)^2 b^{1+\eta}, T_1 \leqq N) + P(T_3 \leqq N) . \end{aligned}$$

By choice of η and Lemma 1, $P(T_3 \leqq N) = o(b^{-(1+\eta)p})$. As in the proof of Lemma 1,

$$P(X_{T_1} > (1-\varepsilon)^2 b^{1+\eta}, T_1 \leqq N) \lesssim (1-\varepsilon)^{-2p}b^{-(1+\eta)p}EN .$$

The term containing μ_b is 0 whenever $\mu_b = 0$. Otherwise, $P(N\mu_b > \varepsilon(1-\varepsilon)b^{1+\eta}) = o(|\mu_b|^r b^{-r(1+\eta)})$.

If $0 < |\mu| < \infty$, (6) is evident from the hypothesis $r \geqq p$. If $p \wedge q > 1$ and $\mu = 0$,

$$|\mu_b| = |\mu - \mu_b| \leqq \textstyle\int_{|x|>b} |x|\, dF = O(b^{1-(p\wedge q)}) .$$

If $p \wedge q < 1$,

$$|\mu_b| \leqq \textstyle\int_{|x|\leqq b} |x|\, dF = O(b^{1-p\wedge q}) .$$

In these two cases, $P(N|\mu_b| > \varepsilon(1-\varepsilon)b^{1+\eta})$ is $o(b^{-r(\eta+p\wedge q)})$. Since $r(p \wedge q) > p$, one can choose η so small that $r(\eta + p \wedge q) > (\eta + 1)p$, and (6) follows. If $p \wedge q = 1$, then $|\mu_b| = O(\log b)$ and $P(N|\mu_b| > \varepsilon(1-\varepsilon)b^{1+\eta})$ is $o(b^{-r(1+\eta)}(\log b)^r)$ which is $o(b^{-(1+\eta)p})$ since $p = 1 < r$ or $p/q = p < r$.

Finally, we need a lower asymptotic bound like (6). For each $\delta > \varepsilon$,

$$\begin{aligned} P(\textstyle\sum_{i=1}^N Z_i &> (1+\varepsilon)b^{1+\eta}) \\ &\geqq P(X_{T_1} > (1+\delta)b^{1+\eta}, T_1 < N) - P(N\mu_b < -(\delta-\varepsilon)b^{1+\eta}) \\ &\quad - P(T_1 \vee T_2 < N) \\ &\gtrsim (1+\delta)^{-p}b^{-(1+\eta)p}EN - o(b^{-(1+\eta)p}) - o(b^{-p-q+(p\vee q)/r}) . \end{aligned}$$

The same arguments as above apply to the first and second terms. If $p \leqq q$, choose $\eta > 0$ so that $-p - q + q/r \leqq -p(1+\eta)$. If $p > q$, choose η so that $-p - q + p/r \leqq -p(1+\eta)$. This is possible since we assume $-q + p/r < 0$. We have

(7) $$P(\textstyle\sum_{i=1}^N Z_i > (1+\varepsilon)b^{1+\eta}) \gtrsim (1+\delta)^{-p}b^{-(1+\eta)p}EN - o(b^{-(1+\eta)p}) .$$

The proof is finished by combining statements (3) through (7).

The following result for the case $r = 1$ does not involve Lemma 2. If $p < 2$, much of the information in Theorem 1 for $r > 1$ can be obtained as in the proof of Theorem 2.

THEOREM 2. *Let X_i be a sequence and N a stopping time with associated parameters p, q, and r as defined in Theorem 1. Suppose that $p \leqq q$, $p < 2$, $r = 1$, and $EN < \infty$. In addition suppose that $p < 1$, or that $E(X_i, |X_i| \leqq b) = \mu_b$ is bounded if $p = 1$, or that $EX_i = 0$ if $p > 1$. Then*

$$P(\textstyle\sum_{i=1}^{N} X_i > y) \sim y^{-p} EN \qquad \text{as} \quad y \to \infty .$$

