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STABLE PROCESSES: SAMPLE FUNCTION GROWTH AT A LAST EXIT TIME

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For stable processes of index α, $1 < \alpha < 2$, exact upper functions are determined for the sample function growth at a last exit time.

1. Introduction. Let $\{X_t, t \geqq 0\}$ be a real valued stable process of index α with $1 < \alpha < 2$, i.e., a stochastic process with stationary independent increments and

$$E^0\{\exp(iuX_t)\} = \int e^{iuy}p_t(y)\,dy = \exp\{t\psi(u)\}\,,$$

where $\psi(u) = -|u|^\alpha(1 + i\beta \operatorname{sign}(u) \tan(\pi\alpha/2))$. The skew parameter β satisfies $|\beta| \leqq 1$. If $\beta = 1$, then any right continuous version of $\{X_t\}$ will have no upward jumps, and if $\beta = -1$, then it will have no downward jumps. We shall assume that $\{X_t\}$ has right continuous paths with left limits.

For each real number x, let $T_x = \inf\{t > 0 : X_t = x\}$ denote the first hitting time of $\{x\}$. Then $P^0\{T_0 = 0\} = 1$ and $P^0\{T_x < \infty\} = 1$. By the strong Markov property we know how the process behaves immediately after hitting the point $\{1\}$ (say) for the first time. But how does the process approach $\{1\}$? We know that it is in a continuous manner. It is also well known that $\{X_t\}$ approaches $\{1\}$ for the first time in exactly the same way as the process escapes from $\{0\}$ for the last time before hitting $\{1\}$. In fact, let $\{X_t^1\}$ be the process obtained by killing $\{X_t\}$ at time T_1, i.e. $X_t^1 = X_t$ if $t < T_1$ and $X_t^1 = \Delta$ if $T_1 \leqq t$, where Δ is the usual adjoined point in the general theory of Markov processes. Define

$$L = \sup\{t > 0 : X_t^1 = 0\}$$

and put $Z_t = X_{L+t}^1$, $t \geqq 0$. Then Z is a strong Markov process. (See [6]. The entrance law, however, requires special considerations.)

Let ζ denote the lifetime of Z and put

$$\begin{aligned} \hat{Z}_t &= Z(\zeta - t-) && \text{if } 0 \leqq t < \zeta \\ &= \Delta && \text{if } \zeta \leqq t\,, \end{aligned}$$

i.e., $\hat{Z}$ is Z reversed in time. (See Chapter 1, Section 6 of [5].) Then $1 - \hat{Z}$ is a strong Markov process equivalent to Z.

The goal of this paper is to analyze the sample function growth of the process $\{Z_t\}$ for small t. Most of the basic facts about this process can be found in [7]. In [7] Millar gives necessary and sufficient conditions for

$$\liminf_{t\to 0} |X^1(L + t)|/t^{1/\alpha}f(t)$$

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being 0 or ∞, where f is a nonnegative increasing function. In this paper it is shown that if $|\beta| < 1$ and f is a nonnegative decreasing function, then with probability 1

$$\limsup_{t\to 0} X^1(L+t)/t^{1/\alpha}f(t) = 0 \qquad \text{or} \quad \infty$$

according as $\int_0^1 (tf(t))^{-1}\,dt < \infty$ or $= \infty$. In particular, we see that whereas $\lim_{t\to 0} X(t)/t^{1/\alpha}|\log(t)| = 0$ (see [3]), we have $\limsup_{t\to 0} X^1(L+t)/t^{1/\alpha}|\log(t)| = \infty$.

If $|\beta| = 1$, then the process $\{X^1(L+t),\ t > 0\}$ has the same sign as β for an initial period of time. Furthermore

$$\limsup_{t\to 0} |X^1(L+t)|/t^{1/\alpha}(\log|\log(t)|)^{1-1/\alpha}$$

equals a finite positive constant a.s.

We can summarize by saying that if $|\beta| < 1$, then the process $\{X^1(L+t)\}$ escapes faster from $\{0\}$ than the process $\{X(t)\}$. If $\beta = 1$, then $\{X^1(L+t)\}$ and $\{X(t)\}$ have exactly the same upper envelope at zero.

