The Annals of Probability
1977, Vol. 5, No. 4, 601-607

FORMULAS FOR STOPPED DIFFUSION PROCESSES WITH STOPPING TIMES BASED ON THE MAXIMUM[1]

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The joint Laplace transform of T and $X(T)$ is derived where $X(\cdot)$ is a time homogeneous diffusion process and T is the first time the process falls a specified amount below its current maximum. This generalizes the work of Taylor. The distribution of the maximum at T is shown to be exponential for Brownian motion. Formulas for more general stopping times based on the current maximum are given.

1. Introduction and summary of results. This paper presents a generalization of the stopping problem introduced by Taylor [5]. The derivation is also more intuitive than that presented in [5]. Let $\{X(t), t \geqq 0\}$ be a Brownian motion process with $X(0) = 0$, drift parameter μ, and variance parameter σ^2. Let $a > 0$ be given and consider the Markov time $T = T_a = \inf\{t \mid M(t) - X(t) \geqq a\}$ where $M(t) = \sup\{X(u), 0 \leqq u \leqq t\}$. T represents the first time the X-process drops a units below its current maximum. Taylor calculates the bivariate Laplace transform of $X(T)$ and T and finds

$$E(\exp(\alpha X(T) - \beta T)) = \frac{\delta \exp(-(\alpha + \gamma)a)}{\delta \cosh(a\delta) - (\alpha + \gamma)\sinh(a\delta)}$$

with $\beta > 0$, $\theta = \delta \coth(a\delta) - \gamma$, $\theta > 0$, $\alpha < \theta$, $\gamma = \mu/\sigma^2$, and $\delta = [\gamma^2 + 2\beta/\sigma^2]^{\frac{1}{2}}$.

From the above transform, the marginal transforms are calculated as well as the moments and asymptotic distributions of $X(T)$ and T. It is unfortunate that Taylor did not point out that the random variable $X(T) + a$ has an exponential distribution with parameter $2\gamma/(\exp(2\gamma a) - 1)$ or $1/a$ for $\gamma \neq 0$ or $\gamma = 0$ respectively. Using this observation (which is directly obtainable from (3.1) in [5]), all of the marginal behavior of $X(T)$ or equivalently $M(T)$ becomes obvious including equations (3.4), (3.5) and (3.6).

In this paper the same Markov time T is studied; however, the underlying process is allowed to be much more general. The process $\{X(t), t \geqq 0\}$ is assumed to be a stochastic process satisfying the time homogeneous Itô stochastic differential equation

$$dX(t) = a(X(t))\,dt + \sigma(X(t))\,dW(t)\,, \quad t \geqq 0 \tag{1}$$

with $X(0) = 0$ a.s., $\{W(t), t \geqq 0\}$ a standard Wiener process, and $\sigma(x) > 0$. It is further assumed that $a(x)$ and $\sigma(x)$ are measurable and defined for x in $[-a, \infty)$

Received June 9, 1975; revised July 8, 1976.

[1] Work supported in part by Air Force Office of Scientific Research under Grant AFOSR 74-2642.

AMS 1970 *subject classifications.* Primary 60J60; Secondary 60G40, 60H10.

Key words and phrases. Diffusion process, stopping time, Laplace transform, stochastic differential equation.

and satisfy the conditions of the existence and uniqueness theorem for stochastic differential equations, namely there exists a constant K such that for all x, y in $[-a, \infty)$

$$|a(x) - a(y)| + |\sigma(x) - \sigma(y)| \leqq K|x - y| \qquad (2)$$
$$a^2(x) + \sigma^2(x) \leqq K^2(1 + x^2)$$

(see [3], page 40). The special case $a(x) = \mu$ and $\sigma(x) = \sigma$ gives the Brownian motion process considered by Taylor.

The main results of this paper are the following:

I. $M(T)$ is a random variable with distribution function given by

$$P(M(T) \geqq x) = \exp[-\textstyle\int_0^x (\Phi(z)/(\int_{z-a}^z \Phi(u)\, du))\, dz] \qquad \text{for} \quad x \geqq 0\,, \qquad (3)$$

with $\Phi(x) = \exp(-\int_0^x 2\gamma(z)\, dz)$, and $\gamma(z) = a(z)/\sigma^2(z)$.

