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ERGODICITY CONDITIONS FOR A DISSONANT VOTING MODEL

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Call a Markov process "ergodic" if the following conditions hold: (a) The process has a unique invariant measure ν. (b) If μ_0 is any initial distribution for the process, then the resulting distribution μ_t at time t will converge weakly to ν as $t \to \infty$. In this paper, necessary and sufficient conditions are obained for the ergodicity of a certain infinite particle process. This process models a dissonant voting system, and is similar to one treated in Holley and Liggett (1975).

1. Introduction. Let S be a countable set, and let $p(x, y)$ be a stochastic matrix defined on $S \times S$. Think of the elements of S as "voters," each of which is either in favor of or against a certain issue. Each voter $x \in S$ will at random times reevaluate his position on the issue, according to the following procedure: x chooses a vector $y \in S$ at random according to the probabilities $p(x, \cdot)$, and then takes on the position opposite to that of y. The process continues indefinitely in this manner. This model is similar to one considered in Holley and Liggett (1975), except that in the reevaluation procedure in the latter model, x takes on the *same* position as y.

In this paper we find necessary and sufficient conditions for our process to be ergodic, i.e., to be such that the distribution of the process at time t is guaranteed to converge weakly to a unique invariant measure ν, which does not depend on the initial distribution of the process. The organization of the paper is as follows:

In Section 2 we give a precise definition of the process, and then derive the structure of a certain Markov chain which will be the basis for our analysis. This Markov chain is of independent interest, and involves particles which "change color" when they move, and "die" when they collide. Section 2 also contains a theorem which for a certain class of transition functions $p(x, y)$, relates our process to the consonant voting model of Holley and Liggett which was mentioned above.

Section 3 then contains the ergodicity results. For example, we find that the process will be ergodic if $p(x, y)$ corresponds to an aperiodic random walk on the integer lattice $\mathbb{Z}^d$. We also investigate several properties of the unique invariant measure μ, such as asymptotic independence between voters.

2. Interrelations between three Markov processes. As above, let S be a countable set, and let $p(x, y)$ be the transition matrix for an irreducible discrete time Markov chain on S. We will always assume that $p(x, x) \equiv 0$, since it will

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simplify notation, and since the intuitive description of our process makes it unreasonable to have $p(x, x) > 0$ for any x (however, most of our results would still hold without this restriction on p). We will also assume that $p(x, y)$ has odd period; see Proposition 2.2. The state space for our process will be $K = \{0, 1\}^S$. Let $C(K)$ be the usual Banach space of continuous, real-valued functions on K; this space is well defined, since K is compact in the product topology. By the Stone-Weierstrass theorem, the class $\mathscr{F}$ of functions on K which depend on only finitely many coordinates is dense in $C(K)$.

For any $f \in \mathscr{F}$ and any $\eta \in K$, define $(\Omega f)(\eta)$ to be

$$\sum_{x \in S} \sum_{y \in S, \eta(x) = \eta(y)} p(x, y)[f(\eta_x) - f(\eta)] ,$$

$$\text{where} \quad \eta_x(z) = \eta(z) , \qquad z \neq x$$
$$= 1 - \eta(x) , \quad z = x .$$

Ω describes the behavior of the process during short ("infinitesimal") periods of time, and its definition reflects our intuitive notions of the process. An application of Theorem 4.2 of Liggett (1972) shows that an extension of Ω generates a strongly continuous semigroup $S(t)$ ($t \geqq 0$) of positive contractions on $C(K)$, so that there exists a strong Markov process η_t on K such that $[S(t)f](\eta) = E^\eta f(\eta_t)$.

We will now motivate the definition of the Markov chain mentioned in Section 1. This Markov chain will be the basic tool for our analysis of our original process η_t, and is in a sense "reciprocal" to η_t. This idea of finding a Markov chain which is reciprocal to a given infinite particle system is due to Spitzer (1970), and has been used in several studies of infinite particle systems; see for instance Holley and Liggett (1975).

Let $\mathscr{M}$ denote the set of all probability measures on K, and fix $\mu \in \mathscr{M}$ as the initial distribution of the process η_t. Let $(\mathbf{a}; \mathbf{x})_t = \mu_t(\eta(x_1) = a_1, \cdots, \eta(x_n) = a_n)$, where μ_t is the distribution of the process η_t at time t, $\mathbf{a} = (a_1, \cdots, a_n) \in \{0, 1\}^n$, and $\mathbf{x} = (x_1, \cdots, x_n)$ is a vector of distinct sites in S. Let $f(\eta)$ be the indicator function of the set $\{\eta(x_1) = a_1, \cdots, \eta(x_n) = a_n\}$ (note that the class of all such functions spans $\mathscr{F}$). Now

$$(\mathbf{a}; \mathbf{x})_t = \int S(t) f \, d\mu , \qquad \text{so} \quad \frac{d}{dt}(\mathbf{a}; \mathbf{x})_t$$

exists and is equal to $\int \Omega S(t) f \, d\mu$, from standard semigroup theory. Let $A = \{x_1, \cdots, x_n\}$. We then have

$$\begin{aligned} \frac{d}{dt}(\mathbf{a}; \mathbf{x})_t = \sum_{y \notin A} \sum_i p(x_i, y)[&(1 - a_i, a_1, \cdots, 1 - a_i, \cdots, a_n; \\ &y, x_1, \cdots, x_i, \cdots, x_n)_t \\ &- (a_i, a_1, \cdots, a_i, \cdots, a_n; y, x_1, \cdots, x_i, \cdots, x_n)_t] \\ &+ \sum_{i,j: a_i \neq a_j} p(x_i, x_j)(a_1, \cdots, a_j, \cdots, a_j, \cdots, a_n; \\ &x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n)_t \\ &- \sum_{i,j: a_i = a_j} p(x_i, x_j)(\mathbf{a}; \mathbf{x})_t . \end{aligned}$$

By appropriate additions and subtractions this expression can be rewritten as

$$\begin{aligned}\textstyle\sum_{y\notin A}\sum_i & p(x_i, y)[(1-a_i, a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n;\\ &\quad y, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)_t - (\mathbf{a}; \mathbf{x})_t]\\ &+ \textstyle\sum_{i,j:a_i\neq a_j} p(x_i, x_j)[(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n;\\ &\quad x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)_t - (a; x)_t] - \textstyle\sum_{i,j:a_i=a_j} p(x_i, x_j)(\mathbf{a}; \mathbf{x})_t ,\end{aligned}$$

so that

$$\begin{aligned}\frac{d}{dt}(\mathbf{a}; \mathbf{x})_t = \textstyle\sum_{y\notin A}\sum_i & p(x_i, y)(1-a_i, a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n;\\ & y, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)_t\\ &+ \textstyle\sum_{i,j:a_i\neq a_j} p(x_i, x_j)(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n;\\ & x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)_t - n(\mathbf{a}; \mathbf{x})_t .\end{aligned} \tag{2.1}$$

Equation (2.1) will be the motivation for the construction of two Markov chains:

(a) *The chain* $\alpha(t)$. Let $S_1 = S \times \{0, 1\}$, and define the stochastic matrix Q on $S_1 \times S_1$ by

$$\begin{aligned} Q(\alpha, \beta) &= p(x, y) , && \text{if } a + b = 1 ,\\ &= 0 , && \text{if not} ,\end{aligned}$$

where $\alpha = (x, a)$ and $\beta = (y, b)$. Intuitively, Q describes the motion of a particle which has a "position" (in S) and a "color" (either 0 or 1). The particle moves on S according to $p(x, y)$, and it changes color each time it moves. Now let $\alpha(t)$ be a continuous time Markov chain on S_1 whose paths are governed by Q and whose holding times at each state have mean one. Finally, extend $\alpha(t)$ to $\bigcup_{i=1}^n S_1^i$ by letting $\boldsymbol{\alpha}(t) = [\alpha_1(t), \cdots, \alpha_i(t)]$ consist of i independent copies of $\alpha(t)$ (here S_1^i denotes that i-fold cartesian product of S_1 with itself).

