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HAUSDORFF MEASURE PROPERTIES OF THE ASYMMETRIC CAUCHY PROCESSES[1]

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The function $\varphi(h) = h/|\log h|$ is shown to be an exact Hausdorff measure function for the range of all strictly asymmetric Cauchy processes in R^k, $k \geqq 2$. The same function is also shown to correctly measure the graph of any strictly asymmetric Cauchy process.

1. Introduction. The strictly asymmetric Cauchy processes in R^k have many properties which distinguish them from other stable processes. These differences are usually related to the fact that these processes are not strictly stable, i.e., they do not satisfy the simple "scaling" property that is so useful in obtaining estimates which lead to various sample path properties. For this reason, these processes have usually been specifically excluded from general arguments. In particular, the correct measure function for the graph was not obtained in [2] or [6] nor was it obtained in [10] for the range.

Let $X(t)$ be a strictly asymmetric Cauchy process in R^k. (This class of processes is defined in the next section.) Let

$$\begin{aligned} \varphi(h) &= h/|\log h| \qquad \text{for} \quad 0 < h < e^{-1}, \\ &= h \qquad \text{for} \quad h \geqq e^{-1}, \end{aligned}$$

and write the corresponding Hausdorff measure as $\varphi - m(\ )$. Denote the graph of the process up to time t by

$$G(0, t) = \{y \in R^{k+1} : y = (X(s), s) \text{ for some } s \in [0, t]\}$$

and the range by

$$R(0, t) = \{y \in R^k : y = X(s) \text{ for some } s \in [0, t]\}.$$

Our main object in the present note is to prove the following

THEOREM. *If $X(t)$ is a strictly asymmetric Cauchy process in R^k, then*

(a) *for all k, there is a positive finite c such that $\varphi - m(G(0, t)) = ct$ for all* $t \geqq 0$ a.s.;

(b) *for $k \geqq 2$, there is a positive finite C such that $\varphi - m(R(0, t)) = Ct$ for all* $t \geqq 0$ a.s.

Note that for $k = 1$, there is no Hausdorff measure problem for $R(0, t)$ since the range has positive Lebesgue measure [1]. We did obtain information about the distribution of $|R(0, t)|$ for large t in [8].

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The proof of the theorem does not follow the pattern of the previous proofs about the correct measure functions for the graph and range of Lévy processes. In all cases previously studied, more economical fine coverings resulted from using covering sets of very different sizes, whereas in both problems considered in this paper, bounded φ-covers can be found using spheres all having the same radius.

In Section 2 we collect the necessary detailed estimates; in Section 3 we complete the proof of the part of the theorem about the graph and the result about the range is proved in Section 4.

2. Preliminaries. A Cauchy process in R^k is a Markov process with stationary independent increments and continuous transition density $p(t, y - x)$ defined by its characteristic function

$$\int_{R^k} \exp\{i(z, u)\} p(t, u)\, du = \exp\{-t\psi(z)\} ,$$

where

$$\psi(z) = |z| \int_{C^k} w(z, \theta) m(d\theta) \tag{1}$$

and m is a probability measure on C^k, the unit sphere in R^k, such that the support of m is not contained in any proper subspace. In (1), w is given by

$$w(z, \theta) = |(\hat{z}, \theta)| + \frac{2i}{\pi} (\hat{z}, \theta) \log |(z, \theta)|$$

where $\hat{z} = z/|z|$. Note that, for convenience, we have omitted a linear term $i(z, z_0)$ in (1) which corresponds to a deterministic linear drift. The inclusion of such a term would necessitate minor adjustments in our arguments but would not affect any of the results for the strictly asymmetric case. We have also omitted the arbitrary multiplicative factor in (1) which clearly makes no difference in the results. We will assume that $X(t)$ is a Hunt process so that it satisfies the strong Markov property and has sample paths that are right continuous with left limits.

In (1), if m is the uniform measure on C^k, the resulting process is called the symmetric Cauchy process. In this case the correct measure function for $R(0, t)$ is given in [10] and for $G(0, t)$ in [2] or [6]. In fact, the results for the symmetric processes remain valid provided

$$\int_{C^k} \theta m(d\theta) = 0$$

and only minor modifications in the existing proofs are required to establish this. In the present paper, therefore, we shall only concern ourselves with the strictly asymmetric Cauchy processes, i.e., those for which m satisfies