II.

$$E(\exp(\alpha M(T) - \beta T)) = \textstyle\int_0^\infty \exp(\alpha x - \int_0^x b(z)\, dz) c(x)\, dx \qquad (4)$$

where

$$b(z) = \frac{g(z-a)h'(z) - h(z-a)g'(z)}{g(z-a)h(z) - g(z)h(z-\alpha)} \qquad \text{and}$$
$$c(x) = \frac{g(x)h'(x) - g'(x)h(x)}{g(x-a)h(x) - g(x)h(x-a)}$$

and g and h are any two independent solutions of the ordinary differential equation

$$\tfrac{1}{2}\sigma^2(x)f''(x) + a(x)f'(x) = \beta f(x) \qquad \text{for} \quad x \quad \text{in} \quad [-a, \infty)\,.$$

In the special case $\gamma(x) = \gamma$ for x in $[-a, \infty)$ equation (3) takes on the exponential form mentioned earlier,

$$\begin{aligned} P(M(T) \geqq x) &= \exp(-2\gamma x/(\exp(2\gamma a) - 1)) \qquad & \gamma \neq 0\,, \quad x \geqq 0 \\ &= \exp(-x/a) & \gamma = 0\,, \quad x \geqq 0\,. \end{aligned} \qquad (5)$$

This agrees with the results in [5], but is much more general, because it requires only the ratio of $a(x)$ and $\sigma^2(x)$ to be constant, not each of the functions to be constant. It also shows that $M(T)$ has a surprisingly simple distribution (exponential $(1/a)$) for any driftless process ($a(x) = 0$). The exponential distribution is intuitively correct in the Brownian motion case, because the spatial homogeneity insures $M(T)$ has a memoryless distribution. It is also intuitive in the $a(x) = 0$ case, because $\sigma^2(x)$ affects only the speed of the diffusion, hence $M(T)$ is still memoryless.

In the Brownian motion case, the functions defined in (4) become $g(x) = \exp(-(\gamma - \delta)x)$, $h(x) = \exp(-(\gamma + \delta)x)$ with $\gamma = \mu/\sigma^2$ and $\delta = [\gamma^2 + 2\beta/\sigma^2]^{\frac{1}{2}}$, $b(z) = \delta \coth(a\delta) - \gamma = \theta > 0$. Substitution in (4) and simplification yields $\delta \exp(-a\gamma)/[\delta \cosh(a\delta) - (\alpha + \gamma)\sinh(a\delta)]$, $\alpha < \theta$ for the right-hand side. Multiplication by $\exp(-a\alpha)$ gives Taylor's result.

In the next section two lemmas are stated which are used in the derivation. Sections 3 and 4 present derivations of the main results. Section 5 provides discussion of possible generalizations.

2. Fundamental lemmas. Let $\{X(t), t \geqq 0\}$ be a stochastic process taking values in a possibly infinite interval I satisfying (1) and (2) with initial condition $X(0) = x$ a.s. and $\sigma(x) > 0$ both for x in I. Let $T_{a,b} = \inf\{t \mid X(t) = a \text{ or } b\}$ for $a \leqq x \leqq b$ and a, b in I, the two barrier first passage time.

LEMMA 1. *Under the above conditions*

$$P(X(T_{a,b}) = a) = q(a, b, x) = \int_x^b \Phi(z)\,dz / \int_a^b \Phi(z)\,dz$$
$$P(X(T_{a,b}) = b) = p(a, b, x) = \int_a^x \Phi(z)\,dz / \int_a^b \Phi(z)\,dz$$

where $\Phi(z) = \exp(-\int_a^z 2\gamma(u)\,du)$ *and* $\gamma(u) = a(u)/\sigma^2(u)$.

PROOF. [3], page 110.

The next lemma gives the Laplace transform of $T_{a,b}$ both unconditionally and conditional on hitting either a or b. While these results are well known (for example, see [1], Chapter 16 or [2]), a simple proof is presented.

Let g and h be any two independent solutions of the ordinary differential equation

$$\tfrac{1}{2}\sigma^2(x)f''(x) + a(x)f'(x) = \beta f(x)\,. \tag{6}$$

LEMMA 2. *Under the above conditions*

$$E(\exp(-\beta T_{a,b}) \mid X(T_{a,b}) = b) = u(a, b, x)/p(a, b, x)$$
$$E(\exp(-\beta T_{a,b}) \mid X(T_{a,b}) = a) = v(a, b, x)/q(a, b, x)$$
$$E(\exp(-\beta T_{a,b})) = u(a,b,x) + v(a,b,x)$$

with

$$u(a, b, x) = \frac{g(a)h(x) - g(x)h(a)}{g(a)h(b) - g(b)h(a)} \quad \text{and} \quad v(a, b, x) = \frac{g(x)h(b) - g(b)h(x)}{g(a)h(b) - g(b)h(a)}$$