(b) *The chain* σ_t. Define L_n to be

$$\{\Delta\} \cup [\textstyle\bigcup_{i=1}^n S_1^i] ,$$

where Δ is a "death point" which will be explained below, and let $D_n = \{[(x_1, a_1), \cdots, (x_i, a_i)] : x_r = x_s \text{ for some } r \neq s \text{ and some } i \leqq n\}$. Now let σ_t be the continuous time Markov chain on $L_n \backslash D_n$ whose infinitesimal parameters are as follows: Δ is absorbing, and for $\sigma = [(x_1, a_1), \cdots, (x_i, a_i)] \in L_n\backslash D_n$ let

$$\begin{aligned} q_{\sigma\tau} &= -i , && \text{if } \tau = \sigma ,\\ &= p(x_r, y) , && \text{if } y \notin \{x_1, \cdots, x_i\} \text{ and}\\ & && \tau = [(x_1, a_1), \cdots, (x_{r-1}, a_{r-1}), (y, 1-a_r),\\ & && \qquad (x_{r+1}, a_{r+1}), \cdots, (x_i, a_i)] ,\\ &= \textstyle\sum_{r,s:a_r=a_s} p(x_r, x_s) , && \text{if } \tau = \Delta ,\\ &= p(x_r, x_s) + p(x_s, x_r) , && \text{if } a_r \neq a_s , \ r > s \text{ and}\\ & && \tau = [(x_1, a_1), \cdots, (x_{r-1}, a_{r-1}),\\ & && \qquad (x_{r+1}, a_{r+1}), \cdots, (x_i, a_i)] ,\\ &= 0 , && \text{otherwise.}\end{aligned}$$

To describe σ_t intuitively, let $\sigma_0 = \boldsymbol{\alpha}(0)$ be in $S_1{}^i\backslash D_n$, and let T_{D_n} be the hitting time of D_n for $\boldsymbol{\alpha}(t)$; T_{D_n} is allowed to be infinite, and of course it definitely will be infinite if i is equal to one. Now let $\sigma_t = \boldsymbol{\alpha}(t)$ for $0 \leqq t < T_{D_n}$. At time T_{D_n} there is a "collision" (in S) between two of the particles $\alpha_1(t), \cdots, \alpha_i(t)$, which allows the following interpretation of $\sigma_{T_{D_n}}$: If the two particles in the collision are of opposite color when they hit, then all of the particles in σ_t die, and thus we set $\sigma_{T_{D_n}} = \Delta$. If the two colliding particles are of the same color, then the particle of higher index dies, and the other particles continue as before, until the time of the next collision. The process continues in this manner.

The matrix $(q_{\sigma\tau})_{\sigma,\tau \in L_n\backslash D_n}$ is nonnegative off the diagonal, nonpositive on the diagonal, and has its row sums equal to zero. Also, q is a bounded linear operator on $l_\infty(L_n/D_n)$, since the absolute values of the elements in any row sum to at most $2n$. Thus we know that the Markov chain σ_t does exist, and that for any $f \in l_\infty(L\backslash D_n)$ we have $E^\sigma f(\sigma_t) = (e^{qt}f)(\sigma)$. We will be interested in the class C of all bounded functions $L_n\backslash D_n$ which vanish at Δ. This class is invariant under q and thus also under e^{qt}, since C is a closed subspace of $l_\infty(L_n\backslash D_n)$.

Now let μ be in $\mathscr{M}$, and define f on $L_n\backslash D_n$ by $f([(x_1, a_1), \cdots, (x_i, a_i)]) \equiv \mu(\eta(x_1) = a_1, \cdots, \eta(x_i) = a_i)$ and $f(\Delta) = 0$. Also, define f_t in the same manner as f, only with μ replaced by μ_t. From semigroup theory we know that $e^{qt}f$ is the unique solution in C of

$$\frac{du_t}{dt} = qu_t\,, \qquad u_0 = f\,. \tag{2.2}$$

But equations (2.1) imply that f_t is also a solution of (2.2), so f_t must equal $e^{qt}f$ $(t \geqq 0)$. Thus we have established the following result, which will be essential in our analysis:

PROPOSITION 2.1. *Fix $\mu \in \mathscr{M}$, and let f correspond to μ as in the last paragraph. Then for all $t > 0$ we have*

$$(\mu_t)(\eta(x_1) = a_1, \cdots, \eta(x_k) = a_k) = E^{\sigma_0} f(\sigma_t)$$

for all $\sigma_0 = [(x_1, a_1), \cdots, (x_k, a_k)] \in L_n\backslash D_n$.

Thus we can study the mathematically difficult η_t process indirectly, by studying the Markov chain σ_t. As we mentioned before, the span of the functions of the form

$$1^{(\cdot)}_{\{\eta(x_1)=a_1,\cdots,\eta(x_k)=a_k\}}$$

is dense in $C(K)$; thus the use of Proposition 2.1 leads to results which concern the weak-* convergence of $\mu_t \in C(K)^*$.

We conclude this section with a result which explains why we assume that $p(x, y)$ has odd period (this of course includes the case in which $p(x, y)$ is aperiodic). In the rest of this section, denote our process and its associated operators by $\mathscr{Y}_-$, Ω_- and $S_-(t)$, and define $\mathscr{Y}_+$, Ω_+ and $S_+(t)$ similarly for the Holly–Liggett model mentioned in Section 1.

PROPOSITION 2.2. *Suppose that $p(x, y)$ has even period and is irreducible. Then the $\mathscr{V}_-$ and $\mathscr{V}_+$ processes are equivalent in the sense that there exists a bijective map T on K, such that*

$$E^{\eta}f(\eta_t^-) = E^{T\eta}f(T\eta_t^+)$$

for all $t > 0$ and for all f in $C(K)$, where η_t^- and η_t^+ are realizations of the $\mathscr{V}_-$ and $\mathscr{V}_+$ processes, respectively.

PROOF. Fix $x_0 \in S$. Let $T_0 = \{y \in S: p^{(2k)}(x_0, y) > 0$ for some $k \geqq 0\}$ and $T_1 = \{y \in S: p^{(2k+1)}(x_0, y) > 0$ for some $k \geqq 0\}$. The irreducibility of $p(x, y)$ implies that $S = T_0 \cup T_1$, and furthermore we claim that $T_0 \cap T_1 = \emptyset$. To see this, suppose $y \in T_0 \cap T_1$, and let n and m be such that $p^{(n)}(y, x_0) > 0$ and $p^{(m)}(x_0, y) > 0$. Then $p^{(n+m)}(y, y) > 0$, and since $y \in T_0 \cap T_1$, we may choose m so that $n + m$ is odd, thus contradicting the even periodicity of y.

