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ON THE INVARIANCE PRINCIPLE FOR NONSTATIONARY MIXINGALES

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In an earlier paper, the author proves an invariance principle for mixingales, a generalization of the concepts of mixing sequences and martingale differences, under the condition that the variance of the sum of n random variables is asymptotic to $\sigma^2 n$ where $\sigma^2 > 0$. In this note we relax further the required degree of stationarity, requiring only that the squared variables properly normalized form a uniformly integrable family, and the partial sums have variances consistent with the Wiener process.

1. Introduction. In [2], the author proves a weak invariance principle for nonstationary sequences of dependent variables which do not have variances that fluctuate too wildly. These variables, a hybrid of the notion of martingales and mixing sequences of rv, are called mixingales. In this note we relax further the required degree of stationarity, simplify the conditions on the mixing rates and show that the results continue to hold. This improves on the mixing rates required by Philipp and Webb (1973).

2. Results. Let $\{X_{n,i}: i = 1, 2, \cdots, n = 1, 2, \cdots\}$ be a double array of zero mean random variables defined on the probability space $(\Omega, \mathscr{F}, P)$. Let $k_n(t)$ be a sequence of nonrandom integer valued, nondecreasing, right continuous functions on $[0, \infty)$. We form a random function

$$W_n(t) = \sum_{i=1}^{k_n(t)} X_{n,i} \tag{2.1}$$

and we wish to show weak convergence of W_n to W a standard Wiener process in the space $D[0, \infty)$ endowed with Stone's (1963) topology. Suppose there exists a double array of positive constants $\{\sigma_{ni}^2\}$ such that the following conditions hold for each $T < \infty$:

$$\text{(a)} \qquad \sup_{s<t<T} \limsup_{n\to\infty} \frac{\sum_{k_n(s)}^{k_n(t)} \sigma_{ni}^2}{t-s} < \infty\,.$$

$$(2.2) \qquad \text{(b)} \qquad \left\{\frac{X_{n,i}^2}{\sigma_{ni}^2}\,;\ n = 1, 2, \cdots,\ i \leqq k_n(T)\right\} \quad \text{is a uniformly integrable set.}$$

$$\text{(c)} \qquad \max_{i \leqq k_n(T)} \sigma_{n,i} \to 0 \qquad \text{as} \quad n \to \infty\,.$$

These conditions are sufficient for the Lindeberg condition to hold for the array $\{X_{n,i}\}$ and are closely related to same.

Often, $\sigma_{n,i}^2$ is the approximate variance added to a sum by the inclusion of $X_{n,i}$: thus in the uncorrelated case, $\sigma_{n,i}^2 = EX_{n,i}^2$ and in the weakly stationary

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case (e.g., Billingsley's (1968) ϕ-mixing Theorem 20.1)

$$\sigma_{n,i}^2 = EX_{n,i}^2 + \sum_{j \neq i} EX_{n,i}X_{n,j} .$$

We will also require that the sequence $\{X_{n,i}; i = 1, 2, \cdots\}$ constitutes a mixingale (cf. [2], [3]); namely that for some double array of σ-fields $\mathscr{F}_{n,i}$, $\mathscr{F}_{n,i} \uparrow$ in i, a sequence of positive constants $\psi_n \downarrow 0$ as $n \to \infty$, and for all $n, i \geqq 1, k \geqq 0$,

$$\text{(2.3)} \quad \begin{aligned} &\text{(a)} \quad \|E(X_{ni} \mid \mathscr{F}_{n,i-k})\|_2 \leqq \psi_k \sigma_{ni} , \quad \text{and} \\ &\text{(b)} \quad \|X_{ni} - E(X_{ni} \mid \mathscr{F}_{n,i+k})\|_2 \\ &\qquad \leqq \psi_{k+1}\sigma_{ni} \quad \text{where } \|\cdot\|_2 \text{ is the } L_2(\Omega) \text{ norm.} \end{aligned}$$

We now introduce our main theorem:

(2.4) THEOREM. *Suppose conditions* (2.2) *and* (2.3) *are in force and*

$$\text{(2.5)} \qquad \sum_{k=1}^{\infty} \left(\sum_{n=0}^{k} \frac{1}{\psi_n^2} \right)^{-\frac{1}{2}} < \infty .$$

Then the sequence $\{W_n\}$ *is tight in Stone's topology on* $D[0, \infty)$. *Moreover if for each* $s < t < u$,

$$\text{(2.6)} \qquad E|E\{(\textstyle\sum_{i=k_n(t)}^{k_n(u)} X_{ni})^2 \mid \mathscr{F}_{n,k_n(s)}\} - (u - t)| \to 0 \quad \text{as} \quad n \to \infty ,$$

then W_n *converges weakly to a standard Wiener process.*

(2.7) REMARK. (2.5) differs markedly from the usual summability type of conditions

$$\text{(2.8)} \qquad \sum \psi_i^\theta < \infty .$$

However, (2.5) implies (2.8) with $\theta = 2$ and is implied by (2.8) with $\theta < 2$. Thus, as a consequence of (2.5) $n\psi_n^2 \to 0$ as $n \to \infty$. (Cf. Section 3 for a proof.)

(2.9) COROLLARY. *Let* $\{(X_{ni}, \mathscr{F}_{ni}) : i = 1, 2, \cdots\}$ *be a sequence of martingale differences for each* n *(so* X_{ni} *is* $\mathscr{F}_{ni}$ *measurable and* $E(X_{ni} \mid \mathscr{F}_{n,i-1}) = 0$ a.s.). *Assume* $\sigma_{ni}^2 = EX_{ni}^2 < \infty$ *and* $k_n(t) = \sup\{j; \sum_{i=1}^{j} \sigma_{ni}^2 \leqq t\}$. *Then if* X_{ni} *satisfies* (2.2) (b), (c) *where* $T = \liminf_{n\to\infty} \sum_i \sigma_{ni}^2$ *and if*

$$\text{(2.10)} \qquad \sum_{k_n(t)}^{k_n(u)} E(X_{ni}^2 \mid \mathscr{F}_{n,k_n(s)}) \to_p u - t \qquad \text{for each} \quad s < t < u < T ,$$

then W_n *converges weakly to* W *on* $D[0, T)$.

For the remaining assume each X_{ni} is $\mathscr{F}_{ni}$ measurable: let

$$\begin{aligned} \phi_m &= \sup \phi(\mathscr{F}_{nk}, \sigma(\textstyle\sum_{i=k+m}^{j} X_{ni})) \qquad \text{and}^1 \\ \alpha_m &= \sup \alpha\{\mathscr{F}_{nk}, \sigma(\textstyle\sum_{i=k+m}^{j} X_{ni})\} , \end{aligned}$$

where the sup is over all $n, k, j \geqq k + m$ and where α, ϕ are defined in Section 3 of [2].

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[1] As remarked in [2], page 170, we may replace $\mathscr{F}_{nk}$ by $\sigma(\sum_{i=1}^{k} X_{ni})$ and still retain the central limit theorem.

(2.11) COROLLARY (ϕ-mixing). *Let $\{X_{ni}\}$ be centered at expectations: Suppose $\sigma_{ni} = \|X_{ni}\|_\beta$, $\beta \geqq 2$ satisfies* 2.2 *and the sequence $\psi_m = \phi_m^{1-1/\beta}$, $m \geqq 0$ satisfies* 2.5. *Finally, assume*

(2.12) $$\mathrm{Var}\,(W_n(t)) \to t \quad \textit{for each} \quad t \quad \textit{as} \quad n \to \infty\,.$$

Then W_n converges weakly to a standard Brownian motion process.

(2.13) REMARKS. The same theorem holds for strongly mixing random variables if $\beta > 2$ and $\phi_m^{1-1/\beta}$ replaced by $\alpha_m^{\frac{1}{2}-1/\beta}$.

