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CAN A NONSTABLE STATE BECOME STABLE BY SUBORDINATION?

BY MICHAEL RUBINOVITCH[1]

University of Utrecht

Let $Z_0(t)$ be a Markov chain and $X(t)$ a subordinator. Set $Z(t) = Z_0(X(t))$ and let α be a nonstable state of Z_0. It is shown, via an example, that it is possible for α to be a stable state of $Z(t)$ even when the total mass of the Lévy measure of X is unbounded.

Let $\{Z_0(t); t \geqq 0\}$ be a standard Markov chain with stationary transition probabilities on a countable state space I.

Let $\{X(t); t \geqq 0\}$ be a subordinator, i.e., a process with stationary independent increments on $[0, \infty)$. We assume that X is defined on the same probability space as Z_0, that it is independent of Z_0 and that $X(0) = 0$ surely. (See Fristedt [3] for a full account of subordinators, their Lévy measures and sample functions properties.)

Define now a new process $\{Z(t); t \geqq 0\}$ by

$$Z(t) = Z_0(X(t)),$$

so that Z is a subordinate of Z_0 directed by X. It is not hard to check that Z is standard Markov chain with stationary transition probabilities on I.

We pose now the following problem: *Can an instantaneous state of Z_0 be stable for Z in nontrivial situations?*

By a trivial situation we mean the following. Let $\alpha \in I$ be an instantaneous state of Z_0 and suppose that X is a compound Poisson process with no drift. Then a.s. there exists a $t_0 > 0$ such that $X(t) = 0$ for all $t < t_0$. Thus if $Z(0) = Z_0(0) = \alpha$ then $Z(t) = \alpha$ for all $t < t_0$ and α is stable for Z.

The interesting situation is when X is not a compound Poisson process and its sojourn time at zero is 0 a.s. Could α then be a stable state of Z? The answer is yes, and the objective of this note is to produce an example that proves this assertion.

Let $p_{ij}(t)$ and $p_{ij}^0(t)$ denote, respectively, the transition probabilities of Z and Z_0 and set, as usual,

$$q_{ij} = \lim_{t\to 0} \frac{1 - p_{ij}(t)}{t}; \quad q_{ij}^0 = \lim_{t\to 0} \frac{1 - p_{ij}^0(t)}{t}.$$

Also let L denote the Lévy measure of X and a be its drift.

An extensive study of subordinates of Markov chains is given in Cohen [2].

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[1] On summer leave from the Technion-Israel Institue of Technology.

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Among much else Cohen gives an explicit expression for the q_{ij} elements in terms of the parameters of Z_0 and X. For $i = j$ this expression reads

$$q_{ii} = aq_{ii}^0 + \int_0^\infty [1 - p_{ii}^0(t)]L(dt) \leqq \infty ,$$

where $aq_{ii}^0 = 0$ if $a = 0$ (see also Stam [7]). An immediate consequence of this is that when α is instantaneous for Z_0 then $a > 0$ will imply that α is instantaneous for Z. So, from here on we take $a = 0$, and, since the compound Poisson situation is no longer of interest, we assume that $L(0, \infty] = \infty$. Under these assumptions we wish to produce an example where $q_{\alpha\alpha}^0 = \infty$ and

$$q_{\alpha\alpha} = \int_0^\infty [1 - p_{\alpha\alpha}^0(t)]L(dt) < \infty . \tag{1}$$

Define a measure μ on $(0, \infty]$ by

$$\mu(dx) = x^{-2+c}\, dx$$

with $0 < c < 1$. This measure was given by Kingman [4] as an example for a canonical measure of a regenerative phenomenon. He showed that near the origin the corresponding p-function behaves according to

$$1 - p(t) \sim At^c \qquad t \to 0 \tag{2}$$

with some constant A. From the Markov characterization theorem ([5], page 125) it now follows that this p-function is a member of the class $\mathscr{PM}$, namely that there exists a Markov chain Z_0 on a countable state space I, and a state $\alpha \in I$ such that $p(t) = p_{\alpha\alpha}^0(t)$. This state is clearly instantaneous for this Markov chain. We have thus produced the process Z_0 and the state α of the desired example. Now let X be defined by its Lévy measure

$$L(dt) = t^{-1-\beta}\, dt \qquad 0 < t < \infty ,$$

where $0 < \beta < 1$. Clearly $L(0, \infty] = \infty$.