Now define $T: K \to K$ by

$$\begin{aligned} T\eta(x) &= \eta(x)\,, & x &\in T_0\,,\\ &= 1 - \eta(x)\,, & x &\in T_1\,, \end{aligned}$$

for any $x \in S$. An argument similar to that in the last paragraph shows that if $x \in T_i$ and $p(x, y) > 0$, then $y \in T_{1-i}$ ($i = 1, 2$). Thus for any f in the domain of Ω_- we have

$$\begin{aligned} \Omega_- f(\eta) &= \sum_x \sum_{y:\eta(x)=\eta(y)} p(x, y)[f(\eta_x) - f(\eta)] \\ &= \sum_x \sum_{y:T(x)\neq T\eta(y)} p(x, y)[f(\eta_x) - f(\eta)]\,. \end{aligned} \tag{2.3}$$

For any $h \in C(K)$, let $Wh(\eta) = h(T\eta)$; then $Wh \in C(K)$, since T is continuous. Noting that $T(\eta_x) = (T\eta)_x$, we have from (2.3) that

$$\begin{aligned} \Omega_- f(\eta) &= \sum_x \sum_{y:T\eta(x)\neq T\eta(y)} p(x, y)[Wf((T\eta)_x) - Wf(T\eta)] \\ &= \Omega_+ Wf(T\eta) = W\Omega_+ Wf(\eta)\,. \end{aligned} \tag{2.4}$$

As a linear operator on $C(K)$, W is positive, has norm equal to one, and is its own inverse. $WS_+(t)W$ ($t \geqq 0$) is a positive semigroup of contractions on $C(K)$, which by (2.4) has generator Ω_-. However, $S_-(t)$ is the unique semigroup having these properties, so we have $S_-(t) = WS_+(t)W$. Thns, for any f in $C(K)$ we have

$$\begin{aligned} E^{\eta}f(\eta_t^-) &= S_-(t)f(\eta) = WS_+(t)Wf(\eta) = S_+(t)Wf(T\eta) = E^{T\eta}\{Wf(\eta_t^+)] \\ &= E^{T\eta}f(T\eta_t^+)\,, \end{aligned}$$

as desired.

As an example of the last result, consider the simple random walk on $\mathbb{Z}$ defined by

$$\begin{aligned} p(x, y) &= \tfrac{1}{2}\,, & &\text{if } |x - y| = 1\,,\\ &= 0\,, & &\text{otherwisie.} \end{aligned}$$

From Holley and Ligget (1975), we know that the extremal invariant measures for the $\mathscr{V}_+$ process are the point masses on the configurations $(\cdots, 0, 0, 0, 0, \cdots)$ and $(\cdots, 1, 1, 1, 1, \cdots)$. Thus since $p(x, y)$ has period two, we know that the extremal invariant measures for the $\mathscr{V}_-$ process are the point masses on the configurations $(\cdots, 0, 1, 0, 1, \cdots)$ and $(\cdots, 1, 0, 1, 0, \cdots)$.

3. Conditions for ergodicity of the process. Let $\mathscr{P}$ be the transition matrix for a discrete time Markov chain on a countable set $\mathscr{S}$, and let $\{c_x\}_{x \in \mathscr{S}}$ be a bounded sequence of positive numbers. Define the matrix R on $\mathscr{S} \times \mathscr{S}$ by letting $R_{xy} = c_x \mathscr{P}(x, y)$ for $x \neq y$ and $R_{xx} = -c_x[1 - \mathscr{P}(x, x)]$. Then R is the infinitesimal generator of the gemigroup $\mathscr{P}(t) = e^{Rt}$ of transition operators for a continuous time Markov chain $Z(t)$ on $\mathscr{S}$. We say that a function $f: \mathscr{S} \to [0, 1]$ is harmonic for $\mathscr{P}$ (respectively $\mathscr{P}(t)$) if $\mathscr{P}f = f (\mathscr{P}(t)f = f)$ for all $t \geqq 0$. These two notions of harmonicity are actually equivalent: If $Rf = 0$, then since $\mathscr{P}(0)f = f$ and $d/dt\, [\mathscr{P}(t)f] = \mathscr{P}(t)Rf$, we have $\mathscr{P}(t)f = f$ for all $t \geqq 0$. On the other hand, if $\mathscr{P}(t)f = f$ for all $t \geqq 0$, then $0 = d/dt\, [\mathscr{P}(t)f] = \mathscr{P}(t)Rf$, so that $0 = \mathscr{P}(0)Rf$. Thus $\mathscr{P}(t)f = f$ for all $t \geqq 0$ if and only if $Rf = 0$, that is, if and only if

$$\textstyle\sum_{y \neq x} c_x \mathscr{P}(x, y)f(y) - c_x[1 - \mathscr{P}(x, x)]f(x) = 0$$

for all x in $\mathscr{S}$. But this last assertion is equivalent to $\sum_y \mathscr{P}(x, y)f(y) = f(x)$, that is, $\mathscr{P}f = f$.

LEMMA 3.1. *Let $f: \mathscr{S} \to [0, 1]$, and suppose that for some $t_i \to \infty$ we have that*

$$k(x) = \lim_{i \to \infty} \mathscr{P}_{t_i} f(x)$$

exists for each $x \in \mathscr{S}$. Then k is harmonic.

PROOF. See Appendix.

This lemma will now be used to prove a related result for the η_t process. It will be clear from the proof that the same result is true for other processes for which some suitable version of Proposition 2.1 holds.

COROLLARY 3.2. *Suppose μ is an initial distribution for the η_t process and $\mu_{t_i} \to \nu$ for some $t_i \to \infty$. Then $\nu \in \mathscr{I}$, the set of invariant measures for the process.*

PROOF. Both here and in the future, we used the notation $f \leftrightarrow \mu$ to mean that the function f corresponds to the measure μ in the following way: For any $[(x_1, a_1), \cdots, (x_k, a_k)] \in L_n$ we have

$$f((x_1, a_1), \cdots, (x_k, a_k)) = \mu(\eta(x_1) = a_1, \cdots, \eta(x_k) = a_k)\,, \qquad \text{and} \qquad f(\Delta) = 0\,.$$

If we let $U_t = e^{qt}$ denote the semigroup for the process σ_t, then by Proposition 2.1 we have $U_t f \leftrightarrow \mu_t\ \forall t \geqq 0$. Thus our hypothesis implies that $U_{t_i} f$ converges to some $\bar{f}$ as $i \to \infty$, and that $\bar{f} \leftrightarrow \nu$. By Lemma 3.1 we have for all $t \geqq 0$, $U_t \bar{f} = \bar{f}$ on each $L_n \backslash D_n$, so that $\nu_t = \nu$ from Proposition 2.1 again.

REMARK 3.3. As is well known, $l_1(S)$ is separable, so that the weak-* closed unit ball of $l_\infty(S)$ is metrizable and thus sequentially compact, by Alaoglu's theorem; a similar statement can be made regarding $C(K)$ and $C(K)^*$. These facts will often be used in conjunction with Lemma 3.1 and Corollary 3.2.

In our case, we will be interested in three classes of harmonic functions. First, let $\mathscr{H}'$ be the set of all functions which are harmonic for $p(x, y)$, and define $\mathscr{H}$ similarly for $Q(\alpha, \beta)$ (see Section 2). Note that $\mathscr{H}'$ is naturally embedded in $\mathscr{H}$, in the sense that if $k \in \mathscr{H}'$, then the function K defined by $K(x, a) \equiv k(x)$ is in $\mathscr{H}$. If any function in $\mathscr{H}$ can be represented in this way, we will say that $\mathscr{H}' = \mathscr{H}$.

We also define the set $\mathscr{H}_0 = \{f \in \mathscr{H}: f(x, 0) + f(x, 1) \equiv 1\}$; our main interest will be in this set (a theorem which gives a more intuitive characterization of $\mathscr{H}_0$ may be found in Matloff (1975)).

In order the state the next lemma, we now define U_t and V_t to the semigroups corresponding to the Markov chains σ_t and $\boldsymbol{\alpha}(t)$ in Section 2 (thus, for instance, $U_t f(\sigma) = E^\sigma f(\sigma_t)$ for any bounded function f on $L_n \backslash D_n$). Also, if $\boldsymbol{\alpha}(t)$ consists only of one component $\alpha(t)$, we write $\alpha(t) = (X(t), a(t))$.

Lemma 3.4. *The following conditions are equivalent*:

(a) *$\mathscr{H}_0$ consists only of the constant function $\frac{1}{2}$.*

(b) $\mathscr{H}' = \mathscr{H}$.

