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# ASYMPTOTIC BEHAVIOR OF STABLE MEASURES

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It is proved that if $\mu$ is a stable measure of index $p$ on a vector space and $q$ is a seminorm, then $\lim_{t\to\infty} t^p\mu\{x: q(x) > t\}$ exists.

**1. Introduction.** A classical theorem of P. Lévy asserts that if $\sigma$ is a stable distribution of index $p$ on the real line, then $\lim_{t\to\infty} t^p\sigma\{x: |x| > t\}$ exists and is strictly positive if $p < 2$ and $\sigma$ is nondegenerate. Kuelbs and Mandrekar [5] have generalized this result for the case of a stable measure on Hilbert space; they obtain it as a consequence of the theory of domains of attraction. In [1] we proved that if $\mu$ is a stable measure of index $p$ on a vector space $E$ and $q$ is a measurable seminorm on $E$, then

$$\limsup_{t\to\infty} t^p\mu\{x: q(x) > t\} < \infty .$$

A similar, though somewhat weaker result has been obtained independently by Kanter [4] using quite different methods. We further showed in [1] that if $p < 2$ and $\mu$ satisfies a nondegeneracy condition with respect to $q$, then

$$\liminf_{t\to\infty} t^p\mu\{x: q(x) > t\} > 0 .$$

In the present work we refine the methods developed in [1] and prove that

$$\lim_{t\to\infty} t^p\mu\{x: q(x) > t\}$$

exists and is strictly positive if $p < 2$ and $\mu$ satisfies the nondegeneracy condition, thus obtaining a complete generalization of P. Lévy's result. It should be mentioned that when the seminorm is not a Euclidean norm, the existence of this limit does not seem to have been proved even for stable measures on finite-dimensional vector spaces.

**2. Preliminaries.** We refer to [1] for the definitions and results that we use. In particular, throughout the paper our basic framework will be a measurable space $(E, \mathscr{B})$, with $E$ a real vector space and $\mathscr{B}$ the $\sigma$-algebra induced on $E$ by a real vector space $F$ in duality with $E$.

**3. Basic inequalities.** For convenient reference, we state some elementary inequalities which are explicitly or implicitly proved in [1]. They are essential for the rest of the paper.

LEMMA 3.1. *Let $X$, $Y$ be independent $E$-valued random vectors. Let $q$ be a measurable seminorm on $E$.*

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(a) *Let* $s > 0$, $\varepsilon > 0$. *Then*

$$P\{q(X+Y) > s\} \geqq P\{q(X) > s(1+\varepsilon)\}P\{q(Y) \leqq s\varepsilon\} + P\{q(Y) > s(1+\varepsilon)\}P\{q(X) \leqq s\varepsilon\} .$$

(b) *Let* $s > 0$, $0 < a < 1$, $b = 1 - a$. *Then*

$$P\{q(X+Y) > s\} \leqq P\{q(X) > as\} + P\{q(Y) > as\} + P\{q(X) > bs\}P\{q(Y) > bs\} .$$

LEMMA 3.2. *Let* $\mu$ *be a strictly stable measure of index* $p$, *and let* $\gamma = 2^{1/p}$. *Let* $X$ *be an* $E$*-valued random vector with* $\mathscr{L}(X) = \mu$.

(a) *Let* $s > 0$, $\varepsilon_n > 0$ *for* $n = 1, 2, \cdots$. *Let*

$$\delta_1 = P\{q(X) \leqq \gamma s \varepsilon_1\} ,$$
$$\delta_n = P\{q(X) \leqq \gamma^n s \textstyle\prod_{i=1}^{n-1} (1+\varepsilon_i)\varepsilon_n\} \qquad \text{for} \quad n \geqq 2 .$$

*Then for all* $n \geqq 1$,

$$P\{q(X) > s\} \geqq 2^n P\{q(X) > \gamma^n s \textstyle\prod_{i=1}^{n} (1+\varepsilon_i)\} \prod_{i=1}^{n} \delta_i$$