2. The last exit process. It turns out that instead of studying the process Z it is easier to analyze the related process Y defined below. Put

$$L = \sup\{t < T_1 : X(t) = 0\}$$
$$L_1 = \sup\{t < T_0 \circ \theta_{T_1} : X(t) = 1\}$$

and define for $t \geqq 0$

$$Y(t) = X(L+t) \qquad \text{if} \quad L + t < L_1$$
$$= \Delta \qquad \text{if} \quad L + t \geqq L_1 .$$

Then Y is given by the path of X from its last 0 before hitting 1, until its last 1 before returning to 0 again. Obviously, Y has right continuous paths with left limits. We will show that Y is a strong Markov process with respect to the σ-fields

$$\mathscr{F}_t = \bigcap_{s>t} \sigma\{Y(u) : 0 \leqq u \leqq s\}, \qquad t \geqq 0 .$$

First, let Q_t denote the probability distribution with density

$$q_t(y) = p_t(y) + (1 - 1/\alpha)t^{1-1/\alpha} \int_0^t (p_t(y) - p_{t-s}(y))s^{1/\alpha - 2}\,ds .$$

We note that $q_t(y) = t^{-1/\alpha}q_1(t^{-1/\alpha}y)$.

Next, put $h(x) = P^x\{T_1 < T_0\}$.

THEOREM 2.1. *$\{Y_t, \mathscr{F}_t, t \geqq 0\}$ is a strong Markov process with transition functions*

$$H_t(x, f) = E^x[(fh)(X_t)I\{t < T_0\}]/h(x) \qquad \textit{if} \quad x \neq 0$$
$$H_t(0, f) = C_0 t^{1/\alpha - 1}Q_t(fh),$$

where $C_0^{-1} = -p_1(0)\Gamma(\alpha)\Gamma(1/\alpha)\Gamma(1 - 1/\alpha)\cos(\pi\alpha/2)[1 + \beta^2\tan^2(\pi\alpha/2)]$.

PROOF. Let $F : (R \cup \{\Delta\})^m \to R$ and $f : R \cup \{\Delta\} \to R$ be bounded and continuous. Let $0 < t_1 < \cdots < t_m$ and put $\varphi(\omega) = F(\omega(t_1), \cdots, \omega(t_m))$ for any function

$\omega : R_+ \to R \cup \{\Delta\}$. Let $t_m < s$. Then

$$E^0\{\varphi(Y)f(Y(s))\}$$
$$= E^0 \lim_{n\to\infty} \sum_k I\left\{\frac{k}{n} < L \leqq \frac{k+1}{n},\ \frac{k+1}{n} + s < L_1\right\}$$
$$\times \varphi\left(X \circ \theta\left(\frac{k+1}{n}\right)\right) f\left(X\left(\frac{k+1}{n} + s\right)\right)$$
$$= E^0 \lim \sum I\left\{T_1 > \frac{k+1}{n},\ T_0 \circ \theta\left(\frac{k}{n}\right) \leqq \frac{1}{n},\ T_0 \circ \theta\left(\frac{k+1}{n}\right) > t_m\right\}$$
$$\times \varphi\left(X \circ \theta\left(\frac{k+1}{n}\right)\right)$$
$$\times I\left\{T_0 \circ \theta\left(\frac{k+1}{n} + t_m\right) > s - t_m\right\} f\left(X\left(\frac{k+1}{n} + s\right)\right)$$
$$\times I\left\{T_0 \circ \theta\left(\frac{k+1}{n} + s\right) > T_1 \circ \theta\left(\frac{k+1}{n} + s\right)\right\}$$
$$= \lim E^0 \sum I\left\{T_1 > \frac{k+1}{n},\ T_0 \circ \theta\left(\frac{k}{n}\right) \leqq \frac{1}{n},\ T_0 \circ \theta\left(\frac{k+1}{n}\right) > t_m\right\}$$
$$\times \varphi\left(X \circ \theta\left(\frac{k+1}{n}\right)\right) H_{s-t_m}\left(X\left(\frac{k+1}{n} + t_m\right), f\right)$$
$$\times P^{X((k+1)/n+t_m)}\{T_0 > T_1\}$$
$$= \lim E^0 \sum I\left\{T_1 > \frac{k+1}{n},\ T_0 \circ \theta\left(\frac{k}{n}\right) \leqq \frac{1}{n},\ T_0 \circ \theta\left(\frac{k+1}{n}\right) > t_m\right\}$$
$$\times \varphi\left(X \circ \theta\left(\frac{k+1}{n}\right)\right) H_{s-t_m}\left(X\left(\frac{k+1}{n} + t_m\right), f\right)$$
$$\times I\left\{T_0 \circ \theta\left(\frac{k+1}{n} + t_m\right) > T_1 \circ \theta\left(\frac{k+1}{n} + t_m\right)\right\}$$
$$= E^0\{\varphi(Y)H_{s-t_m}(Y(t_m), f)\}\,.$$