3. Proofs. We will need the following elementary lemmas.

(3.1) LEMMA. *Let $\{(X_i, \mathscr{F}_i)\}$ be a sequence of martingale differences* (*so* $E(X_n \mid \mathscr{F}_{n-1}) = 0$ a.s. *for all n*) *and assume $|X_i| \leqq K\sigma_i$* a.s. *for all i*, (*$\{\sigma_i\}$ is any sequence of positive constants*). *Then*

$$E(\textstyle\sum_{i=1}^n X_i)^4 \leqq 10K^4(\sum_{i=1}^n \sigma_i^2)^2\,.$$

PROOF. Put

$$S_n = \textstyle\sum_{i=1}^n X_i\,, \qquad v_n^2 = \sum_{i=1}^n \sigma_i^2\,.$$

Then:

$$\begin{aligned} ES_n^4 &= \textstyle\sum_i EX_i^4 + 4\sum_{i<j} EX_iX_j^3 + 6\sum_{i<j} EX_i^2X_j^2 + 12\sum_{i<j<k} EX_iX_jX_k^2 \\ &\leqq 4\textstyle\sum_{j=1}^n ES_jX_j^3 + 6\sum_{j=2}^n ES_{j-1}^2X_j^2 \end{aligned}$$

where the first term on the majorant side is bounded by $4K^4 \sum_{j=1}^n v_j\sigma_j^3 \leqq 4K^4v_n^4$, and the second, by $6K^4 \sum_{j=1}^n v_{j-1}^2\sigma_j^2 \leqq 6K^4v_n^4$. □

For the next two lemmas, we drop the subscript n from X_{ni}, S_{ni} and σ_{ni}, and put $S_n = \sum_{i=1}^n X_i$ and $v_n^2 = \sum_{i=1}^n \sigma_i^2$. Replacing (6.3) of [2], we have the following

(3.2) LEMMA. *Suppose X_i satisfies* 2.3 *and $\psi_j > 0$ for all j. Then for all n,*

(3.3) $$E\{\max_{j\leqq n} S_j^2\} \leqq 16v_n^2\{\textstyle\sum_{k=0}^\infty (\sum_{i=0}^k \psi_i^{-2})^{-\frac{1}{2}}\}^2\,.$$

PROOF. Replacing 6.4 of [2] we have, with v_n^2 substituted for n,

(3.4) $$E\{\max_{j\leqq n} S_j^2\} \leqq 4v_n^2(\textstyle\sum_i a_i)\left\{\dfrac{\psi_0^2+\psi_1^2}{a_0} + 2\sum_{k=1}^\infty \psi_k^2(a_k^{-1} - a_{k-1}^{-1})\right\}.$$

Let $a_0 = \psi_0$ and define a_k recursively for $k \geqq 1$ by:

$$a_k^{-1} - a_{k-1}^{-1} = \frac{a_k}{\psi_k^2}\,.$$

Then

$$\frac{1}{\psi_k^2} \leqq \left(\frac{1}{a_k} + \frac{1}{a_{k-1}}\right)\left(\frac{1}{a_k} - \frac{1}{a_{k-1}}\right) = \frac{1}{a_k^2} - \frac{1}{a_{k-1}^2}\,,$$

so that

$$\frac{1}{a_n^2} \geqq \textstyle\sum_{k=0}^n \dfrac{1}{\psi_k^2}\,,$$

$$\textstyle\sum_{k=0}^\infty a_k \leqq \sum_{k=0}^\infty \left(\sum_{n=0}^k \dfrac{1}{\psi_n^2}\right)^{-\frac{1}{2}} = x\,, \quad \text{say.}$$

Finally $(\psi_0^2 + \psi_1^2)/a_0 \leqq 2a_0$. Substituting these values in the majorant of 3.4 yields $16v_n^2x^2$.

