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# AN ENTRANCE LAW WHICH REACHES EQUILIBRIUM

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This paper constructs a collection of probability vectors $\varphi_n$ for all $n \in Z$ and a stochastic matrix $Q$ on a countable state space so that

(1) $Q(i, j) > 0$ for all $i, j$,
(2) $\varphi_n Q = \varphi_{n+1}$ for all $n \in Z$,
(3) $\varphi_n = \varphi_{n+1}$ for all $n \geqq 0$; $\varphi_{-1} \neq \varphi_0$.

**0. Introduction.** Let $Z^+$ be the positive integers. Let $Q = \{Q(i, j)\}_{i \in Z^+, j \in Z^+}$ be a positive recurrent transition matrix. An entrance law for $Q$ is a collection of vectors $\{\varphi_n\}_{n \in Z}$, for which $\sum_{i=1}^{\infty} \varphi_n(i) = 1$, $\varphi_n(i) \geqq 0$ for all $n$, $i$ and $\varphi_n Q = \varphi_{n+1}$ for all $n \in Z$. By Kolmogorov's consistency theorem, such entrance laws determine a Markov chain $\{X_n\}_{n \in Z}$ with stationary transition probabilities and time parameter set $Z$. They are discussed in detail by Cox [1]. For a more general formulation relating the existence of entrance laws to nonuniqueness of Markov random fields, see also [2] and [3].

There is a unique probability vector $\varphi$ such that $\varphi Q = \varphi$ and a trivial entrance law $\varphi_n = \varphi$ for all $n$. The problem under consideration is whether or not there can be other entrance laws which reach equilibrium at a finite time. Kesten (Section 1) easily answered the question affirmatively. In Section 2 I show that this can be done with a strictly positive transition matrix. The significance of this is that many of the theorems of [1], [2] and [3] refer only to strictly positive transition matrices Hence, we have two distinct Markov chains, one stationary, with the same transition probabilities and which look identical from time zero onward.

**1. Some examples.**

EXAMPLE 1. Let $0 < p < 1$. Let $Q(1, j) = p(1 - p)^{j-1}$. For all $i \geqq 1$, $n \geqq 1$ let $Q(i + 1, j) = \delta_{i,j}$, $\varphi_0(j) = \varphi_n(j) = p(1 - p)^{j-1}$, and $\varphi_{-n}(j) = \delta_{n,j}$. Here a $\delta$-distribution marches in from infinity until it is concentrated at 1 in time $-1$ and then it goes into equilibrium at time 0.

EXAMPLE 2. We would like to impose the additional condition that $Q(i, j) > 0$ for all $i, j$. It is easy to obtain this condition at the cost of not being able to construct a complete entrance law. Let $0 < p, t < 1$. Let $L > 0$ be a fixed positive integer. Let $Q(1, j) = p(1 - p)^{j-1}$. For $i \geqq 1$, $n \geqq 1$, let $Q(i + 1, j) = t(\delta_{i,j}) + (1 - t)p(1 - p)^{j-1}$, $\varphi_0(j) = \varphi_n(j) = p(1 - p)^{j-1}$. For $1 \leqq n \leqq L$ let $\varphi_{-n}(j) = (t^{L-n})\delta_{n,j} + (1 - t^{L-n})p(1 - p)^{j-1}$.

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Here a $\delta$-distribution starts in position $L$ at time $-L$ and at each time a fixed portion, $t$, of the distribution moves one step backwards and the rest goes into equilibrium. Note that $\varphi_{-n}$ is undefined for $n > L$. While $L$ can be arbitrarily large, a complete entrance law is impossible. The difficulty is the fact that at each time the proportion going into equilibrium remains fixed. We shall now show how to adjust this example by varying this proportion in order to obtain a complete entrance law.

## 2. Example for strictly positive transition matrix.

NOTATION. Here $n$, $i$ and $j$ only range over positive integers.

$$\varphi(1) = \tfrac{1}{2}$$

$$\varphi(i+1) = 1/3\varphi(i) + \frac{1}{2^{(2i+1)}}$$

$$Q(1, j) = \varphi(j)$$

$$Q(i+1, j) = \frac{\varphi(i)}{3\varphi(i+1)}\,\delta_{i,j} + \frac{\varphi(j)}{2^{2i+1}\varphi(i+1)}$$

$$e_n = \textstyle\prod_{i=n}^{\infty} \dfrac{\varphi(i)}{3\varphi(i+1)}$$

$$\varphi_n(j) = \varphi_0(j) = \varphi(j) \qquad \text{for all} \quad j$$

$$\varphi_{-n}(j) = e_n\delta_{n,j} + (1-e_n)\varphi_{(j)}\,.$$

CLAIMS.

(1) $\varphi$ is a probability vector.
(2) $Q$ is a strictly positive transition matrix.
(3) $0 < e_n < 1$ for all $n \geqq 1$, and therefore $\varphi_{-1} \neq \varphi_0$.
(4) $\varphi_n Q = \varphi_{n+1}$ for all $n \in Z$.

PROOF. (1) By induction $\varphi(i) < (\frac{2}{3})^i$ so that $\sum_{i=1}^{\infty} \varphi(i) < \infty$. $\sum_{i=1}^{\infty} \varphi(i) = \frac{1}{2} + \sum_{i=2}^{\infty} \varphi(i) = \frac{1}{2} + \sum_{i=1}^{\infty} (1/3\varphi(i) + 1/2^{2i+1}) = \frac{1}{2} + \frac{1}{3}(\sum_{i=1}^{\infty} \varphi(i)) + \frac{1}{6}$ so that $\sum_{i=1}^{\infty} \varphi(i) = 1$.

(2) Trivial from 1.

(3) For all $i \geqq 1$, $\varphi(i)/3\varphi(i+1) < 1$ so $e_n < 1$. Since $\varphi(1) = \frac{1}{2} > \frac{1}{3}$ and $\varphi(n+1) > \frac{1}{3}\varphi(n)$, $\varphi(n) > (\frac{1}{3})^n$ for all $n$. Hence $\sum_{i=1}^{\infty} 1/2^{2i+1}\varphi(i+1) < \sum_{i=1}^{\infty} 3^{i+1}/2^{2i+1} < \infty$ and thus $e_n = \prod_{i=n}^{\infty} \varphi(i)/3\varphi(i+1) = \prod_{i=n}^{\infty} (1 - 1/2^{2i+1}\varphi(i+1)) > 0$.

(4) Straightforward computation.

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