The Annals of Probability
1977, Vol. 5, No. 2, 298-302

ON THE DECOMPOSITION OF A SUBADDITIVE STOCHASTIC PROCESS

BY ANDRÉS DEL JUNCO
University of Toronto

We give an elementary proof of the decomposition of a subadditive stochastic process as an additive process plus a positive subadditive process with time constant 0. The proof is based on two ideas. The first is a general idea for obtaining a kind of weak limit point for L_1-bounded sequences of random variables, based on the martingale convergence theorem. The second is a general result about martingales which seems to be new and is of independent interest.

The proof of the ergodic theorem for a subadditive stochastic process, as originally given by Kingman in [2], depends on the following decomposition.

THEOREM 1. *If x_{st} is a subadditive stochastic process with time constant $\gamma > -\infty$ then $x_{st} = y_{st} + z_{st}$ where y_{st} is additive and z_{st} is a positive subadditive process with time constant* 0.

Kingman proves Theorem 1 by choosing a weak limit point $\mu \in L_1^{**}$ for the sequence $\{f_m\}$ defined by (6) below, and then showing that the finitely additive measure μ is actually countably additive. This is done by writing

$$\mu = \mu_c - \mu_f \tag{1}$$

where μ_c is countably additive and μ_f is purely finitely additive (see [5] for the definition and for the proof of the existence of the decomposition (1)), and then showing that μ_f is 0. This in turn depends on the fact that the sum of purely finitely additive measures is again purely finitely additive ([5], Theorem 1.17). Theorem 1 has also been proved by Burkholder [1] by applying a theorem of Komlós [4] to the sequence f_m.

Both of these proofs of Theorem 1 depend on rather deep results which are not widely known. The purpose of this paper is to give a more elementary proof of Theorem 1 based on the martingale convergence theorem. The basic idea is as follows. Let $\{\mathscr{F}_k\}$ be an increasing sequence of finite σ-algebras in the sample space $(\Omega, \mathscr{F}, P)$ which generate $\mathscr{F}$ (up to null sets) and choose a subsequence $\{f_{m(j)}\}$ of $\{f_m\}$ such that for all k, $E(f_{m(j)} \mid \mathscr{F}_k)$ converges as $j \to \infty$, say to η_k. Then $\{\eta_k\}$ is an L_1-bounded martingale which converges to y, say. Then y can be regarded as a sort of weak limit of $f_{m(j)}$ and it turns out that if the $\mathscr{F}_k$ are chosen with a bit of care then y has enough good properties to carry through the proof. In fact is easy to see that y is just μ_c in the decomposition 1, but our argument avoids L_1^{**} altogether.

Received March 15, 1976.
AMS 1970 *subject classifications.* Primary 60G10; Secondary 60G45, 28A65.
Key words and phrases. Subadditive process, martingale.

Before proceeding to the proof of Theorem 1 we shall state and prove a lemma which will be needed and which is of interest in its own right. It seems likely that a more general result is true but we shall just prove the minimum that we require.

LEMMA 1. *Let η_k be an L_1-bounded martingale with respect to a sequence $\{\mathscr{F}_k\}$ of finite σ-algebras on a probability space $(\Omega, \mathscr{F}, P)$ and $\eta = \lim_k \eta_k$. Let $\mathscr{G}_k$ be an increasing family of σ-algebras and $l(k)$ and $b(k)$ increasing unbounded integer sequences such that $\mathscr{F}_{l(k)} \subset \mathscr{G}_k \subset \mathscr{F}_{b(k)}$. Then $E(\eta_{b(k)} \mid \mathscr{G}_k) \to \eta$ almost surely.*

PROOF. Suppose the result has been proved in case $\eta = 0$. Then, in the general case, $\xi_k = \eta_k - E(\eta \mid \mathscr{F}_k)$ is an L_1-bounded martingale with respect to $\mathscr{F}_k$ which converges to 0. Thus $E(\xi_{b(k)} \mid \mathscr{G}_k) = E(\eta_{b(k)} \mid \mathscr{G}_k) - E(\eta \mid \mathscr{G}_k)$ converges to 0. Since $\bigvee_{k=1}^{\infty} \mathscr{G}_k = \bigvee_{k=1}^{\infty} F_k$, $E(\eta \mid \mathscr{G}_k) \to \eta$, so this would establish the result in general.