(c) $\lim_{t\to\infty} \sum_y |P^{(x,a)}(\alpha(t) = (y, a)) - P^{(x,a)}(\alpha(t) = (y, 1-a))| = 0$ *for each* $x \in S$.

(d) *If $\mu \in \mathscr{M}$ and $f \leftrightarrow \mu$, then*

$$\lim_{t\to\infty} V_t f(\alpha_1, \cdots, \alpha_n) = (\tfrac{1}{2})^n$$

on S_1^n, for each $n \geqq 1$.

Remark 3.5. If the absolute value signs are removed from the expression in (c), the resulting sum becomes

$$P^\alpha(a(t) = a) - P^\alpha(a(t) = 1 - a)\,,$$

which converges to 0 as $t \to \infty$ (to see this, note that $a(t)$ can be considered to be a continuous time Markov chain on $\{0, 1\}$ with a stationary distribution which is uniform on that set). However, the absolute value signs make condition (c) less trivial. Roughly, (c) says that our particle $X(t)$ is about equally likely to reach y after an even number of jumps as it is after an odd number of jumps.

Proof of Lemma 3.4. Condition (b) says that for each $f \in \mathscr{H}$ we have $f(x, 0) \equiv f(x, 1)$, and thus (b) immediately implies (a).

Next we show that (a) implies (c). The latter assertion may be rewritten as

$$\textstyle\sum_y |P^{(x,0)}(\alpha(t) = (y, 0)) - P^{(x,1)}(\alpha(t) = (y, 0))| \to 0$$

as $t \to \infty$, for each $x \in S$. It suffices to show that for any sequence of times tending to ∞, there exists a subsequence along which the desired limit occurs. To avoid cumbersome notation, we will work with a single sequence $\{t_i\}$, and will not relabel when we extract subsequences.

Take any f: $S_1 \to [0, 1]$ such that $f(x, 0) + f(x, 1) \equiv 1$. By Remerk 3.3 there is a subsequence of $\{t_i\}$ and an $\bar{f} \in \mathscr{H}$ (see Lemma 3.1) such that $U_{t_i} f \to \bar{f}$. Furthermore,

$$\begin{aligned} U_{t_i} f(x, 0) + U_{t_i} f(x, 1) &= E^{(x,0)} f(\alpha(t_i)) + E^{(x,1)} f(\alpha(t_i)) \\ &= E^{(x,0)}[f(\alpha(t_i)) + f(\alpha(t_i)^\#)] = 1\,, \end{aligned}$$

$$\text{where} \quad (z, c)^\# \quad \text{means} \quad (z, 1 - c)\,.$$

Thus $\bar{f} \in \mathscr{H}_0$, so that $\bar{f} \equiv \frac{1}{2}$. This says that

$\frac{1}{2} = \lim_i \sum_y \{P^{(x,0)}[\alpha(t_i) = (y, 0)]f(y, 0) + P^{(x,0)}[\alpha(t_i) = (y, 1)]f(y, 1)\}$, and
$\frac{1}{2} = \lim_i \sum_y \{P^{(x,1)}[\alpha(t_i) = (y, 0)]f(y, 0) + P^{(x,1)}[\alpha(t_i) = (y, 1)]f(y, 1)\}$.

Thus after equating the right-hand sides of the last two equations and extracting a further subsequence we get

$$\begin{aligned}
&\lim_i \textstyle\sum_y \{P^{(x,0)}[\alpha(t_i) = (y, 0)] - P^{(x,1)}[\alpha(t_i) = (y, 0)]\}f(y, 0) \\
&\quad = \lim_i \textstyle\sum_y \{P^{(x,1)}[\alpha(t_i) = (y, 1)] - P^{(x,0)}[\alpha(t_i) = (y, 1)]\}f(y, 1) \\
&\quad = \lim_i \textstyle\sum_y \{P^{(x,0)}[\alpha(t_i) = (y, 0)] - P^{(x,1)}[\alpha(t_i) = (y, 0)]\}f(y, 1) .
\end{aligned} \tag{3.1}$$

By equating the first and last terms of (3.1), and recalling that $f(y, 1) = 1 - f(y, 0)$, we have

$$\begin{aligned}
&2 \lim_i \textstyle\sum_y \{P^{(x,0)}[\alpha(t_i) = (y, 0)] - P^{(x,1)}[\alpha(t_i) = (y, 0)]\}f(y, 0) \\
&\quad = \lim_i \textstyle\sum_y \{P^{(x,0)}[\alpha(t_i) = (y, 0)] - P^{(x,1)}[\alpha(t_i) = (y, 0)]\} \\
&\quad = \tfrac{1}{2} - \tfrac{1}{2} = 0 .
\end{aligned}$$

Thus if we fix x, the sequence of functions

$$\{P^{(x,0)}[\alpha(t_i) = (y, 0)] - P^{(x,1)}[\alpha(t_i) = (y, 0)]\}$$

converges weakly in $l_1(S)$ and thus also strongly, since weak and strong convergence of sequences are equivalent in that space. Thus

$$\lim_i \textstyle\sum_y |P^{(x,0)}[\alpha(t_i) = (y, 0)] - P^{(x,1)}[\alpha(t_i) = (y, 0)]| = 0 ,$$

as desired.

To show that (c) $\Rightarrow$ (d), let $f \leftrightarrow \mu$ and write $V_t f(\alpha_1, \cdots, \alpha_n)$ as

$$\begin{aligned}
&\textstyle\sum_{\mathbf{z} \in S^n} \sum_{\mathbf{c} \in \{0,1\}^n} P^{\alpha}(\boldsymbol{\alpha}(t) \\
&\quad = [(z_1, c_1), \cdots, (z_n, c_n)])f[(z_1, c_1), \cdots, (z_n, c_n)] \\
&\quad = \textstyle\sum_{\mathbf{z}} \sum_{\mathbf{c} \neq (1,\cdots,1)} \{P^{\alpha}(\boldsymbol{\alpha}(t) = [(z_1, c_1), \cdots, (z_n, c_n)]) \\
&\qquad - P^{\alpha}(\boldsymbol{\alpha}(t) = [(z_1, 1), \cdots, (z_n, 1)])\}f[(z_1, c_1), \cdots, (z_n, c_n)] \\
&\qquad + \textstyle\sum_{\mathbf{z}} P^{\alpha}(\boldsymbol{\alpha}(t) = [(z_1, 1), \cdots, (z_n, 1)]) ,
\end{aligned} \tag{3.2}$$

since

$$f((z_1, 1), \cdots, (z_n, 1)) = 1 - \textstyle\sum_{\mathbf{c} \neq (1,\cdots,1)} f[(z_1, c_1), \cdots, (z_n, c_n)] .$$

Now

$$\begin{aligned}
\textstyle\sum_{\mathbf{z}} P^{\alpha}(\boldsymbol{\alpha}(t) = [(z_1, 1), \cdots, (z_n, 1)]) &= P^{\alpha}(a_1(t) = 1, \cdots, a_n(t) = 1) \\
&= \textstyle\prod_{i=1}^n P^{\alpha_i}(a_i(t) = 1) ,
\end{aligned}$$

where $\alpha_i(t) = (X_i(t), a_i(t))$. Thus by Remark 3.5, the last term in (3.2) converges to $(\frac{1}{2})^n$ as $t \to \infty$.