(b) *Let* $s > 0$, $a_0 = 1$, $0 < a_j < 1$, $b_j = 1 - a_j$ *for* $j = 1, 2, \cdots$. *Let* $\varphi(t) = P\{q(X) > t\}$. *Then for all* $n \geqq 1$,

$$P\{q(X) > s\} \leqq 2^n P\{q(X) > \gamma^n s a_1 \cdots a_n\} + \textstyle\sum_{j=1}^{n} 2^{j-1}(\varphi(\gamma^j s a_0 \cdots a_{j-1} b_j))^2 .$$

**4. Tail behavior.** Given a Borel set $A$ in $R$, we denote by $\mathscr{M}(A)$ the space of all finite nonnegative measures on the Borel subsets of $A$. We denote the restriction of a measure $\nu$ to the Borel subsets of $A$ by $\nu \mid A$. We write $\nu_i \to_w \nu$ if a net $\{\nu_i\}$ converges to $\nu$ in the weak topology of $\mathscr{M}(A)$.

Let $\mu$ be a stable probability measure on $(E, \mathscr{B})$ and let $q$ be a measurable seminorm on $E$. For each $t > 0$, let $\lambda_t$ be the measure on $\mathscr{M}([0, \infty))$ defined by

$$\lambda_t(B) = t^p \mu\{x : q(x) \in tB\} , \qquad B \quad \text{a Borel set in} \quad [0, \infty) .$$

For each $r > 0$, let $I_r = [r, \infty)$, $I_r^\circ = (r, \infty)$.

LEMMA 4.1. *Let* $\mu$ *be a strictly stable probability measure of index* $p$ *on* $(E, \mathscr{B})$.

(a) *For each* $r > 0$ *the set of measures* $\{\lambda_t \mid I_r\}_{t>0}$ *is relatively compact for the weak topology of* $\mathscr{M}(I_r)$.

(b) *Let* $\{t_k\}$ *be a sequence such that* $t_k \to \infty$ *and* $\{\lambda_{t_k} \mid I_r\}$ *converges weakly in* $\mathscr{M}(I_r)$ *for each* $r > 0$. *Then there exists* $h \geqq 0$ *such that for each* $r > 0$,

$$\lambda_{t_k}(I_r) \to h r^{-p} \qquad \text{as} \quad k \to \infty .$$

PROOF. (a) It is enough to prove that $\{\lambda_t \mid I_r\}_{t>0}$ is uniformly bounded in norm and tight. By Theorem 3.1 (1) of [1], there exists a constant $C$ such that

$$\mu\{x : q(x) \geqq t\} \leqq C t^{-p} \qquad \text{for all} \quad t > 0 .$$

Fix $r > 0$. Then $\lambda_t(I_r) = t^p \mu\{x : q(x) \geqq tr\} \leqq C r^{-p}$ for all $t > 0$. This proves that $\sup_{t>0} \|(\lambda_t \mid I_r)\| < \infty$.

Now choose $s > r$ such that $Cs^{-p} < \varepsilon$. Then $(\lambda_t \mid I_r)([r, s])^c = \lambda_t(I_s{}^\circ) < \varepsilon$ for all $t > 0$. This shows that $\{\lambda_t \mid I_r\}_{t>0}$ is tight.

(b) Let $\lambda_{t_k} \mid I_r \to_w \theta_r$ (a finite measure on $I_r$), and let $F_r(s) = \theta_r((s, \infty))$ for $s \geqq r$. It is clear that the family of measures $\{\theta_r\}_{r>0}$ is consistent in an obvious way and the formula $F(s) = F_r(s)$ for $r \leqq s$ defines uniquely a nonincreasing, right-continuous function on $(0, \infty)$. We proceed to identify the function $F$.

Let $\alpha > 0$, $\beta > 0$ be such that $\alpha^p + \beta^p = 1$. Let $X$, $Y$ be independent $E$-valued random vectors with $\mathscr{L}(X) = \mathscr{L}(Y) = \mu$. Then $\mathscr{L}(\alpha X + \beta Y) = \mu$, and applying Lemma 3.1 (a) we obtain: for all $t > 0$,

$$P\{q(X) > t\} \geqq P\{q(\alpha X) > t(1 + \varepsilon)\}P\{q(\beta X) \leqq t\varepsilon\} + P\{q(\beta X) > t(1 + \varepsilon)\}P\{q(\alpha X) \leqq t\varepsilon\} .$$