Thus we shall assume $\eta = 0$. It is easy to see that $E(\eta_{b(k)} \mid \mathscr{G}_k)$ is an L_1-bounded martingale with respect to $\mathscr{G}_k$ and hence converges almost surely. We have to show the limit is 0.

Fix $\varepsilon > 0$. Choose k so large that

$$(2) \qquad |\eta_{l(k)}| \leqq \varepsilon \quad \text{on a set} \quad G \in \mathscr{F}_{l(k)}\,, \quad P(G) > 1 - \varepsilon\,,$$

and also

$$(3) \qquad E(|\eta_{b(k)}| - |\eta_{l(k)}|) < \varepsilon^2\,.$$

(This can be done since $E|\eta_k| \nearrow \sup_k E|\eta_k|$ by the martingale property.) Now by (3)

$$(4) \qquad \begin{aligned} \varepsilon^2 &> E(|\eta_{b(k)}| - |\eta_{l(k)}|) \\ &= \textstyle\sum_A P(A) E((|\eta_{b(k)}| - |\eta_{l(k)}|)|A)\,, \end{aligned}$$

where the summation is over the atoms A of $\mathscr{F}_{l(k)}$. Since $E((|\eta_{b(k)}| - |\eta_{l(k)}|)|A \geqq 0$ by the martingale property, (4) implies that there is a set $\bar{G} \in \mathscr{F}_{l(k)}$, $P(\bar{G}) > 1 - \varepsilon$ such that if A is an atom of $\mathscr{F}_{l(k)}$ contained in $\bar{G}$

$$(5) \qquad E((|\eta_{b(k)}| - |\eta_{l(k)}|)|A) < \varepsilon\,.$$

If A is an atom of $\mathscr{F}_{l(k)}$ contained in $\bar{G} \cap G$, (2) and (5) imply $E((|\eta_{b(k)}|)|A) < 2\varepsilon$. It follows that $|E(\eta_{b(k)} \mid \mathscr{G}_k)| < (2\varepsilon)^{\frac{1}{2}}$ on a set $A' \subset A$, $P(A' \mid A) > 1 - (2\varepsilon)^{\frac{1}{2}}$. Since $P(\bar{G} \cap G) > 1 - 2\varepsilon$, it follows that $|E(\eta_{b(k)} \mid \mathscr{G}_k)| < (2\varepsilon)^{\frac{1}{2}}$ on a set of probability greater than $(1 - 2\varepsilon)(1 - (2\varepsilon)^{\frac{1}{2}})$. Since ε is arbitrary this completes the proof.

For completeness we shall now recall the definition of and basic facts concerning subadditive processes. A subadditive process x_{st} is a process x_{st} indexed by all pairs (s, t) of nonnegative integers with $s \leqq t$ such that

(a) The process $\{x_{s,t}\}$ is equivalent to the shifted process $\{x_{s+1,t+1}\}$ (stationarity);
(b) $x_{st} \leqq x_{sr} + x_{rt}$ for $s \leqq r \leqq t$ (subadditivity);
(c) $(1/n)E(x_{0n}) > K$ for some constant K.

Set $g_n = E(x_{0n})$. Then $g_n/n \to \gamma > -\infty$. γ is called the time constant of the process.