PROOF. We will choose a function $\eta(b)$ which increases to infinity slowly as $b \to \infty$. Expressions (3) through (7) will hold with $b^{1+\eta}$ replaced by $b\eta(b)$. For (3) and (4) this is immediate. A martingale inequality and Wald's equation give

$$\begin{aligned} P(\sup_n |\textstyle\sum_{i=1}^{N\wedge n} Y_i| > \varepsilon b\eta(b)) &\leqq E(\textstyle\sum_{i=1}^{N} Y_i)^2/(\varepsilon b\eta(b))^2 \\ &= ENEY_1^2/(\varepsilon b\eta(b))^2 . \end{aligned}$$

Under the restriction that $p < 2$, $EY_1^2 = O(b^{2-p})$. For this range of p we have the more general version of (5),

$$(5') \qquad P(|\textstyle\sum_{i=1}^{N} Y_i| > \varepsilon b\eta(b)) = o((b\eta(b))^{-p}) ,$$

for any $\eta(b)$ which increases to ∞ as b goes to ∞.

To obtain (6) and (7) with $b\eta(b)$ replacing $b^{1+\eta}$ we choose $\eta(b)$ so that $P(T_3 \leqq N)$ and $P(T_1 \vee T_2 \leqq N)$ are $o((b\eta(b))^{-p})$. Lemma 1 says this is true if $\eta(b) \leqq ((P(T_3 \leqq N) \vee P(T_1 \vee T_2 \leqq N)) \cdot b^p)^{-1/2p}$ and $A_1(b) = (-\infty, b]$, $A_2(b) = [-b, \infty)$.

To complete the proof of (6) and (7) we must verify that

$$(8) \qquad P(N|\mu_b| > \varepsilon(1-\varepsilon)b\eta(b)) = o((b\eta(b))^{-p}) .$$

Choose $\eta(b)$ so that

$$P(N > b) = o((b\eta(b))^{-1}) .$$

For instance, in addition to the above conditions on $\eta(b)$ suppose $\eta(b) \leqq (bP(N > b))^{-\frac{1}{2}}$. If $p \geqq 1$ and μ_b is bounded then $|\mu_b|^{-1}b\eta(b)$ is eventually $>b$, so that

$$\begin{aligned} P(N > |\mu_b|^{-1}b\eta(b)) &\leqq o(|\mu_b|[b\eta(b)\eta(|\mu_b|^{-1}b\eta(b))]^{-1}) \\ &\leqq o(|\mu_b|b^{-1}\eta(b)^{-2}) . \end{aligned}$$

If $p > 1$ and $EX_i = 0$, we have $|\mu_b| = O(b^{1-p})$ as in the proof of Theorem 1, and (8) follows. If $p = 1$ and μ_b is bounded, (8) follows. If $p < 1$, again $|\mu_b| = O(b^{1-p})$. We need only use $P(N > b) = o(b^{-1})$ to obtain

$$\begin{aligned} P(N|\mu_b| > b\eta(b)) &= o((|\mu_b|^{-1}b\eta(b))^{-1}) \\ &\leqq o((b^p\eta(b))^{-1}) \leqq o((b\eta(b))^{-p}) . \end{aligned}$$

COROLLARY 1. *Let X_i be a sequence of random variables and N a stopping time as in Theorem 1 or 2. Then*

$$\lim_{y\to\infty} P(\sup_n S_{N\wedge n} > y)y^p = EN$$

and

$$P(\sup_{n\leqq N} S_n > y) \sim P(S_N > y) .$$

PROOF. Since $\sup_{n\leq N} S_n \geqq S_N$, we need prove only the $\leqq$ part of the equality, for which we require appropriate versions of (4), (5) and (6). Instead of (4) we have

$$P(\sup_{n\leq N} S_n > y) \leqq P(\sup_{n\leq N} \sum_{i=1}^n Z_i > (1-\varepsilon)y) + P(\sup_{n\leq N} \sum_{i=1}^n Y_i > \varepsilon y) .$$

Replace y by $b^{1+\eta}$ or $b\eta(b)$ if p, q, r are related as in Theorem 1 or 2, respectively. Lemma 2 or the argument for (5′) in Theorem 2 gives

$$P(\sup_{n\leq N} \sum_{i=1}^n Y_i > \varepsilon y) = o(y^{-p})$$

where $y = b^{1+\eta}$ in the former case and $y = b\eta(b)$ in the latter. The argument for (6) also applies intact if $\sup_{n\leq N} \sum_{i=1}^n Z_i$ replaces $\sum_{i=1}^N Z_i$, in both Theorems 1 and 2.