This shows that $\{Y_t, t \geqq 0\}$ is a Markov process. We have to identify the transition function $H_t(0, f) = E^0 f(Y_t)$.

In order to show that $\{Y_t\}$ is a strong Markov process with respect to the σ-fields $\{\mathscr{F}_t\}$, we have to prove that for each $t > 0$ and each bounded continuous function f, the map $s \to H_t(Y_s, f)$ is right continuous a.s. The map $x \to H_t(x, f)$ is clearly continuous everywhere except possibly at 0. We therefore only have to show that

$$\lim_{s\to 0} H_t(Y_s, f) = E^0 f(Y_t)\,.$$

If $|\beta| = 1$, then Y_s has the same sign as β for all sufficiently small s. (See Lemma 4.6 of [7].) In the following analysis of $H_t(x, f)$ for small x we will therefore assume that x has the same sign as β if $|\beta| = 1$.

$$P^x\{T_1 < T_0\} \sim \tfrac{1}{2}(1 + \beta\,\mathrm{sign}\,(x))|x|^{\alpha-1}$$

as $x \to 0$. (See formula (3.1) in [7].) Similarly, for $t^{-1}|x|^\alpha \to 0$

$$P^x\{t < T_0\} \sim Ct^{1/\alpha - 1}(1 + \beta \operatorname{sign}(x))|x|^{\alpha-1}, \qquad \text{where}$$
$$C^{-1} = -2p_1(0)\Gamma(\alpha)\Gamma(1/\alpha)\Gamma(1 - 1/\alpha)\cos(\pi\alpha/2)[1 + \beta^2 \tan^2(\pi\alpha/2)] .$$

(See the proof of Lemma 4.3 of [7].) Consequently, for fixed $t > 0$

$$\lim_{s \to t, x \to 0} P^x\{s < T_0\}/P^x\{T_1 < T_0\} = 2Ct^{1/\alpha - 1} .$$

For $x \neq 0$

$$H_s(x, f) = \frac{E^x[(fh)(X_s)I\{s < T_0\}]}{P^x\{s < T_0\}} \frac{P^x\{s < T_0\}}{P^x\{T_1 < T_0\}} .$$

It therefore follows from Lemma 4.5 of [7] that

$$\lim_{s \to t, x \to 0} H_s(x, f) = 2Ct^{1/\alpha - 1}Q_t(fh) .$$

By dominated convergence and the Markov property

$$E^0 f(Y_t) = \lim_{s \to 0} E^0\{H_{t-s}(Y_s, f)\} = \lim_{s \to 0} H_{t-s}(Y_s, f) = 2Ct^{1/\alpha - 1}Q_t(fh) .$$

This completes the proof of Theorem 2.1.

COROLLARY (Millar). *For $A \in \mathscr{F}_0$ we either have $P^0(A) = 0$ or $P^0(A) = 1$.*

REMARK. The process $\{Z_t\}$ is obtained by killing the process $\{Y_t\}$ the first time it hits 1.

3. Probability estimates. Let $\{P_L{}^x\}$ denote the usual family of measures associated with the transition functions $\{H_t\}$.

We shall need the following asymptotic formulas for stable densities. (See Skorokhod [8].)

$$\text{If} \quad \beta \neq 1 , \quad \text{then} \quad p_1(y) \sim Ay^{-(\alpha+1)} \quad \text{as} \quad y \to \infty .$$
$$\text{If} \quad \beta = 1 , \quad \text{then} \quad p_1(y) \sim Ay^{-1+\lambda/2}\exp(-By^\lambda) \quad \text{as} \quad y \to \infty ,$$
$$\text{where} \quad \lambda = \alpha/(\alpha - 1) .$$

In the following estimates the letters c and C will denote positive constants whose values are unimportant. Their values may even change from line to line.