PROOF OF THEOREM 1. x_{st} is a process indexed by $\Lambda^+ = \{(s, t) : s, t \in Z^+, s \leqq t\}$ and thus is equivalent to a canonical process $\bar{x}_{st}$ with sample space $R^{\Lambda+}$ in the same way that a process indexed by Z^+ is equivalent to a process with sample space R^{Z^+} (see, e.g., [1], Chapter 2). Furthermore $\bar{x}_{st}$ has a canonical stationary extension $\bar{\bar{x}}_{st}$ to a process indexed by $\Lambda = \{(s, t) : s,t \in Z, s \leqq t\}$ with sample space R^{Λ}, just as in the one parameter case ([1], Proposition 6.5), which has the same joint distributions as x_{st}. Note that $\bar{\bar{x}}_{st}$ is necessarily subadditive. We shall assume that x_{st} is itself $\bar{\bar{x}}_{st}$ which allows us to assume the technically convenient facts that, first, the sample space $(\Omega, \mathscr{F}, P)$ is separable and, second, there is an invertible measure preserving transformation σ of Ω such that $x_{st} \circ \sigma = x_{s+1,t+1}$. (Concerning this assumption see the remark at the end of the paper.) For any measurable function f let $Sf = f \circ \sigma$. Note now that the proof of the theorem is reduced to showing that there is a $y \in L_1$ such that $E(y) = \gamma$ and $\sum_{i=0}^{n-1} S^i y \leqq x_{0n}$. Indeed this would imply that $\sum_{i=s}^{t-1} S^i y \leqq x_{st}$ for $s \leqq t$ and one can then set $y_{st} = \sum_{i=s}^{t-1} S^i y$ and $z_{st} = x_{st} - y_{st}$.

Now, as in [2], Section 6, set

$$f_m = \frac{1}{m} \sum_{j=1}^{m} (x_{0j} - x_{1j}) \,. \tag{6}$$

For $m \geqq n$ we have

$$\begin{aligned}
\sum_{i=0}^{n-1} S^i f_m &= \frac{1}{m} \sum_{i=0}^{n-1} \sum_{j=1}^{m} (x_{i,j+1} - x_{i+1,j+i}) \\
&= \frac{1}{m} \sum_{s=1}^{m+n-1} \sum_{t=a}^{b-1} (x_{ts} - x_{t+1,s}) \\
&\qquad\qquad (\text{where } a = \max(0, s-m),\ b = \min(s, n)) \\
&= \frac{1}{m} \sum_{s=1}^{m+n-1} x_{as} - x_{bs} \\
&\leqq \frac{1}{m} \sum_{s=1}^{m+n-1} x_{ab} \quad (\text{by subadditivity}) \\
&= \frac{1}{m} \left[\sum_{s=1}^{n} x_{0s} + (m-n)x_{0n} + \sum_{s=1}^{n-1} x_{sn}\right] \\
&= R_n^{\,m}\,, \quad \text{say.}
\end{aligned}$$

Note that as $m \to \infty$, $R_n^{\,m} \to x_{0n}$ a.s. and in mean. Furthermore $f_m \leqq x_{01}$ for all m and $E(f_m) = (1/m) \sum_{i=1}^{m} (g_i - g_{i-1}) = g_m/m$ is bounded, so that $E|f_m|$ must be bounded, say by M.

Choose an increasing sequence of finite σ-algebras $\mathscr{F}_k$ which generate $\mathscr{F}$ and such that for each i there are two increasing unbounded integer sequences $l_i(k)$ and $b_i(k)$ such that

$$\mathscr{F}_{l_i(k)} \subset \sigma^i(\mathscr{F}_k) \subset \mathscr{F}_{b_i(k)} \,.$$

(One way to do this is to let $\mathscr{H}_1, \mathscr{H}_2, \cdots$ be a sequence of finite σ-algebras which generate $\mathscr{F}$, choose a bijection ξ from Z^+ to $Z \times Z^+$ and set

$$\mathscr{F}_k = \bigvee_{m \leq k} \sigma^{\xi_1(m)} \mathscr{H}_{\xi_2(m)} ,$$