Therefore, for all $s > 0$

$$\lambda_{t_k}(I_s{}^\circ) \geqq \lambda_{t_k}(I^\circ_{\alpha^{-1}s(1+\varepsilon)})P\{q(\beta X) \leqq st_k\varepsilon\} + \lambda_{t_k}(I^\circ_{\beta^{-1}s(1+\varepsilon)})P\{q(\alpha X) \leqq st_k\varepsilon\} .$$

Assume that $s$, $\alpha^{-1}s(1 + \varepsilon)$ and $\beta^{-1}s(1 + \varepsilon)$ are continuity points of $F$. Then letting $k \to \infty$, we obtain

$$F(s) \geqq F(\alpha^{-1}s(1 + \varepsilon)) + F(\beta^{-1}s(1 + \varepsilon)) .$$

Letting $\varepsilon \downarrow 0$ through an appropriate sequence, we get

$$F(s) \geqq F(\alpha^{-1}s) + F(\beta^{-1}s) . \tag{1}$$

By right continuity (1) is true whenever $s > 0$, $\alpha > 0$, $\beta > 0$ and $\alpha^p + \beta^p = 1$. To obtain an inequality in the opposite direction, we apply Lemma 3.1 (b):

$$P\{q(X) > t\} \leqq P\{q(\alpha X) > at\} + P\{q(\beta X) > at\} + P\{q(\alpha X) > bt\}P\{q(\beta X) > bt\} .$$

This implies: for all $s > 0$,

$$\lambda_{t_k}(I_s{}^\circ) \leqq \lambda_{t_k}(I^\circ_{\alpha^{-1}sa}) + \lambda_{t_k}(I^\circ_{\beta^{-1}sa}) + t_k{}^pP\{q(\alpha X) > bst_k\}P\{q(\beta X) > bst_k\} .$$

Choose $s$, $\alpha^{-1}sa$ and $\beta^{-1}sa$ to be continuity points of $F$. Letting $k \to \infty$, and using the fact that $P\{q(X) > t\} \leqq Ct^{-p}$, we obtain:

$$F(s) \leqq F(\alpha^{-1}sa) + F(\beta^{-1}sa) .$$

Proceeding as above, we get

$$F(s) \leqq F(\alpha^{-1}s) + F(\beta^{-1}s) \tag{2}$$

valid for $s > 0$, $\alpha > 0$, $\beta > 0$ and $\alpha^p + \beta^p = 1$. Now (1) and (2) yield the functional equation for $F$

$$F((u^p + v^p)^{-1/p}) = F(u^{-1}) + F(v^{-1}) , \qquad u > 0 , \quad v > 0 .$$

Therefore there exists $h \geqq 0$ such that $F(u) = hu^{-p}$ for all $u > 0$. □

THEOREM 4.1. *Let $\mu$ be a stable probability measure of index $p$ on $(E, \mathscr{B})$. Then*

*$l = \lim_{t\to\infty} t^p\mu\{x : q(x) > t\}$ exists and for each $r > 0$*

$$\lambda_t \mid I_r \to_w \lambda \mid I_r \qquad \textit{as} \quad t \to \infty$$

*where $\lambda(B) = l \int_B ps^{-p-1}\,ds$ for $B$ Borel in $(0, \infty)$.*

PROOF. It is enough to prove the first part of the statement. The rest follows then from

$$\lim_{t\to\infty} \lambda_t(I_r) = \lim_{t\to\infty} r^{-p}(rt)^p\mu\{x : q(x) > tr\} = lr^{-p} = \lambda(I_r)\,.$$

We prove the existence of the limit for strictly stable measures first. Let $\mu$ be a strictly stable probability measure, and let $X$ be an $E$-valued random vector with $\mathscr{L}(X) = \mu$.