COROLLARY 2. *Let X_t be a process with stationary, independent increments with Lévy measure ν such that $\nu(y, \infty) \sim y^{-p}$, some $p > 0$, and $\nu(-\infty, -y) = O(y^{-q})$, some $q > 0$. Let T be a stopping time for the process X_t such that $P(T > s) = o(s^{-r})$, and p, q, r are related as in Theorem* 1 *or* 2. *Then*

$$\lim_{y\to\infty} P(X_T > y)y^p = ET .$$

PROOF. Suppose that X_t is adapted to the increasing family of σ-fields $\mathscr{F}_t$. Let $\varepsilon > 0$, $Y_n = X_{n\varepsilon}$, $N =$ integral part of $(1 + T/\varepsilon)$. Then Y_n is a sum of independent random variables distributed like X_ε. By a theorem of Feller (1969), $P(X_\varepsilon > y) \sim \varepsilon y^{-p}$. An examination of Feller's proof reveals that also, $P(X_\varepsilon < -y) = O(y^{-q})$. (A minor correction is necessary.) Further, N is a stopping time for $\{Y_n, \mathscr{F}_{n\varepsilon}\}$. If $\xi = N\varepsilon - T$, then $0 < \xi \leqq \varepsilon$. Under the condition that T takes a certain value t, $W_\xi = Y_N - X_T$ has the same distribution as X_ξ where $\xi = \varepsilon[1 + t/\varepsilon] - t \leqq \varepsilon$. The hypotheses on T and ν allow us to obtain from Theorem 1 or 2,

$$\lim_{y\to\infty} P(X_{N\varepsilon} > y)/P(X_1 > y) = EN\varepsilon . \tag{9}$$

As $\varepsilon \to 0$, $N\varepsilon$ converges to T and $EN\varepsilon \to ET$. Since $X_{N\varepsilon} = W_\xi + X_T$, if $\delta > 0$,

$$P(X_{N\varepsilon} > y) \leqq P(W_\xi > y\delta) + P(X_T > y(1-\delta)) . \tag{10}$$

From the above information about ξ and W_ξ we obtain

$$\begin{aligned}\limsup_{y\to\infty} P(W_\xi > y\delta)y^p &= \limsup_{y\to\infty} \textstyle\int_0^\infty P(W_\xi > y\delta \mid T = t)P(T \in dt)y^p \\ &\leqq \varepsilon\delta^{-p} .\end{aligned}$$

Multiply (10) by y^p and let $y \to \infty$ to see that

$$EN\varepsilon \leqq \varepsilon\delta^{-p} + (1-\delta)^{-p} \liminf_{y\to\infty} P(X_T > y)y^p .$$

First let $\varepsilon \to 0$ and then $\delta \to 0$ to conclude that

$$ET \leqq \liminf_{y\to\infty} P(X_T > y)y^p .$$

By a conditioning argument similar to the above,

$$\begin{aligned}P(X_{N\varepsilon} > y) &\geqq P(X_T > (1+\delta)y)P(\sup_{0\leq t\leq\varepsilon} |X_t| < \delta y) ; \\ ET &\geqq (1+\delta)^{-p} \limsup_{y\to\infty} P(X_T > y)y^p .\end{aligned}$$

4. Applications. The stopping times which arise most frequently are boundary-crossing times. We state here asymptotic results for S_N which are immediate corollaries of Theorems 1 and 2 using known asymptotic rates for the distribution of N, where N is the exit time at a fixed one-sided or two-sided boundary or at a two-sided square-root boundary. Similar results for $\sup_{n \leqq N} S_n$ and for Lévy processes can be obtained from Corollaries 1 and 2.