Now rewrite the first term on the right-hand side of (3.2) as

$$\textstyle\sum_{\mathbf{z}} \sum_{\mathbf{c} \neq \mathbf{1}} [r(\mathbf{z}, \mathbf{c}) - r(\mathbf{z}, \mathbf{1})]f((z_1, c_1), \cdots, (z_n, c_n)) .$$

Since the range of values of $\mathbf{c}$ is finite, we will be done if we show that

$$\lim_{t\to\infty} \sum_{\mathbf{z}} |(r(\mathbf{z}, \mathbf{c}) - r(\mathbf{z}, \mathbf{1})| = 0 .$$

However, by adding and subtracting at most $2^n - 2$ terms within the absolute value signs in the last expression, we see that we only need to show that

$$\sum_{\mathbf{z}} |(r(\mathbf{z}, \mathbf{d}^{(1)}) - r(\mathbf{z}, \mathbf{d}^{(2)})| \to 0 ,$$

where $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$ differ in only one coordinate, say the first. But then

$$\begin{aligned}
\sum_{\mathbf{z}} &|r(\mathbf{z}, \mathbf{d}^{(1)}) - r(\mathbf{z}, \mathbf{d}^{(2)})| \\
&= \sum_{\mathbf{z}} |P^{\boldsymbol{\alpha}}(\boldsymbol{\alpha}(t) = [(z_1, 0), (z_2, d_2), \cdots, (z_n, d_n)]) \\
&\qquad - P^{\boldsymbol{\alpha}}(\boldsymbol{\alpha}(t) = [(z_1, 1), (z_2, d_2), \cdots, (z_n, d_n)])| \\
&= \sum_{\mathbf{z}} |P^{\alpha_1}(\alpha_1(t) = (z_1, 0)) - P^{\alpha_1}(\alpha_1(t) = (z_1, 1))| \\
&\qquad \times P^{(\alpha_2, \cdots, \alpha_n)}([\alpha_2(t), \cdots, \alpha_n(t)] = [(z_2, d_2), \cdots, (z_n, d_n)]) \\
&= (\textstyle\sum_{z_1} |P^{\alpha_1}(\alpha_1(t) = (z_1, 0)) - P^{\alpha_1}(\alpha_1(t) = (z_1, 1))|) \\
&\qquad \times P^{(\alpha_2, \cdots, \alpha_n)}([a_2(t), \cdots, a_n(t)] = (d_2, \cdots, d_n)) \to 0
\end{aligned}$$

as $t \to \infty$, by (c).

Finally, suppose (d) holds. Take any $f_0 \in \mathscr{H}_0$, and let μ be the product measure such that $f \leftrightarrow \mu$ for an extension of f_0 to $\bigcup_1^\infty S_1^n$.

Then $f_0 = V_t f_0 \to \frac{1}{2}$, by (d), so that $\mathscr{H}_0 = \{\frac{1}{2}\}$. To show that this implies $\mathscr{H}' = \mathscr{H}$, take any $f \in \mathscr{H}$. Define $\tilde{f}$ on S_1 by

$$\begin{aligned}
\tilde{f}(x, a) &= \tfrac{1}{2}[1 + f(x, 0) - f(x, 1)] , \quad a = 0 , \\
&= \tfrac{1}{2}[1 + f(x, 1) - f(x, 0)] , \quad a = 1 .
\end{aligned}$$

Then $0 \leqq \tilde{f} \leqq 1$, and

$$\begin{aligned}
(Q\tilde{f})(x, 0) &= \textstyle\sum_y p(x, y)\tilde{f}(y, 1) = \frac{1}{2} \sum_y p(x, y)[1 + f(y, 1) - f(y, 0)] \\
&= \tfrac{1}{2}[1 + (Qf)(x, 0) - (Qf)(x, 1)] \\
&= \tfrac{1}{2}[1 + f(x, 0) - f(x, 1)] = \tilde{f}(x, 0) ,
\end{aligned}$$

and similarly $(Q\tilde{f})(x, 1) = \tilde{f}(x, 1)$. From this and the fact that $\tilde{f}(x, 0) + \tilde{f}(x, 1) \equiv 1$, we have $\tilde{f} \in \mathscr{H}_0$ and thus $\tilde{f} \equiv \frac{1}{2}$. Hence $f(x, 0) \equiv f(x, 1)$ and $\mathscr{H}' = \mathscr{H}$.

In order to state our main ergodicity theorem, we introduce the following notation: Consider the noninteracting particle system $\boldsymbol{\alpha}(t) = [(X_1(t), a_1(t)), \cdots, (X_n(t), a_n(t))]$ in Section 2 and Lemma 3.4. For each $\boldsymbol{\alpha} \in S_1^n$, let $g(\boldsymbol{\alpha}) = P^{\boldsymbol{\alpha}}(X_i(t) = X_j(t)$ for some $i \neq j$ and some $t > 0)$. $g(\boldsymbol{\alpha})$ may in a sense be thought of as a measure of distances between voters in S, and is similar to functions used in Liggett (1973 and 1974a), Spitzer (1974), and Holley and Liggett (1975).

THEOREM 3.6. *The η_t process has a unique invariant measure ν if and only if $\mathscr{H}_0 = \{\frac{1}{2}\}$, and in this case each of the following holds*:

(a) *$\mu_t \to \nu$ as $t \to \infty$ for every initial distribution μ.*
(b) $\nu(\eta(x) = a) \equiv \frac{1}{2}$.

(c) $|\nu(\eta(x_1) = a_1, \cdots, \eta(x_n) = a_n) - (\frac{1}{2})^n| \leqq g(\alpha_1, \cdots, \alpha_n)$ *for all* $[\alpha_1, \cdots, \alpha] = [(x_1, a_1), \cdots, (x_n, a_n)] \in S_1^n \backslash D_n$.

PROOF. We first show that $\mathscr{I} = \{\nu\}$ implies (a) and (b). To establish (a), let μ be any initial distribution for the process, and recall that it suffices to show that if $t_i \to \infty$ and $\lim_i \mu_{t_i}$ exists, then the limit must be ν (see Remark 3.3). But this is immediate from the fact that $\mathscr{I} = \{\nu\}$ and Corollary 3.2.

(b) is also an easy consequence of $\mathscr{I} = \{\nu\}$, since the measure $\tilde{\nu}$ defined by $\tilde{\nu}(\eta(x_1) = a_1, \cdots, \eta(x_n) = a_n) \equiv \nu(\eta(x_1) = 1 - a_1, \cdots, \eta(x_n) = 1 - a_n)$ must be invariant (from symmetry considerations). Then $\tilde{\nu} = \nu$, so that $\nu(\eta(x) = 1 - a) = \nu(\eta(x) = a)$. The last two quantities sum to one, and thus they each must be $\frac{1}{2}$.

Now continue to assume $\mathscr{I} = \{\nu\}$, and take any $f_0 \in \mathscr{H}_0$. Let μ be the product measure which corresponds to an extension f of f_0, as in the proof of Lemma 3.4. Let $\bar{f} \leftrightarrow \nu$, and note that by (a) and Proposition 2.1, we have $\lim_{t\to\infty} U_t f = \bar{f}$ on $L_n \backslash D_n$ $(n \geqq 1)$. By (b), $\bar{f} = \frac{1}{2}$ on S_1, and since $f_0 \in \mathscr{H}_0$ we also have $U_t f = f$ on S_1. Thus we have proved that if $\mathscr{I} = \{\nu\}$, then $\mathscr{H}_0 = \{\frac{1}{2}\}$. Now by applying parts (a) and (d) of Lemma 3.4, along with the fact that $|U_t f - V_t f| \leqq g$ on $L_n \backslash D_n$, we have (c).