$$\int_{C^k} \theta m(d\theta) = \xi \neq 0 . \tag{2}$$

For $k = 1$, any asymmetric Cauchy process is strictly asymmetric, while for $k \geqq 2$, ξ is a point in the open unit ball of R^k. It turns out that the line through 0 and ξ is, in a sense, the "preferred direction" for $X(t)$ both for small t and large t—but we do not need to make this precise in the present paper. We shall, however, need to use the modified scaling property which can be proved directly from the definition (1), i.e., for $r > 0$, $t > 0$

$$rX(t) \quad \text{and} \quad X(rt) - \left(\frac{2}{\pi} rt \log r\right) \xi$$

have the same distribution. We will frequently use the special case of this that

$$X(t) \quad \text{and} \quad tX(1) + \left(\frac{2}{\pi} t \log t\right) \xi \tag{3}$$

have the same distribution, or equivalently

$$p(t, x) = p\left(1, xt^{-1} - \left(\frac{2}{\pi} \log t\right) \xi\right) t^{-k} . \tag{4}$$

We will now prove some lemmas which give the necessary preliminary estimates. Here and in the rest of the paper we adopt the convenient practice of letting c and C denote finite positive constants whose values are not important and depend only on the specific process $X(t)$ being considered. The values of these constants may even change from line to line within a proof.

First we need a rather crude estimate for the large tail of a Cauchy distribution in R^k.

LEMMA 1. *If $X = X(1)$ is a random vector in R^k having a Cauchy distribution, there are constants c and C such that for $\lambda \geqq 1$,*

$$c\lambda^{-1} \leqq P\{|X| \geqq \lambda\} \leqq C\lambda^{-1} .$$

PROOF. For $k = 1$, the exact asymptotic expansion for the tails of the distribution is known [9]. The general result follows by considering the individual coordinates of X each of which has a 1-dimensional Cauchy distribution.

We can now use (3) to deduce estimates for both tails of the distribution of $X(t)$ when t is small.

LEMMA 2. *If $X(t)$ is a strictly asymmetric Cauchy process in R^k, there are positive constants a_0, c, and t_0 such that*

(a) *if $0 < t \leqq a|\log a|^{-1}$ and $0 < a \leqq a_0$, then*

$$P\{|X(t)| \geqq a\} \leqq cta^{-1} ;$$

(b) *if $ca|\log a|^{-1} \leqq t \leqq t_0$, then*

$$P\{|X(t)| \leqq a\} \leqq \frac{ca}{t \log^2 t} .$$

PROOF. (a) By (3), we have

$$\begin{aligned} P\{|X(t)| \geqq a\} &= P\left\{\left|X(1) + \left(\frac{2}{\pi} \log t\right) \xi\right| \geqq \frac{a}{t}\right\} \\ &\leqq P\left\{|X(1)| \geqq \left(1 - \frac{2}{\pi} \frac{t|\log t|}{a}\right) \frac{a}{t}\right\} \end{aligned}$$

and the result follows from Lemma 1.

(b) Let $Y(t)$ be the projection of $X(t)$ on the line through 0 and ξ. Then $Y(t)$ is a 1-dimensional asymmetric Cauchy process which satisfies the scaling property (3) with ξ replaced by some $h \in R^1$, $h \neq 0$. The density of $Y(t)$, $p_1(t, x)$, has a known asymptotic expansion for large x but all we need to know is that

$$\max_{u \leqq |x| \leqq v} p_1(1, x) \leqq cu^{-2} \quad \text{for} \quad u \geqq u_0 . \tag{5}$$

Then

$$\begin{aligned} P\{|X(t)| \leqq a\} &\leqq P\{|Y(t)| \leqq a\} \\ &= P\left\{\left|Y(1) + \left(\frac{2}{\pi}\log t\right)h\right| \leqq at^{-1}\right\} \\ &\leqq 2at^{-1}\max_{u \leqq |x| \leqq v} p_1(1, x) \end{aligned}$$

where $u = 2\pi^{-1}|h \log t| - at^{-1} \geqq \pi^{-1}|h \log t|$ provided that $t \geqq \pi|h|^{-1}a|\log a|^{-1}$ and a is sufficiently small. An application of (5) now completes the proof.