Let us consider the sequence $t_n = \gamma^n$, where $\gamma = 2^{1/p}$. Since by Lemma 3.1(a), $\{\lambda_t \mid I_r\}_{t>0}$ is relatively compact for $r = k^{-1}$, $k = 1, 2, \cdots$, we can apply the standard diagonal method to obtain a subsequence $\{t_{n_j}\}$ such that $\{\lambda_{t_{n_j}} \mid I_r\}$ converges weakly in $\mathscr{M}(I_r)$ for each $r > 0$. By Lemma 4.1(b), there exists $l \geqq 0$ such that for each $r > 0$,

$$\lim_{j\to\infty} 2^{n_j}P\{q(X) > r\gamma^{n_j}\} = lr^{-p}\,.$$

We shall prove: $\lim_{s\to\infty} s^pP\{q(X) > s\} = l$.

Let $\rho > 1$, $\eta < 1$ be given. Choose

(i) a real number $\alpha \in (\gamma^{-1}, 1)$

(ii) a natural number $m$ such that

$$\prod_{k=m+1}^{\infty} P\{q(X) \leqq (\gamma\alpha)^k\} > \eta$$

and

$$\prod_{k=m+1}^{\infty} (1 + \alpha^k) < \rho$$

(iii) a real number $\beta > 0$ such that

$$\prod_{k=1}^{m} (1 + \beta^k) < \rho$$

(iv) a real number $s_0 \geqq 1$ such that

$$\prod_{k=1}^{m} P\{q(X) \leqq (\gamma\beta)^k s_0\} > \eta\,.$$

The possibility of choice (ii) is guaranteed by Lemma 3.3 of [1]. Applying Lemma 3.2(a) with $\varepsilon_k = \beta^k$ for $k \leqq m$, $\varepsilon_k = \alpha^k$ for $k \geqq m + 1$, we have

$$\begin{aligned} P\{q(X) > s\} &\geqq 2^{n_j}P\{q(X) > \gamma^{n_j}s \textstyle\prod_{k=1}^{m}(1+\beta^k)\prod_{k=m+1}^{n_j}(1+\alpha^k)\} \\ &\quad \times \textstyle\prod_{k=1}^{m} P\{q(X) \leqq (\gamma\beta)^k s\} \prod_{k=m+1}^{n_j} P\{q(X) \leqq (\gamma\alpha)^k s\} \\ &\geqq 2^{n_j}P\{q(X) > \gamma^{n_j}s\rho^2\}\eta^2 \end{aligned}$$

for all $s \geqq s_0$, $n_j \geqq m$. Letting $j \to \infty$, we obtain for $s \geqq s_0$:

$$s^pP\{q(X) > s\} \geqq l\rho^{-2p}\eta^2\,. \tag{1}$$

In order to obtain an inequality in the opposite direction we proceed as follows. Let $\delta < 1$, $\varepsilon > 0$ be given. Choose

(i) a real number $\alpha \in (\gamma^{-\frac{1}{2}}, 1)$,

(ii) a natural number $m$ such that

$$\prod_{k=m+1}^{\infty}(1-\alpha^k) > \delta\,,$$

(iii) a real number $\beta > 0$ such that

$$\prod_{k=1}^{m}(1-\beta^k) > \delta\,,$$

(iv) a real number $s_1$ such that

$$\tfrac{1}{2}C^2\delta^{-2p}\sum_{j=1}^{m}(2(\gamma\beta)^{-2p})^j < \varepsilon s_1^{\,p}\,.$$

Applying Lemma 3.2(b) with $a_k = 1-\beta^k$ for $k \leqq m$, $a_k = 1-\alpha^k$ for $k \geqq m+1$ and using Theorem 3.1(1) of [1], we have

$$\begin{aligned} P\{q(X) > s\} &\leqq 2^{n_j}P\{q(X) > \gamma^{n_j}s\delta^2\} + \textstyle\sum_{k=1}^{m} 2^{k-1}(\varphi((\gamma\beta)^k s\delta))^2 \\ &\quad + \textstyle\sum_{k=m+1}^{n_j} 2^{k-1}(\varphi((\gamma\alpha)^k s\delta^2))^2 \\ &\leqq 2^{n_j}P\{q(X) > \gamma^{n_j}s\delta^2\} + \tfrac{1}{2}C^2\delta^{-2p}\textstyle\sum_{k=1}^{m}(2(\gamma\beta)^{-2p})^k s^{-2p} \\ &\quad + \tfrac{1}{2}C^2\delta^{-4p}\textstyle\sum_{k=m+1}^{\infty}(2(\gamma\alpha)^{-2p})^k s^{-2p} \\ &\leqq 2^{n_j}P\{q(X) > \gamma^{n_j}s\delta^2\} + \varepsilon s^{-p} + Ms^{-2p} \end{aligned}$$