Finally, we will show that if $\mathscr{H}_0 = \{\frac{1}{2}\}$, then $\mathscr{I} = \{\nu\}$. Suppose that ν_1 and ν_2 are invariant measures for the process, and let $f_i \leftrightarrow \nu_i$ $(i = 1, 2)$. Then both f_1 and f_2 are fixed by U_t, and thus their restrictions to S_1 are in $\mathscr{H}_0$. Hence, since we are assuming $\mathscr{H}_0 = \{\frac{1}{2}\}$, we see that f_1 and f_2 agree on S_1. Now suppose that $f_1 = f_2$ on S_1^k, for all $k < n$ $(n \geqq 2)$, and fix $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) \in S_1^n \backslash D_n$. Then

$$\begin{aligned} f_1(\boldsymbol{\alpha}) - f_2(\boldsymbol{\alpha}) &= U_t f_1(\boldsymbol{\alpha}) - U_t f_2(\boldsymbol{\alpha}) = E^{\alpha}[f_1(\sigma_t) - f_2(\sigma_t)] \\ &= E^{\alpha}[f_1(\sigma_t) - f_2(\sigma_t), t < \tau], \end{aligned}$$

by the induction hypothesis; here τ is the hitting time of $\{\Delta\} \cup S_1^{n-1}$ for σ_t. Thus

$$f_1(\boldsymbol{\alpha}) - f_2(\boldsymbol{\alpha}) = E^{\alpha}[f_1(\sigma_t) - f_2(\sigma_t), t < \tau < \infty] + E^{\alpha}[f_1(\sigma_t) - f_2(\sigma_t), \tau = \infty].$$

The first term vanishes as $t \to \infty$, and the second term is equal to $E^{\alpha}[f_1(\boldsymbol{\alpha}(t)) - f_2(\boldsymbol{\alpha}(t)), \tau = \infty]$, where $\boldsymbol{\alpha}(t)$ moves according to V_t. Now rewrite the last expression as

$$\begin{aligned} V_t f_1(\boldsymbol{\alpha}) - V_t f_2(\boldsymbol{\alpha}) &- E^{\alpha}[f_1(\boldsymbol{\alpha}(t)) - f_2(\boldsymbol{\alpha}(t)), \tau < \infty] \\ &= V_t f_1(\boldsymbol{\alpha}) - V_t f_2(\boldsymbol{\alpha}) - E^{\alpha}[f_1(\boldsymbol{\alpha}(t)) - f_2(\boldsymbol{\alpha}(t)), t \geqq \tau] \\ &\quad - E^{\alpha}[f_1(\boldsymbol{\alpha}(t)) - f_2(\boldsymbol{\alpha}(t)), t < \tau < \infty]. \end{aligned}$$

Again the last term converges to zero as $t \to \infty$, and from Lemma 3.4(d) we have $\lim_{t\to\infty} V_t f_i(\boldsymbol{\alpha}) = (\frac{1}{2})^n$ $(i = 1, 2)$. Thus we will be done if we show that

$$\lim_{t\to\infty} E^{\alpha}[f_1(\boldsymbol{\alpha}(t)) - f_2(\boldsymbol{\alpha}(t)), t \geqq \tau] = 0.$$

But

$$\begin{aligned} &E^{\alpha}[f_1(\boldsymbol{\alpha}(t)) - f_2(\boldsymbol{\alpha}(t)), t \geqq \tau] \\ &\quad = \textstyle\sum_{\boldsymbol{\beta} \in S_1^n} \int_0^t P^{\alpha}(\tau \in ds, \boldsymbol{\alpha}(\tau) = \boldsymbol{\beta}) \cdot [V_{t-s} f_1(\boldsymbol{\beta}) - V_{t-s} f_2(\boldsymbol{\beta})], \end{aligned}$$

which converges to zero by bounded convergence and Lemma 3.4(d).

Thə following lemma will be used in the proof Proposition 3.8, and is true without the assumption that $\mathscr{H}_0 = \{\frac{1}{2}\}$.

LEMMA 3.7. *Q is irreducible.*

PROOF. Take any $\alpha = (x, a)$ and $\beta = (y, b)$ in S_1. By the irreducibility and odd periodicity of p, there exist nonnegative integers k and m such that both $p^{(k)}(x, y)$ and $p^{(2m+1)}(x, x)$ are positive. Thus $p^{(2n_1+1)}(x, y)$ and $p^{(2n_2)}(x, y)$ are positive for some n_1 and n_2 so that α leads to β in S_1.

PROPOSITION 3.8. *$\mathscr{H}_0 = \{\frac{1}{2}\}$ under any of the following conditions:*

(a) *S is an Abelian group, and $p(x, y) \equiv p(0, y - x)$.*
(b) *$p(x, y)$ is recurrent.*
(c) *The space-time process for $p(x, y)$ has no nonconstant bounded harmonic functions.*

PROOF. (a) $S_1 = S \times \mathbb{Z}_2$ is an Abelian group and it is easily verified that $Q(\alpha, \beta) \equiv Q(0, \beta - \alpha)$. Since Q is irreducible, a well-known result of Choquet and Deny (1960) implies that $\mathscr{H}$ consists only of constants, and thus $\mathscr{H}_0 = \{\frac{1}{2}\}$, by Lemma 3.4(b).

(b) Another well-known condition which implies that $\mathscr{H}$ consists only of constants is that Q is recurrent. To see why the recurrence suffices, take $f \in \mathscr{H}$ and let $\alpha_n = (X_n, a_n)$ move on S_1 according to Q. The harmonicity of f implies that $f(\alpha) = E^\alpha f(\alpha_n)$ identically in α and n. Now fix α and β in S_1 and let T_α be the hitting time of α for α_n. We then have

$$f(\beta) = E^\beta f(\alpha_n) = \textstyle\sum_{m=1}^n E^\beta[f(\alpha_n) \mid T_\alpha = m]P^\beta(T_\alpha = m) + a \text{ term bounded by } P^\beta(T_\alpha > n)\,.$$

Now the strong Markov property implies that $E^\beta[f(\alpha_n) \mid T_\alpha = m] = E^\alpha f(\alpha_{n-m}) = f(\alpha)$ for $m \leqq n$. Thus the recurrence of Q would imply that the right-hand side of the above equation for $f(\beta)$ converges to $f(\alpha)$ as $n \to \infty$, so that f is constant.

Now to establish the recurrence of Q, note that we need to show that $P^x(X_{2k} = x \text{ for some } k > 0) \equiv 1$. Let X_n start at x, and let $T_1, T_1 + T_2, \cdots$ be the successive return times to x (they are finite a.s., by hypothesis). Then $T_1, T_2, \cdots$ are i.i.d., and since $P(T_i \text{ is even}) < 1$ (p is of odd period) we have $P(T_1 \text{ odd}, T_n \text{ even for all } n \geqq 2) = 0$. Thus with probability one, at least one sum $T_1 + \cdots + T_n$ is even, which is the desired result.

(c) Take any $f \in \mathscr{H}$. Define $k: \{0, 1, 2, \cdots\} \times S \to [0, 1]$ by $k(m, x) = f(x, a(m))$ where $a(m) \in \{0, 1\}$ and $a(m) \equiv m \pmod 2$. Then since $f \in \mathscr{H}$, k is harmonic for the space-time process for $p(x, y)$. Thus k is constant, so that $\mathscr{H}$ consists only of constants and $\mathscr{H}_0 = \{\frac{1}{2}\}$.

THEOREM 3.9. *Suppose $S = \mathbb{Z}^d$ and $p(x, y) \equiv p(o, y - x)$. Then the η_t process has a unique invariant measure ν. Moreover, ν is translation invariant, and it has the following asymptotic independence property:*

Let

$$A_1 = \{\eta(x_1) = a_1, \cdots, \eta(x_k) = a_k\}, \qquad A_2 = \{\eta(y_1) = b_1, \cdots, \eta(y_l) = b_l\}$$

and

$$A_2 + nz = \{\eta(y_1 + nz) = b_1, \cdots, \eta(y_l + nz) = b_l\},$$

where the sites $x_1, \cdots, x_k\, y_1, \cdots, y_l$ *are distinct, and where* z *is nonzero. Then*

$$\lim_{n\to\infty} \nu(A_1 \cap (A_2 + nz)) = \nu(A_1)\nu(A_2).$$

PROOF. The unicity of ν is an immediate consequence of Theorem 3.6 and Proposition 3.8. To see that ν is translation invariant, let $\mathscr{J}$ be the set of all translation invariant measure in $\mathscr{M}$, and take any $\mu \in \mathscr{J}$. The translation invariance of p implies that $\mu_t \in \mathscr{J}$ for all $t \geqq 0$, and since $\mathscr{J}$ is weak-* closed, part (a) of Theorem 3.6 tells us that $\nu = \lim_{t\to\infty} \mu_t$ must also be in $\mathscr{J}$.