COROLLARY 3. *Let S_n be a random walk such that $P(X_1 > y) \sim y^{-p}$ and $P(X_1 < -y) = O(y^{-q})$ for some p, $q > 0$. Let $a < 0 < b$, and $N = \min(n: S_n \notin [a, b])$. Then*

$$\lim_{y\to\infty} P(S_N > y)y^p = EN .$$

PROOF. All moments of N are finite (Feller, XVIII). Consequently $P(N > y) = o(y^{-r})$ for all $r > 0$, and the hypotheses $r > p/q$, $r \geqq p/2$, or $r \geqq p$ of Theorem 1 are satisfied.

COROLLARY 4. *Let S_n be a random walk such that $\sum_{n=1}^{\infty} (1/n)P(S_n \leqq 0) < \infty$, $P(X_1>y) \sim y^{-p}$, $0 < p < 1$ and $P(X_1 < -y) = O(y^{-p})$. Let $N = \min(n: S_n > 0)$. Then $EN < \infty$ and $\lim_{y\to\infty} P(S_N > y)y^p = EN$.*

PROOF. From $EN = \exp(\sum_{n=1}^{\infty}(1/n)P(S_n \leqq 0))$ (Feller, XII), we have $P(N>n) = o(n^{-r})$ with $r = 1$. Theorem 2 applies.

COROLLARY 5. *Let S_n be a random walk such that $P(X_1 > y) \sim y^{-p}$, $P(X_1 < -y) = O(y^{-q})$ for some p, $q < 3$, and $EX_1 = 0$. There exists $c_0 > 0$ such that if $c < c_0$ and if $N_c = \min(n: |S_n| > cn^{\frac{1}{2}})$ then*

$$\lim_{y\to\infty} P(S_{N_c} > y)y^p = EN_c .$$

PROOF. Breiman (1967) showed that for each $c > 0$ either there is an n such that $P(N_c > n) = 0$ or $P(N_c > n) \sim \alpha n^{-\beta(c)}$, where α is a constant depending on the distribution of X_1, and $\beta(c)$ is a universal constant for each c such that $\lim_{c\to 0} \beta(c) = \infty$. We can choose c_0 so that $\beta(c_0) > p/2$. Theorem 1 applies.

5. Some problems. It will be observed that Lemma 2 is a more sensitive estimator of the asymptotic property of a randomly stopped truncated sequence than is the method used to obtain (5′) in Theorem 2. Lemma 2 gives a rate in terms of $P(N > y)$ whereas the method using Wald's equation produces a bound with EN as a factor. Possibly Lemma 2 could be proved with $b\eta(b)$, $\eta(b)$ a slowly varying function, replacing $b^{1+\eta}$.

If $r < 1$ then $EN = \infty$, and we would expect for some range of p and q that

$$P(S_N > y)y^p \to \infty . \tag{11}$$

One sees from the proof of Lemma 1 that $P(T_1 \leqq N)b^p \to \infty$ in this case. Lemma 2 holds for any $r > 0$. To complete the proof of (11) one would need to show that $P(T_1 \vee T_2 \leqq N)b^p \to \infty$ more slowly than $P(T_1 \leqq N)b^p$. An example which suggests that (11) holds if $r < 1$, for appropriate p and q, is obtained from results of Rogozin (1971). He showed (Theorem 9) that if the process distribution of a random walk is in the domain of attraction of a stable process of index α and $\alpha u < 1$, $u = P(X_1 > 0)$, then its ascending ladder variable is in the domain of attraction of a stable law of index αu. The ladder variable is S_N where $N = \min(n: S_n > 0)$. Rogozin also showed (Theorem 4) that N is in the domain of attraction of a stable law of index v, $0 < v < 1$ if $\lim_{n\to\infty} (1/n) \sum_{k=1}^{n} P(S_k > 0) = v$, and $\sum_{k=1}^{\infty} P(S_k > 0)/k = \infty$. If a random walk is stable and symmetric of index p, then $P(X_1 > y) \sim cy^{-p}$. Rogozin's results imply $P(S_N > y)y^p \sim c'y^{p/2}$, and $P(N > y) \sim y^{-\frac{1}{2}}\psi(y)$ where $\psi(y)$ is slowly varying.

The combination of Corollaries 1 and 2 leads us to believe that under the hypotheses of Corollary 2,

$$\lim_{y\to\infty} P(\sup_{t\leq T} X_t > y)y^p = ET .$$

We have not found a proof.

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