where $M$ is a constant, for all $s \geqq s_1$, $n_j \geqq m$. Letting $j \to \infty$, we obtain for $s \geqq s_1$,

$$s^pP\{q(X) > s\} \leqq l\delta^{-2p} + \varepsilon + Ms^{-p}\,. \tag{2}$$

From (1) and (2) we conclude: $\lim_{s\to\infty} s^pP\{q(X) > s\} = l$.

Now let $\mu$ be a stable probability measure of index $p$. Let $X$, $Y$ be independent $E$-valued random vectors with $\mathscr{L}(X) = \mathscr{L}(Y) = \mu$. Then $\mathscr{L}(X-Y) = \mu * \bar{\mu}$ is a symmetric stable probability measure of index $p$, hence a strictly stable probability measure of index $p$.

From Lemma 3.1(a) we have: for each $t > 0$, $\varepsilon > 0$

$$t^pP\{q(X-Y) > t(1+\varepsilon)^{-1}\} \geqq 2t^pP\{q(X) > t\}P\{q(Y) \leqq t(1+\varepsilon)^{-1}\varepsilon\}\,.$$

Letting $t \to \infty$ we obtain:

$$l(1+\varepsilon)^p \geqq 2\limsup_{t\to\infty} t^pP\{q(X) > t\}\,,$$

where $l = \lim_{t\to\infty} t^p(\mu * \bar{\mu})\{x : q(x) > t\}$. Since $\varepsilon$ is arbitrary it follows that

$$\limsup_{t\to\infty} t^pP\{q(X) > t\} \leqq l/2\,. \tag{3}$$

Applying Lemma 3.1(b) we have for each $t > 0$, $0 < a < 1$

$$t^pP\{q(X-Y) > ta^{-1}\} \leqq 2t^pP\{q(X) > t\} + t^p(P\{q(X) > ta^{-1}b\})^2\,.$$

Letting $t \to \infty$, the second term on the right tends to 0 and we get

$$al \leqq 2\liminf_{t\to\infty} t^pP\{q(X) > t\}\,.$$

Making $a \uparrow 1$, we obtain

$$l/2 \leqq \liminf_{t\to\infty} t^pP\{q(X) > t\}\,. \tag{4}$$

Finally (3) and (4) yield: $\lim_{t\to\infty} t^p\mu\{x : q(x) > t\} = l/2$. □

Let us write, for a seminorm $q$ and a stable probability measure $\mu$ of index $p$,

$$l_q(\mu) = \lim_{t\to\infty} t^p \mu\{x : q(x) > t\} .$$

If $p = 2$, then $\mu$ is Gaussian and Fernique's [3] result on tail bounds implies that $l_q(\mu) = 0$. On the other hand, we have

THEOREM 4.2. *Let $\mu$ be a stable probability measure of index $p < 2$ on $(E, \mathscr{B})$. Suppose that there exists a measurable linear form $f$ on $E$ such that* (a) $|f| \leqq Kq$ *for some constant $K$ and* (b) $\mu \circ f^{-1}$ *is nondegenerate. Then* $l_q(\mu) > 0$.

PROOF. This follows at once from Theorem 3.1 of [1]. □

In the next proposition we consider the properties of $l_q$ when restricted to the class of stable probability measures of a fixed index and generalize partially a result of Feller ([2], page 271). We omit the proof; the argument is based on Lemma 3.1 and is analogous to other arguments in the paper.

Given a measure $\mu$ and $a > 0$, let $T_a \mu$ be the measure defined by $(T_a \mu)(B) = \mu(a^{-1}B)$, $B \in \mathscr{B}$.

THEOREM 4.3. *Let $\mu$, $\nu$ be stable probability measures of index $p$ on $(E, \mathscr{B})$. Then*

(a) $l_q(\mu * \nu) = l_q(\mu) + l_q(\nu)$

(b) $l_q(T_a \mu) = a^p l_q(\mu)$.

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