Now in order to prove the ergodicity of ν, first note that because of the irreducibility of p, either g is identically less than one on $S_1^2\backslash D_2$ or it is identically equal to one. The ergodicity proof in the former case uses part (c) of Theorem 3.6, is identical to the proof of Theorem 5.8(c) of Holley and Liggett (1975). We omit this proof and instead concentrate on the case in which $g(\alpha, \beta) \equiv 1$.

First suppose $k = l = 1$. Let $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and σ_t move independently, each according to U_t, with initial values (x_1, a_1), $(x_1, 1 - a_1)$, $(y_1 + nz, b_1)$ and $[(x_1, a_1), (y_1 + nz, b_1)]$, respectively. Let τ_1 and τ_2 be the hitting times of D_2 for $[\alpha(t), \gamma(t)]$ and $[\beta(t), \gamma(t)]$. Noting that $\alpha(t)$ and $\beta(t)$ initially have the same "position" in S, but are initially of the opposite "color," we let τ_3 be the first time that these two particles have a same-color collision; that is, we let τ_3 be the hitting time of the diagonal of S_1^2 for the process $[\alpha(t), \beta(t)]$. τ_1 and τ_2 have the same distribution, and by assumption they are finite a.s. τ_3 is also finite a.s., by an argument similar to the proof of Proposition 3.8(b). Also, note that $\lim_{n\to\infty} P(\tau_1 < \tau_3) = 0$.

Let $f \leftrightarrow \nu$, and recall that $f(\Delta) = 0$ and $f(\alpha) \equiv \frac{1}{2}$ on S_1. Thus since $g(\alpha, \beta) \equiv 1$, $\lim_{t\to\infty} f(\sigma_t)$ exists and is equal to either zero or $\frac{1}{2}$. Thus

$$\begin{aligned}\nu(A_1 \cap (A_2 + nz)) &= f(\sigma_0) = U_t f(\sigma_0) = \lim_{t\to\infty} U_t f(\sigma_0) \\ &= E \lim_{t\to\infty} f(\sigma_t) = \tfrac{1}{2} P(\alpha(\tau_1) = \gamma(\tau_1)),\end{aligned}$$

so that we need to show that $\lim_{n\to\infty} P(\alpha(\tau_1) = \gamma(\tau_1)) = \frac{1}{2}$. But the strong Markov property implies that

$$P(\alpha(\tau_1) = \gamma(\tau_1), \tau_3 \leqq \tau_1) = P(\beta(\tau_2) = \gamma(\tau_2), \tau_3 \leqq \tau_2)$$

so that

$$|P(\alpha(\tau_1) = \gamma(\tau_1)) - P(\beta(\tau_2) = \gamma(\tau_2))| \leqq P(\tau_1 < \tau_3). \tag{3.3}$$

Now by symmetry considerations and the fact that $\beta(0)^{\#} = \alpha(0)$, we have

$$P(\beta(\tau_2) = \gamma(\tau_2)) = 1 - P(\beta(\tau_2)^{\#} = \gamma(\tau_2)) = 1 - P(\alpha(\tau_1) = \gamma(\tau_1)).$$

Thus since the right-hand side of (3.3) converges to zero as $n \to \infty$, we see that $\lim_{n\to\infty} P(\alpha(\tau_1) = \gamma(\tau_1)) = \frac{1}{2}$, as desired.

Now to consider the general case, let σ_t, $\sigma_t^{(1)}$ and $\sigma_t^{(2)}$ each move according to U_t, starting at

$$[(x_1, a_1), \cdots, (x_k, a_k), (y_1 + nz, b_1), \cdots, (y_l + nz, b_l)], [(x_1, a_1, \cdots, (x_k, a_k)]$$

and $[(y_1 + nz, \cdots, y_l + nz)]$, respectively. We can construct our probability space so that $\sigma_t^{(1)}$ and $\sigma_t^{(2)}$ are independent, and such that $\sigma_t = (\sigma^{(1)}, \sigma_t^{(2)})$ until time τ_4, the time of the first collision between a particle in $\sigma_t^{(1)}$ and one in $\sigma_t^{(2)}$. Now using the notation $|\sigma| = i$ to mean $\sigma \in S_1^i \backslash D_i$ (with $|\sigma| = 0$ if $\sigma = \Delta$) define the following events:

$$E_1 = \{\exists\, c \text{ such that } |\sigma_t| = 1 \text{ for all } t \geqq c\}\,,$$
$$E_2 = \{|\sigma_{t_0}^{(1)}| \leqq 1 \text{ and } |\sigma_{t_0}^{(2)}| \leqq 1\}\,,$$
$$E_3 = \{|\sigma_{t_0}^{(1)}| = 1 \text{ and } |\sigma_{t_0}^{(2)}| = 1\}\,,$$
$$E_4 = \{t_0 < \tau_4\}\,,$$
$$E_5 = \{\text{no particle in } \sigma_{t_0}^{(1)} \text{ is within } M \text{ units of a particle in } \sigma_{t_0}^{(2)}\}\,, \quad \text{and}$$
$$E_6 = \{\exists\, t \geqq 0 \text{ such that } |\sigma_t^{(1)}| = |\sigma_t^{(2)}| = 1\}\,,$$

where t_0 and M will be determined below; the distance function which we are using in E_5 is the Euclidean metric on $S = \mathbb{Z}^d$.

Fix any $\varepsilon > 0$. Since $g(\alpha, \beta) \equiv 1$, we have $\lim_{t_0 \to \infty} P(E_2) = 1$. Also $\lim_{t_0 \to \infty} P(E_3) = (E_6)$; thus we may choose t_0 such that

$$P(E_2) \geqq 1 - \frac{\varepsilon}{3} \qquad \text{and} \qquad |P(E_6) - P(E_3)| \leqq \frac{\varepsilon}{3}\,,$$

independently of n. Let $\boldsymbol{\delta}(t) = [\delta_1(t), \delta_2(t)]$ move on S_1^2 according to V_t, and let τ_5 be the hitting time of D_2 for $\boldsymbol{\delta}(t)$. Then by the translation invariance of $p(x, y)$ and the results in the first part of this proof, there exists a number M such that if the projections of $\delta_1(0)$ and $\delta_2(0)$ on S are at least M units apart, then $P(\delta_1(\tau_5) = \delta_2(\tau_5))$ is in $[\frac{1}{2} - \varepsilon/3, \frac{1}{2} + \varepsilon/3]$. Now choose N such that for all $n \geqq N$ we have $P(E_4) \geqq 1 - \varepsilon/3$ and $P(E_5) \geqq 1 - \varepsilon/3$. Let $E_{i_1, \cdots, i_k}$ denote $E_{i_1} \cap \cdots \cap E_{i_k}$. Then

$$\nu(A_1 \cap (A_2 + nz)) = \tfrac{1}{2}P(E_1) = \tfrac{1}{2}[P(E_{1,2,4,5}) + e_1]\,,$$