(3.5) LEMMA. *If* $\{X_n\}$ *satisfies* (2.2c), (2.3), *and* (2.5), *then the set*

$$\left\{\max_{j\leqq n} \frac{S_j^2}{v_n^2}\,;\, n \geqq 1\right\} \quad \textit{is uniformly integrable.}$$

PROOF. We indicate only the changes to be made in the proof of Lemma 6.5 in [2]. Put

$$X_i^c = X_i I[|X_i| \leqq c\sigma_i]$$

and U_i, Y_i, Z_i are unchanged. Then

$$I = \varepsilon_{y/3}\left(\max_{j\leqq n} \frac{\bar{Y}_j^2}{v_n^2}\right)$$

$$II = E\left(\max_{j\leqq n} \frac{\bar{Z}_j^2}{v_n^2}\right)$$

$$III = E\left(\max_{j\leqq n} \frac{\bar{U}_j^2}{v_n^2}\right).$$

U_i is a mixingale (satisfies (2.3)) with mixing functions $\hat{\psi}_k = \psi_{m\vee k}$. Now,

$$\sum_k \left(\sum_{n=0}^k \frac{1}{\hat{\psi}_n^2}\right)^{-\frac{1}{2}} \leqq \sum_{k=0}^m \left(\frac{k}{\psi_m^2}\right)^{-\frac{1}{2}} + \sum_{m+1}^\infty \left(\sum_{n=0}^k \frac{1}{\psi_n^2}\right)^{-\frac{1}{2}}$$

$$= O(m^{\frac{1}{2}}\psi_m) + \sum_{m+1}^\infty \left(\sum_{n=0}^k \frac{1}{\psi_n^2}\right)^{-\frac{1}{2}}$$

where by 2.7 and 3.2, m may be chosen so that $III \leqq \varepsilon/27$.

Similarly each Z_i is a mixingale with functions $\hat{\psi}_k = 1 \vee \psi_k$ and σ_i^2 replaced by $\sigma_i^2 \sup_j \varepsilon_c(X_j^2/\sigma_j^2)$. Therefore by 3.2 we may pick c sufficiently large that $II \leqq \varepsilon/27$.

Finally, by 6.2 of [2] and Lemma 3.1,

$$E(\max_{j\leqq n} \bar{Y}_j^4) \leqq 10(\tfrac{4}{3})^4(2m+1)^3(4c)^4 v_n^4$$

so for fixed (m, c) we pick y such that $I \leqq \varepsilon/27$. □

Verification of the following is a trivial consequence of Billingsley's Theorem 8.4.

(3.6) LEMMA. *Let* $\{W_n(t)\}$ *be a sequence of random elements of* $D[0, 1]$ *such that*

$$\left\{\max_{t\leqq s\leqq t+\delta} \frac{[W_n(s) - W_n(t)]^2}{\delta}\,;\, n > N(t, \delta),\, 0 \leqq t \leqq 1,\, \delta \in S\right\}$$

is a uniformly integrable set for some sequence S *of* δ *approaching* 0 *and nonrandom finite valued function* $N(t, \delta)$. *Then* $\{W_n\}$ *is tight in the uniform topology on* $D[0, 1]$.

PROOF OF THEOREM 2.4. Tightness of $\{W_n\}$ now follows from 2.2(a), Lemmas (3.5) and (3.6), and the fact that the rate of uniform integrability achieved in (3.5) does not depend on our location in the sequence.

Convergence to Brownian motion will follow from the verification of (6.7) of [2] with the upper limit of summation replaced by $k - 2$ there and in the definition of U_n. Then

$$\begin{aligned} ||U_n - \textstyle\sum_{j=1}^{k-2} u_j W_n(t_j)||_2 & \\ &\leqq (\max |u_j|)(k-2) \textstyle\sum_{i=1}^{k_n(t_{k-2})} ||E(X_{ni} \mid \mathscr{F}_{n,k_n(t_{k-1})}) - X_{ni}||_2 \\ &= O(\textstyle\sum_{i=1}^{k_n(t_{k-2})} \sigma_{ni} \psi_{k_n(t_{k-1})-i+1}) \\ &= O(\textstyle\sum_{i=1}^{k_n(t_{n-2})} \sigma_{ni}^2)^{\frac{1}{2}} (\textstyle\sum_{i=k_n(t_{k-1})-k_n(t_{k-2})+1}^{\infty} \psi_i^2)^{\frac{1}{2}} \\ &\to 0 \quad \text{as} \quad n \to \infty . \end{aligned}$$

The remaining changes required in the proof of Theorem 2.6 of [2] follow along parallel lines.