LEMMA 3. *If $X(t)$ is a strictly asymmetric Cauchy process in R^k, there is a positive constant a_0 such that if $0 \leqq a \leqq a_0$ and $0 \leqq \beta - \alpha \leqq a|\log a|^{-1}$, then*

$$P\{\inf_{\alpha \leqq t < \beta} |X(t)| < a\} \leqq 2P\{|X(\beta)| < 2a\} .$$

PROOF. Let

$$\begin{aligned} \tau &= \inf\{t \geqq \alpha : |X(t)| < a\} , \\ E &= \{\inf_{\alpha \leqq t < \beta} |X(t)| < a\} = \{\tau < \beta\} , \\ F &= \{|X(\beta)| < 2a\} . \end{aligned}$$

Then since $|X(\beta)| \geqq 2a$ implies $|X(\beta) - X(\tau)| \geqq a$, we have

$$\begin{aligned} P(E \cap F^c) &\leqq P\{\tau < \beta, |X(\beta) - X(\tau)| \geqq a\} \\ &\leqq c|\log a|^{-1}P(E) \end{aligned}$$

where the last inequality is a consequence of the strong Markov property and Lemma 2(a). But this is sufficient to prove the lemma since then

$$\begin{aligned} P(E) &= P(E \cap F) + P(E \cap F^c) \\ &\leqq P(F) + \tfrac{1}{2}P(E) \end{aligned}$$

for a sufficiently small.

3. The graph. We note that Hawkes [1] has already proved that $\varphi - m(G(0, t)) < \infty$ a.s. when $k = 1$ and his methods extend with only trivial changes to general k. Part (a) of the theorem will therefore be established if we can show that

$$\varphi - m(G(0, t)) > 0 \quad \text{a.s.} \tag{6}$$

since the method of Section 7 of [11] then shows that the measure must equal ct for some $c > 0$. The graph can be thought of as a process in R^{k+1} with stable components so we can use the methods of [6]. Let $S(a)$ be the ball $\{x : |x| \leqq a\}$ in R^k and I_a its indicator function. Then

$$T(a, \alpha) = \int_0^\alpha I_a(X(t))\, dt$$

is the time spent in $S(a)$ up to time α. If we can prove that

$$\limsup_{a\to 0} \frac{T(a, a)}{\varphi(a)} \leqq c \quad \text{a.s.,} \tag{7}$$

then (6) will follow as in [6] by the application of a density theorem. We will prove a somewhat stronger statement than (7) in the next lemma.

REMARK. φ gives the right normalization in (7). This can be seen by obtaining the lim inf behavior of the maximum process as in Theorem 4 of [3]. Inverting this gives the lim sup behavior of the first passage process which in turn gives the appropriate lower bound for the lim sup of $T(a, a)$.

LEMMA 4. *If $X(t)$ is a strictly asymmetric Cauchy process in R^k, then there is a finite constant c such that*

$$\limsup_{a\to 0} \frac{T(a, a|\log a|^3)}{\varphi(a)} \leqq c \quad \text{a.s.}$$

PROOF. Let $\alpha = 6ca|\log a|^{-1}$ where c is as in Lemma 2 and then $\alpha_i = \alpha + ia|\log a|^{-1}$. Let

$$\tau_1 = \inf\{t \geqq \alpha : |X(t)| \leqq a\}.$$

Then

$$P\{\tau_1 \leqq a|\log a|^3\} \leqq \sum_{i=1}^{|\log a|^4} P(E_i),$$

where

$$E_i = \{\inf_{\alpha_{i-1}\leqq t<\alpha_i} |X(t)| \leqq a\}.$$

By Lemmas 3 and 2(b), we have for a sufficiently small

$$P(E_i) \leqq c|\log a|^{-1}i^{-1}$$

so that for small a

$$P\{\tau_1 \leqq a|\log a|^3\} = O\left(\frac{\log|\log a|}{|\log a|}\right).$$

Now if $\tau_1 < \infty$, we define

$$\tau_2 = \inf\{t \geqq \tau_1 + \alpha : |X(t)| \leqq a\}$$

and note that by the strong Markov property

$$\begin{aligned} P\{\tau_2 \leqq a|\log a|^3\} &= P\{\tau_1 \leqq a|\log a|^3, \tau_2 \leqq a|\log a|^3\} \\ &= O\left(\left(\frac{\log|\log a|}{|\log a|}\right)^2\right). \end{aligned} \tag{8}$$

The only times in $[0, \tau_2]$ for which it is possible to have $|X(t)| \leqq a$ are in $[0, \alpha]$ and $[\tau_1, \tau_1 + \alpha]$. Therefore

$$P\{T(a, a|\log a|^3) > 2\alpha\} \leqq P\{\tau_2 < a|\log a|^3\}.$$

Letting $a_r = 2^{-r}$ and using (8) we then have for r sufficiently large and $a_{r+1} \leqq a \leqq a_r$,

$$T(a, a|\log a|^3) \leqq T(a_r, a_r|\log a_r|^3) \leqq 12c\varphi(a_r) \leqq 48c\varphi(a).$$

This completes the proof of the lemma and thus the proof of part (a) of the theorem.