where $e_1 \leqq 3(\varepsilon/3) = \varepsilon$. Now since $\sigma_t = (\sigma_t^{(1)}, \sigma_t^{(2)})$ until time τ_4, we have $P(E_{1,2,4,5}) = P(E_{1,3,4,5}) = EW$, where $W = E(1_{E_{1,3,4,5}} \mid 1_{E_{3,4,5}})$. On the complement of $E_{3,4,5}$, W is zero almost surely, while on $E_{3,4,5}$ we have $W \in [\frac{1}{2} - \varepsilon/3, \frac{1}{2} + \varepsilon/3]$ a.s. Thus $P(E_{1,2,4,5})$ is within $(\varepsilon/3)P(E_{3,4,5})$ of $\frac{1}{2}P(E_{3,4,5})$. From all of these facts we get

$$\begin{aligned}
&|\nu(A_1 \cap (A_2 + nz)) - \nu(A_1)\nu(A_2)| \\
&\qquad = |\tfrac{1}{2}P(E_1) - \tfrac{1}{4}P(E_6)| \\
&\qquad \leqq \tfrac{1}{2}[|P(E_1) - P(E_{1,2,4,5})| + |P(E_{1,2,4,5}) - \tfrac{1}{2}P(E_{3,4,5})| + |\tfrac{1}{2}P(E_{3,4,5,}) - \tfrac{1}{2}P(E_6)|] \\
&\qquad \leqq \tfrac{1}{2}\left[\varepsilon + \frac{\varepsilon}{3}P(E_{3,4,5}) + \tfrac{1}{2}\left(3 \cdot \frac{\varepsilon}{3}\right)\right] < \varepsilon
\end{aligned}$$

for all $n \geqq N$, which completes the proof.

4. Appendix.

PROOF OF LEMMA 3.1. We first prove the result for the case in which $c_x \equiv 1$. By bounded convergence we have

$$\begin{aligned} \mathscr{P}k(x) - k(x) &= \textstyle\sum_y \mathscr{P}(x, y) \lim_i \mathscr{P}(t_i)f(y) - \lim_i \mathscr{P}(t_i)f(x) \\ &= \lim_i \textstyle\sum_y \mathscr{P}(x, y)[\mathscr{P}(t_i)f(y) - \mathscr{P}(t_i)f(x)] \\ &= \lim_i \textstyle\sum_y \mathscr{P}(x, y)\Big[\sum_{n=0}^{\infty} \frac{e^{-t_i}t_i^n}{n!} \sum_z \mathscr{P}^{(n)}(y, z)f(z) \\ &\qquad - \textstyle\sum_{n=0}^{\infty} \frac{e^{-t_i}t_i^n}{n!} \sum_z \mathscr{P}^{(n)}(x, z)f(z)\Big]. \end{aligned} \tag{4.1}$$

Since the terms involved are nonnegative, we can write the last expression as

$$\begin{aligned} &\lim_i \textstyle\sum_{n=0}^{\infty} \left\{\frac{e^{-t_i}t_i^n}{n!} \sum_z [\mathscr{P}^{(n+1)}(x, z) - \mathscr{P}^{(n)}(x, z)]f(z)\right\} \\ &\quad = \lim_i \textstyle\sum_{n=0}^{\infty} W(n, x, i) \\ &\quad = \lim_i [\textstyle\sum_{n=0}^{N_1-1} W(n, x, i) + \sum_{n=N_1}^{N_2} W(n, x, i) + \sum_{n=N_2+1}^{\infty} W(n, x, i)], \end{aligned} \tag{4.2}$$

where N_1 and N_2 will be determined below.

Fix $\varepsilon > 0$ and $x \in \mathscr{S}$, and let $N(t)$ be a Poisson process with intensity one. Since $N(t)/t \to 1$ as., we also have convergence in probability. Also,

$$\lim_{t\to\infty} \frac{e^{-t}t^n}{n!} = 0, \tag{4.3}$$

uniformly in n. To see this, write

$$\frac{e^{-t}t^{k+1}}{(k+1)!} - \frac{e^{-t}t^k}{k!} \quad \text{as} \quad \frac{e^{-t}t^k}{(k+1)!}[t - (k+1)],$$

and conclude that the maximum value $e^{-t}t^n/n!$ for fixed t occurs at $n = [t] - 1$; then use Stirling's approximation to show that this maximum value goes to zero as $t \to \infty$. In view of these facts we can find I such that for all $i \geqq I$ we have

$$\frac{e^{-t_i}t_i^n}{n!} < \varepsilon \tag{4.4}$$

for all n, and

$$P\left(\left|\frac{N(t_i)}{t_t} - 1\right| > \varepsilon\right) < \varepsilon; \tag{4.5}$$

Take any $i \geqq I$ and let N_1 and N_2 be the extreme elements of the set of all integers n such that $|(n/t_i) - 1| \leqq \varepsilon$ (we may choose I large enough so that this set is nonempty). Then (4.5) implies that

$$\textstyle\sum_{n \notin [N_1, N_2]} \frac{e^{-t_i}t_i^n}{n!} < \varepsilon, \tag{4.6}$$

so that the sum of the first and last terms in (4.2) is less than ε in absolute value;

here we have used the fact that a sum like $\sum_z \mathscr{P}^{(l)}(x, z)f(z)$, being a weighted average of numbers in [0, 1], is itself in [0, 1]. We also must estimate the second term in (4.2), which we may rewrite as

$$\sum_{n=N_1}^{N_2} \frac{e^{-t_i}t_i^n}{n!}\left(\frac{n}{t_i} - 1\right) \sum_z \mathscr{P}^{(n)}(x, z)f(z) + \frac{e^{-t_i}t_i^{N_2}}{N_2!} \sum_z \mathscr{P}^{(N_2+1)}(x, z)f(z) - \frac{e^{-t_i}t_i^{N_1-1}}{(N_1 - 1)!} \sum_z \mathscr{P}^{(N_1)}(x, z)f(z)\,. \tag{4.7}$$

The definitions of N_1 and N_2 imply that the absolute value of the first term in (4.7) is $\leqq \varepsilon$, and the uniformity statement (4.3) implies that each of the second and third terms is in $[0, \varepsilon]$. Thus we see that the limit in (4.2) is less than 3ε in absolute value, and since ε was an arbitrary positive number, the limit must actually be 0. This gives us $\mathscr{P}k(x) - k(x) = 0$ for all $x \in \mathscr{S}$, so that k is harmonic.

We now must treat the general case, in which the numbers c_x are not necessarily 1. Let $\sup_x c_x = c$ (finite by hypothesis), and define the stochastic matrix $\tilde{\mathscr{P}}$ by

$$\tilde{\mathscr{P}}(x, x) = 1 - \frac{c_x}{c}[1 - \mathscr{P}(x, x)]\,, \qquad \text{and}$$

$$\tilde{\mathscr{P}}(x, y) = \frac{c_x}{c}\mathscr{P}(x, y) \quad \text{for} \quad x \neq y\,.$$

Now let $\tilde{\mathscr{P}}(t)$ be the transition function for a continuous time Markov chain on $\mathscr{S}$ whose states each have holding parameter c, and whose paths are governed by $\tilde{\mathscr{P}}(x, y)$. Since $\tilde{\mathscr{P}}(t)$ and $\mathscr{P}(t)$ have the same infinitesimal parameters, we have $\tilde{\mathscr{P}}(t) = \mathscr{P}(t)$. We know from the first part of the proof that $\tilde{\mathscr{P}}k = k$ (we proved the result under the assumption $c_x \equiv 1$, but it is clear that that same proof would work if $c_x \equiv c$). Thus $\tilde{\mathscr{P}}(t)k = k$ for all $t \geqq 0$, by the discussion preceding the statement of the lemma. This implies that $\mathscr{P}(t)k = k$ for all $t \geqq 0$ and $\mathscr{P}k = k$.

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