LEMMA 3.1. *If $\beta \neq 1$, then for $N > 1$*

$$cN^{-\alpha} < \int_N^\infty q_1(y)\,dy < CN^{-\alpha} .$$

If $\beta = 1$, then for $N > 0$

$$c\exp(-(B + \varepsilon)N^\lambda) < \int_N^\infty q_1(y)\,dy < C\exp(-(B - \varepsilon)N^\lambda) ,$$

where c and C depend only on $\varepsilon > 0$.

PROOF. The lower bounds follow from the fact that

$$\int_N^\infty p_1(y)\,dy < \int_N^\infty q_1(y)\,dy .$$

For $\beta \neq 1$ we get

$$\int_N^\infty (p_1(y) - p_{1-s}(y))\, dy = \int_N^{N(1-s)^{-1/\alpha}} p_1(y)\, dy$$
$$< C \int_N^{N(1-s)^{-1/\alpha}} y^{-(\alpha+1)}\, dy = \frac{C}{\alpha} N^{-\alpha} s .$$

The result now follows from Fubini's theorem. The case $\beta = 1$ is handled the same way, thus completing the proof of Lemma 3.1.

LEMMA 3.2. *If either $|\beta| < 1$, or $\beta = 1$ and $0 < x < 1$, then*

$$c|x|^{\alpha-1} \wedge 1 \leqq P^x\{T_1 < T_0\} \leqq C|x|^{\alpha-1} \wedge 1 .$$

PROOF. See Lemma 3.1 of [7].

Combining Theorem 2.1 and Lemma 3.2 we get

LEMMA 3.3. *Assume that $\beta \neq -1$. Then*

$$P_L{}^0\{Y_t > Nt^{1/\alpha}\} < C \int_N^\infty y^{\alpha-1} q_1(y)\, dy$$

for all $N > 0$. And, if in addition $Mt^{1/\alpha} < 1$, then

$$P_L{}^0\{Nt^{1/\alpha} < Y_t < Mt^{1/\alpha}\} > c \int_N^M y^{\alpha-1} q_1(y)\, dy .$$

LEMMA 3.4. *Assume that $\beta \neq -1$. Then we can choose $k > 1$ such that for all sufficiently small t*

$$P_L{}^0\{t^{1/\alpha} < Y_t < kt^{1/\alpha}\} \geqq C > 0 .$$

Fix this k. Then for $t^{1/\alpha} < x < kt^{1/\alpha}$ and $0 < N < (\log(t))^2$

$$c \int_N^\infty y^{\alpha-1} p_1(y)\, dy < P_L{}^x\{Y_t > Nt^{1/\alpha} + x\} < C \int_N^\infty y^{\alpha-1} p_1(y)\, dy .$$

PROOF. The first assertion follows from Lemma 3.3. Consider the identity

$$P_L{}^x\{Y_t > M\} = \frac{E^x[I\{X_t > M,\, t < T_0\} P^{X_t}\{T_1 < T_0\}]}{P^x\{T_1 < T_0\}} .$$

According to Lemma 3.2,

$$ct^{1-1/\alpha} \leqq P^x\{T_1 < T_0\} \leqq Ct^{1-1/\alpha} .$$

By the scaling property

$$P^x\{T_0 \leqq t\} = P^1\{T_0 \leqq tx^{-\alpha}\} \leqq P^1\{T_0 \leqq 1\} .$$

And by the first passage relation and the scaling property

$$P^x\{X_t > y,\, T_0 \leqq t\} \leqq P^x\{T_0 \leqq t\} P^0\{X_t > y\} \leqq P^x\{T_0 \leqq t\} P^x\{X_t > y\}$$

for $y > 0$. We therefore conclude that if we put $c = P^1\{T_0 > 1\}$, then

$$P^x\{X_t > y,\, t < T_0\} = P^x\{X_t > y\} - P^x\{X_t > y,\, T_0 \leqq t\} \geqq cP^x\{X_t > y\}$$

for $y > 0$. Hence for $0 < M$

$$c \int_M^\infty y^{\alpha-1} P^x\{X_t \in dy\} < \int_M^\infty y^{\alpha-1} P^x\{X_t \in dy,\, t < T_0\} < \int_M^\infty y^{\alpha-1} P^x\{X_t \in dy\} .$$