REMARK. For $k = 1$, it is worth noting that the sections of $G(0, t)$ are the level sets

$$Z(x, t) = \{s \leqq t: X(s) = x\}.$$

Millar [4] showed that the correct measure function for $Z(x, t)$ is

$$f(h) = \log\log|\log h|/|\log h|.$$

This would lead one to expect that the correct measure function for $G(0, t)$ would be

$$hf(h) = \varphi(h)\log\log|\log h|$$

rather than $\varphi(h)$. This means that we have shown that the Hausdorff measure of $G(0, t)$ is not obtained by "integrating" the measure of sections. This is related to the fact that $f - m(Z(x, t))$ is essentially the local time at x and Millar and Tran [5] have shown that this is unbounded.

4. The range for $k \geqq 2$. The upper bound for $\varphi - m(R(0, t))$ is implicit in [1] since $R(0, t)$ is a projection of $G(0, t)$ and so cannot have larger φ-measure. Thus we only need to prove the lower bound. This will follow by the standard arguments if we can show that

$$\limsup_{a\to 0}\frac{T(a)}{\varphi(a)} \leqq c \quad \text{a.s.},$$

where $T(a) = T(a, \infty)$ is the total time spent in $S(a)$. By Lemma 4, it will be sufficient to show that for all sufficiently small a

$$T(a) = T(a, a|\log a|^3). \tag{9}$$

To do this we modify an argument used in [7] to estimate hitting probabilities. Let U(a) denote the time spent in $S(3a)$ after time $a|\log a|^3$. Then

$$\begin{aligned} EU(a) &= \int_{a|\log a|^3}^{\infty}\int_{|x|\leqq 3a} p(t, x)\,dx\,dt \\ &= \int_{|x|\leqq 3a} dx\int_{a|\log a|^3}^{\infty} p\left(1, xt^{-1} - \left(\frac{2}{\pi}\log t\right)\xi\right)t^{-k}\,dt \\ &\leqq ca|\log a|^{-3}, \end{aligned} \tag{10}$$

using the scaling property (4), the fact that $p(1, x)$ is bounded, and that $k \geqq 2$. Now consider $c < \pi/8|\xi|$ and $t \leqq ca|\log a|^{-1}$. Then

$$P\{|X(t)| \leqq a\} = P\left\{\left|X(1) + \left(\frac{2}{\pi}\log t\right)\xi\right| \leqq \frac{a}{t}\right\}$$

will be near one for a small since $X(1)$ satisfies this inequality whenever $X(1)$ is in $S(|\log a|/2c)$. Thus

$$ET(a) \geqq \int_0^{ca|\log a|^{-1}} P\{|X(t) \leqq a\}\,dt \geqq \tfrac{1}{2}ca|\log a|^{-1} \tag{11}$$

for small a. Now let

$$E(a) = \{|X(t)| \leqq 2a \text{ for some } t \geqq a|\log a|^3\}\,,$$
$$\tau = \inf\{t \geqq a|\log a|^3 : |X(t)| \leqq 2a\}\,.$$

If we restart the process at τ, $X(t)$ will be in $S(3a)$ whenever it is within a of $X(\tau)$. Thus by the strong Markov property

$$EU(a) \geqq P(E(a))ET(a)\,,$$

and (10) and (11) lead to

$$P(E(a)) = O(|\log a|^{-2})\,.$$

Letting $a_r = 2^{-r}$, we have $E(a_r)$ does not occur for all sufficiently large r. Then, if r is large and $a_{r+1} \leqq a \leqq a_r$, we have

$$T(a, a|\log a|^3) \geqq T(a, a_{r+1}|\log a_{r+1}|^3) = T(a)\,.$$

This completes the proof of (9) and hence of part (b) of the theorem.

REMARK. This last argument is really a crude estimate of the "rate of escape" of $X(t)$ from the starting point. Making the obvious improvements yields

COROLLARY. *If $X(t)$ is a strictly asymmetric Cauchy process then for $\varepsilon > 0$*

$$\frac{|X(t)|}{t}|\log t|^{2+\varepsilon} \to \infty \quad \text{a.s.} \qquad as \quad t \to 0\,.$$

However, the actual rate of escape is faster than this and will probably depend on the dimension k. In order to obtain precise results we would require better estimates for delayed hitting probabilities, which would in turn require detailed information about the potential kernel of the process.

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