For these X, Z_0 and α we have from (2) the behavior near the origin of the integrand of (1), namely

$$[1 - p_{\alpha\alpha}^0(t)]L(dt) \sim At^{-1+(c-\beta)}\, dt \qquad t \to 0 .$$

Now take $c > \beta$ and you have an example where the integral in (1) converges, $q_{\alpha\alpha} < \infty$ and α is a stable state of Z. Take $c < \beta$, the integral will diverge, α is an instantaneous state of Z.

We conclude that even in nontrivial situations *an instantaneous state of a Markov chain can become stable via subordination*. It can, of course, also stay instantaneous.

The implications of this result are quite interesting. Consider the random set

$$S_\alpha = \{t \geqq 0 : Z_0(t) = \alpha \mid Z_0(0) = \alpha\}$$

of time points when Z_0 visits α. It is known (Chung [1]) that this set is heavy (Kingman [6]) in the sense that its Lebesgue measure is positive a.s. Next consider the set

$$G(t) = \{x : X(s) = x; 0 \leqq s < t\}$$

of all points hit by X during $[0, t)$. When $a = 0$ this set is light since its Lebesgue measure is zero a.s.

Now let $G = G(\infty)$. Both S_α and G are instantaneous sets and, moreover, they are stochastically independent. However, there are situations when the heavy set, S_α is "so heavy" or the light set G is "so light" that there a.s. exists a $t_0 > 0$ such that

$$G(t_0) \subset S_\alpha \quad \text{a.s.}$$

Since $X(t_0) > 0$ a.s. this also implies that there exists an $x_0 > 0$ (a.s.) such that

$$G \cap [0, x_0) \subset S_\alpha \cap [0, x_0) \quad \text{a.s.}$$

(the inclusion is of course proper).

On the other hand there are situations when G is not sufficiently light or S_α not heavy enough, so that no such t_0 or x_0 exist.

It is important to note that there is nothing special about the example given here. In fact, as pointed out to me by G. E. H. Reuter, one can use the famous Kolmogorov example ([1], page 278) to give a behavior like (2) without having to appeal to Kingman's deep characterization theorem. Let $Z_0(t)$ be the Markov chain of Example 3 on page 278 of [1] and $\alpha = 0$. Then

$$1 - p_{00}^0(t) = \int_0^t p_{00}^0(s)h(t-s)\,ds$$

where

$$h(t) = \sum_1^\infty e^{-q_n t}$$

with $\sum q_n^{-1} < \infty$. Taking $q_n = n^\beta$ ($\beta > 1$) it is easy to see that $h(t) \sim At^{-1/\beta}$ as $t \to 0$, so that

$$1 - p_{00}^0(t) \sim \int_0^t h(s)\,ds \sim Bt^c$$

where

$$c = 1 - \beta^{-1} \qquad \text{and} \qquad 0 < c < 1\,.$$

Another observation due to Reuter is that for each $p_{\alpha\alpha}(t)$ of a nonstable state α, one can produce nontrivial Lévy measures (and hence subordinators) to make the integral in (1) finite or infinite. This will be discussed in a forthcoming paper on subordination of regenerative phenomena and intersections of random sets, where a short proof of Cohen's formula in a more general setup, as well as a complete characterization of subordinates of regenerative phenomena, will be given.

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REFERENCES

[1] CHUNG, K. L. (1967). *Markov Chains with Stationary Transition Probabilites*. Springer-Verlag, Berlin.

[2] Cohen, J. W. (1962). Derived Markov chains I, II, III. *Nederl. Akad. Wetensch. Indag. Math.* **24** 55–92.

[3] Fristedt, B. E. (1973). Sample functions of stochastic processes with stationary, independent increments. *Advances in Probability*, **3**, Dekker, New York.

[4] Kingman, J. F. C. (1964). The stochastic theory of regenerative events. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **2** 180–224.

[5] Kingman, J. F. C. (1972). *Regenerative Phenomena*. Wiley, New York.

[6] Kingman, J. F. C. (1973). Homecomings of Markov processes. *Advances in Appl. Probability* **5** 66–102.

[7] Stam, A. J. (1962). On the Q-matrix of a Markov chain. *Nederl. Akad. Wetensch. Indag. Math.* **24** 578–582.

Department of Industrial Engineering
Northwestern University
Evanston, Illinois 60201