It now follows from Lemma 3.2 that for $0 < M < \frac{1}{2}$

$$c \int_M^\infty y^{\alpha-1} P^x\{X_t \in dy\} < \int_M^\infty P^y\{T_1 < T_0\} P^x\{X_t \in dy, t < T_0\} < C \int_M^\infty y^{\alpha-1} P^x\{X_t \in dy\} .$$

The constants c and C do not depend on t, x or M. To finish the proof we note that

$$t^{1-1/\alpha} \int_N^\infty y^{\alpha-1} p_1(y)\, dy \leqq \int_{Nt^{1/\alpha}+x}^\infty y^{\alpha-1} P^x\{X_t \in dy\} \leqq t^{1-1/\alpha} \int_N^\infty (y+k)^{\alpha-1} p_1(y)\, dy .$$

This completes the proof of Lemma 3.4.

Put $Y_t^* = \sup\{Y_s : 0 < s \leqq t\}$.

LEMMA 3.5. *Assume that* $\beta \neq -1$. *Then*

$$P_L{}^0\{Y_t^* > Nt^{1/\alpha}\} \leqq CP_L{}^0\{Y_t > Nt^{1/\alpha}\}$$

for $1 < N$.

PROOF. Let $t^{1/\alpha} < x$. Arguing as in the proof of Lemma 3.4, we see that there exists a constant C such that $P_L{}^x\{Y_s \geqq x\} \geqq C > 0$ for all $s < t$. This implies that

$$P_L{}^0\{Y_t^* > Nt^{1/\alpha}\} \leqq C^{-1} P_L{}^0\{Y_t > Nt^{1/\alpha}\}$$

by the first passage relation.

4. The case $|\beta| \leqq 1$. Combining Lemmas 3.1, 3.3 and 3.5 we get

LEMMA 4.1. *For all t sufficiently small and for* $1 < N < (\log(t))^2$

$$cN^{-1} < P_L{}^0\{Y_t > Nt^{1/\alpha}\} \leqq P_L{}^0\{Y_t^* > Nt^{1/\alpha}\} < CN^{-1} .$$

Using these estimates we can determine the rate of escape from zero of the last exit process $\{Y_t, t \geqq 0\}$.

THEOREM 4.1. *Let* $f(t)$, $0 < t < 1$, *be a nonnegative decreasing function.*

If $\int_0^1 (tf(t))^{-1}\, dt < \infty$, *then* $\lim_{t\to 0} Y_t/t^{1/\alpha} f(t) = 0$ a.s.

If $\int_0^1 (tf(t))^{-1}\, dt = \infty$, *then* $\limsup_{t\to 0} Y_t/t^{1/\alpha} f(t) = \infty$ a.s.

PROOF. Assume that the integral is finite. Then $\sum f(2^{-n})^{-1} < \infty$. Put

$$A_n = \{Y^*(2^{-n}) > \varepsilon f(2^{-n}) 2^{-n/\alpha}\} .$$

By Lemma 4.1 we have $\sum P^0(A_n) < \infty$. By Borel–Cantelli, this implies

$$\limsup_{t\to 0} Y_t^*/t^{1/\alpha} f(t) \leqq 2\varepsilon \quad \text{a.s.}$$

This proves the first part of the theorem. Next, assume that the integral is infinite. Then $\sum f(2^{-n})^{-1} = \infty$. It follows that $\sum^* f(2^{-n})^{-1} = \infty$, where $\sum^*$ denotes summation over those indices n for which $f(2^{-n}) < n^2$. We shall only consider such indices.