where $\xi(m) = (\xi_1(m), \xi_2(m))$.) Since $\mathscr{F}_k$ is a finite σ-field and $||E_k f_m||_\infty$ is bounded for fixed k, we may choose by a diagonal selection process a subsequence $\{f_{m(j)}\}$ of $\{f_m\}$ such that $E_k f_{m(j)}$ converges to some $\eta_k \varepsilon L_1$, for all k. (The sense of convergence here does not need to be specified since it amounts simply to convergence of an n-tuple of real numbers.) Obviously n_k will be $\mathscr{F}_k$-measurable and it is easy to check that η_k is a martingale with respect to $\mathscr{F}_k$ and that $E|\eta_{0k}| \leqq M$ since $E|f_m| \leqq M$. Thus $\eta_k \to y \in L_1$. Now, using the fact that $E(Sg\,|\,\mathscr{G}) = SE(g\,|\,\sigma G)$ for any σ-algebra $\mathscr{G}$ and $g \in L_1$, we have

$$\begin{aligned} E_k(S^i f_{m(j)}) &= S^i E(f_{m(j)}\,|\,\sigma^i \mathscr{F}_k) \\ &= S^i E[E(f_{m(j)}\,|\,\mathscr{F}_{b_i(k)})\,|\,\sigma^i \mathscr{F}_k] \\ &\to S^i E(\eta_{b_i(k)}\,|\,\sigma^i \mathscr{F}_k) \qquad \text{as } j \to \infty . \end{aligned}$$

Now since $\sum_{i=0}^{n-1} S^i f_{m(j)} \leqq R_n^{m(j)}$ we have $\sum_{i=0}^{n-1} E_k S^i f_{m(j)} \leqq E_k R_n^{m(j)}$ and since $R_n^{m(j)} \to x_{0n}$ in mean, letting $j \to \infty$ we get

$$\sum_{i=0}^{n-1} S^i E(\eta_{b_i(k)}\,|\,\sigma^i \mathscr{F}_k) \leqq E_k x_{0n} . \tag{7}$$

As $k \to \infty$ the left-hand side of (7) converges to $\sum_{i=0}^{n-1} S^i y$ a.s. by Lemma 1 and the right-hand side converges to x_{0n} a.s. Thus we have $\sum_{i=0}^{n-1} S^i y \leqq x_{0n}$. In particular $nE(y) \leqq g_n$ for all n so $E(y) \leqq \gamma$. It remains only to show that $E(y) \geqq \gamma$. Note that $E(E_k f_m) = g_m/m$, so $E(\eta_k) = \gamma$. Also since $x_{01} \geqq f_m$, $E_k x_{01} \geqq E_k f_m$ so $E_k x_{01} \geqq \eta_k$. Thus applying Fatou's lemma to $E_k x_{01} - \eta_k$ we get

$$\begin{aligned} E(x_{01}) - E(y) &= E \liminf (E_k x_{01} - \eta_k) \\ &\leqq \liminf E(E_k x_{01} - \eta_k) \\ &= E(x_{01}) - \gamma . \end{aligned}$$

Thus $E(y) \geqq \gamma$.

REMARK. It may seem unnatural to assume that σ is invertible. However it appears that this assumption is also necessary in Kingman's original proof ([2]). The equation $S\kappa = \kappa T$ on page 509 is not correct unless S is defined by $S\mu(A) = \mu(\theta A)$ which requires at least that θ take measurable sets to measurable sets. Furthermore on the same page one needs to know that if π is a purely finitely additive measure then $S\pi$ is also, which seems to require the invertibility of S. (Note that σ and S in this paper correspond to θ and T respectively in [2], Section 6.)

Acknowledgment. I would like to thank John Baxter for several very helpful discussions concerning Lemma 1.

REFERENCES

[1] BREIMAN, L. (1968). *Probability*. Addison-Wesley, Reading, Mass.
[2] BURKHOLDER, D. L. (1968). Discussion following [3].

[3] KINGMAN, J. F. C. (1968). The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B* **30** 499–510.

[4] KINGMAN, J. F. C. (1973). Subadditive ergodic theory. *Ann. Probability* **1** 883–909.

[5] KOMLOS, J. (1967). A generalization of a problem of Steinhaus. *Acta Math. Acad. Sci. Hungar.* **18** 217–229.

[6] YOSIDA, K. and HEWITT, E. (1952). Finitely additive measures. *Trans. Amer. Math. Soc.* **72** 46–66.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO
TORONTO, ONTARIO
M5S 1A1
CANADA