PROOF OF REMARK 2.7. We prove only that (2.5) implies (2.8) with $\theta = 2$; the other implication under monotonicity of ψ_i (assumed w.l.o.g. cf. [2]) is trivial.

Under (2.5), the monotonicity implies $k(\sum_{n=0}^{k} 1/\psi_n^2)^{-\frac{1}{2}} \to 0$ so for k sufficiently large,

$$\frac{1}{\psi_k^2} \geqq \frac{1}{k} \sum_{n=0}^{k} \frac{1}{\psi_n^2} = \frac{1}{k} \left(\sum_{n=0}^{k} \frac{1}{\psi_n^2}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{k} \frac{1}{\psi_n^2}\right)^{\frac{1}{2}} \geqq \left(\sum_{n=0}^{k} \frac{1}{\psi_n^2}\right)^{\frac{1}{2}} .$$

Thus, $\sum \psi_n^2 < \infty$.

PROOF OF 2.9. The $L_1(\Omega)$ convergence of (2.10) follows from Lemma 2.11 of [4].

The proof of (2.11) follows the same line as (3.9) of [2].

(3.7) LEMMA. *Suppose that the conditions of either* (2.11) *or* (2.13) *are satisfied. Then* (2.6) *holds and* Var $(W_n(t) - W_n(s)) \to t - s$ *for any* $s < t$.

PROOF. It follows from Minkowski's inequality and (2.2)(c) that

$$||W_n(t)||_2 \leqq ||W_n(s)||_2 + ||W_n(t) - W_n(s)||_2$$

and the last term $\to 0$ over any subsequence for which $k_n(t) - k_n(s)$ is bounded. This with (2.12) implies $k_n(t) - k_n(s) \to \infty$. Now put $U = (W_n(t) - W_n(s))^2$, $U^c = UI(U \leqq c)$. Now if $E_j(\cdot) = E(\cdot \mid \mathscr{F}_{nk_n(s)-j})$ and $g(c) = ||U - U^c||_1$

$$\begin{aligned} ||E_j U - EU||_1 &\leqq 2||U - U^c||_1 + ||E_j U^c - EU^c||_1 \\ &\leqq 2g(c) + \max(2\phi_i, 5\alpha_i)c \end{aligned}$$

by Lemma (3.5) of [2] which, by Lemma 3.5, can be made arbitrarily small for c and j sufficiently large, the convergence to 0 being uniform in $n \geqq N(s, t)$. Now choose $\delta_n \to 0$ such that $j_n = k_n(s) - k_n(s - \delta_n) \to \infty$ but $||W_n(s) - W_n(s - \delta_n)||_2 \leqq \psi_0(k_n(s) - k_n(s - \delta_n))$ $(\max_i \sigma_{ni}) \to 0$. Then if $V = W_n(s - \delta_n)$, $Z = (W_n(s) - W_n(s - \delta_n))$ and $Y = W_n(t) - W_n(s)$,

$$\begin{aligned} (3.8) \qquad ||W_n(t)||_2^2 &= ||V + Z + Y||_2^2 \\ &= EZ^2 + EV^2 + EY^2 + 2E(VE_{j_n} Y) + 2E[Z(V + Y)] . \end{aligned}$$

Using the Minkowski and Schwartz inequalities and the fact that the random variables satisfy 2.3 with $\sum_i \psi_i^2 < \infty$, we can show

$$||E_{j_n} Y||_2 \leqq (\sum_{i=j_n}^{\infty} \psi_i^2)^{\frac{1}{2}} (\sum_{i=k_n(s)+1}^{k_n(t)} \sigma_{ni}^2)^{\frac{1}{2}} \to 0 \quad \text{as} \quad n \to \infty .$$

Moreover, since V and $(V + Y)$ are bounded in $L_2(\Omega)$, the last two summands on the right side of (3.8) converge to 0 as $n \to \infty$. Thus, taking limits on both sides, $\lim_{n\to\infty} EY^2 = t - s$.

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