To simplify notation we will assume that $f(2^{-n}) < n^2$ for all n. Let $k > 1$ be chosen as in Lemma 3.4, let K be a fixed large constant and put

$$B_n = \{2^{-(n+1)/\alpha} < Y(2^{-(n+1)}) < k2^{-(n+1)/\alpha},\ K2^{-n/\alpha}f(2^{-n}) < Y(2^{-n})\}\,.$$

By Lemma 3.4 and the estimates for stable densities given in the beginning of Section 3,

$$P^0(B_n) > cf(2^{-n})^{-1}\,,$$

where the constant c does not depend on n. Hence $\sum P^0(B_n) = \infty$. The events B_n are not independent. But for $m \neq n$ we have

$$P^0(B_m \cap B_n) \leqq CP^0(B_m)P^0(B_n)$$

in virtue of Lemma 3.4. By a generalization of the Borel–Cantelli lemma for dependent events (see page 317 of [9]),

$$P^0(\limsup B_n) > 0\,.$$

By the zero-one law of Section 2 this implies that

$$P^0(\limsup B_n) = 1\,.$$

Thus $\limsup_{t\to 0} Y_t/t^{1/\alpha}f(t) \geqq K$ a.s. This completes the proof.

REMARK. If $\int_0^1 (tf(t))^{-1}\,dt = \infty$, then $\liminf Y_t/t^{1/\alpha}f(t) = -\infty$ a.s.

5. The case $|\beta| = 1$. If $\beta = -1$, then according to Lemma 4.6 of [7] the process $\{Y_t, t > 0\}$ is negative for all sufficiently small t. In fact, if $\beta = -1$, then by the zero-one law the initial behaviour of the process $\{-Y_t, t > 0\}$ is identical to the initial behaviour of the last exit process with $\beta = 1$ (and the same value of α). We shall therefore only consider the case $\beta = 1$. If $\beta = 1$, then (as noted in Section 2 of [7]) $0 < Y_t < 1$ for $0 < t$.

Combining Lemmas 3.1, 3.3, and 3.5 we get

LEMMA 5.1. *For all t sufficiently small and for $1 < N < (\log(t))^2$*

$$P_L{}^0\{Y_t{}^* > Nt^{1/\alpha}\} < C\exp(-B(1-\varepsilon)N^\lambda)$$
$$P_L{}^0\{Y_t{}^* > Nt^{1/\alpha}\} > c\exp(-B(1+\varepsilon)N^\lambda)\,.$$

The constants c and C depend only on $\varepsilon > 0$.

THEOREM 5.1. $\limsup_{t\to 0} Y_t/t^{1/\alpha}(B^{-1}\log|\log(t)|)^{1-1/\alpha} = 1$ a.s.

PROOF. Let $\frac{1}{2} < b < 1$ and let $\varepsilon > 0$. Put for large n

$$A_n = \{Y^*(b^n) > b^{n/\alpha}((1+\varepsilon)B^{-1}\log|\log(b^n)|)^{1-1/\alpha}\}\,.$$

By Lemma 5.1, $P^0(A_n) < C\exp(-(1+\varepsilon/2)\log(n))$. So $\sum P^0(A_n) < \infty$. By Borel–Cantelli, this implies

$$\limsup_{t\to 0} Y_t/t^{1/\alpha}(B^{-1}\log|\log(t)|)^{1-1/\alpha} < 1+\varepsilon \quad \text{a.s.}$$

provided we have chosen b close enough to 1.

The second part of the assertion is proved the same way the second part of Theorem 4.1 was proved.

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REFERENCES

[1] BLUMENTHAL, R. M. (1957). An extended Markov property. *Trans. Amer. Math. Soc.* **85** 52–72.

[2] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory.* Academic Press, New York.

[3] FRISTEDT, B. (1974). Sample functions of stochastic processes with stationary, independent increments. *Adv. Probability* **3**, 241–396, Dekker, New York.

[4] GIKHMAN, I. I. and SKOROKHOD, A. V. (1969). *Introduction to the Theory of Random Processes.* Saunders, Philadelphia.

[5] MEYER, P. A. (1968). Processes de Markov: la frontière de Martin. *Lecture Notes in Mathematics* **77**. Springer, Berlin.

[6] MEYER, P. A., SMYTHE, R. T. and WALSH, J. B. (1972). Birth and death of Markov processes. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **3** 295–306. Univ. of California Press.

[7] MILLAR, P. W. (1976). Sample functions at a last exit time. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **34** 91–111.

[8] SKOROKHOD, A. V. (1961). Asymptotic formulas for stable distribution laws. *Selected Transl. Math. Statist. Prob.* **1** 157–161.

[9] SPITZER, F. (1964). *Principles of Random Walk.* Van Nostrand, Princeton.

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