PROOF. Let $\{X(t), t \geqq 0\}$ be defined by (1) and (2) and integrate (1) to find

$$X(t) = x + \int_0^t a(X(s))\,ds + \int_0^t \sigma(X(s))\,dW(s)\,. \tag{7}$$

Let $f(x)$ be any solution of (6) and consider the transformation $Y(t) = \exp(-\beta t)f(X(t))$. Using Itô's lemma ([3], page 24), $Y(t)$ satisfies

$$dY(t) = f'(X(t))\sigma(X(t))\,dW(t) \tag{8}$$

with $Y(0) = f(x)$ a.s. or

$$Y(t) - f(x) = \int_0^t \exp(-\beta s)f'(X_s)\sigma(X_s)\,dW(s)\,. \tag{9}$$

Truncate $T_{a,b}$, forming $\tau_u = T_{a,b} \wedge u$, replace t by τ_u in (9) and take the expectation of both sides. The mean of the right side is 0 ([3], page 29), since the integrand is bounded for $s \leqq T_{a,b}$. Let $u \to \infty$ to find

$$f(x) = E(Y(T_{a,b})) = E(\exp(-\beta T_{a,b})f(X(T_{a,b})))\,. \tag{10}$$

Let g and h be two independent solutions of (6). Equation (10) yields two linear equations

$$\begin{aligned} g(x) &= q(a, b, x)g(a)E(\exp(-\beta T_{a,b}) \mid X(T_{a,b}) = a) \\ &\quad + p(a, b, x)g(b)E(\exp(-\beta T_{a,b}) \mid X(T_{a,b}) = b) \\ h(x) &= q(a, b, x)h(a)E(\exp(-\beta T_{a,b}) \mid X(T_{a,b}) = a) \\ &\quad + p(a, b, x)h(b)E(\exp(-\beta T_{a,b}) \mid X(T_{a,b}) = b) . \end{aligned}$$

The lemma follows easily by solving these two linear equations.

3. The distribution of $M(T)$. To compute $P(M(T) \geqq x)$, partition the interval $[0, x]$ into n subintervals $\{[s_{ni}, s_{ni+1}], 0 \leqq i \leqq n-1\}$ with $0 = s_{n0} < s_{n1} < \cdots < s_{nn} = x$. Let $m_n = \max_{0 \leqq i \leqq n-1}(s_{ni+1} - s_{ni})$ and assume $m_n \to 0$ as $n \to \infty$. For $\{M(T) \geqq x\}$ to occur, the $X(t)$ process must reach x. Consequently, if $M(T) \geqq x$ and $M(t) = y$ for $0 \leqq y < x$ and some $t < T$, then the process must hit $y + dy$ before it hits $y - a$. As a discrete approximation to $P(M(T) \geqq x)$, compute $P_n = P(\bigcap_{i=0}^{n-1} \{X(t) \text{ hits } s_{ni+1} \text{ before } s_{ni} - a\})$. It will be shown that the limit as $n \to \infty$ of P_n is independent of the particular sequence of partitions chosen, hence the limit is $P(M(T) \geqq x)$. Using the strong Markov property and time homogeneity

$$P_n = \prod_{i=0}^{n-1} P(X(t) \text{ hits } s_{ni+1} \text{ before } s_{ni} - a \mid X(0) = s_{ni}) .$$

Using Lemma 1 the ith factor in the product is given by $p(s_{ni} - a, s_{ni+1}, s_{ni})$. After some manipulation

$$\lim_{n\to\infty} P_n = \exp[\lim_{n\to\infty} \sum_{i=0}^{n-1} \log(1 - p(s_{ni} - a, s_{ni+1}, s_{ni}))] .$$

As $n \to \infty$, $m_n \to 0$ and $\sum_{i=0}^{n-1} (s_{ni+1} - s_{ni})^k \to 0$ for $k \geqq 2$. This indicates that only the first term in the Taylor expansion of $\log(1 - u)$ need be kept. Further, by the continuity of $\Phi(z)$,

$$\lim_{n\to\infty} \{\textstyle\int_{s_{ni}}^{s_{ni+1}} \Phi(z)\, dz/(s_{ni+1} - s_{ni}) - \Phi(s_{ni})\} = 0 .$$

As $n \to \infty$ the sum converges to the ordinary Riemann integral $-\int_0^x (\Phi(z)/(\int_{z-a}^z \Phi(u)\, du))\, dz$. This limit is independent of the particular partition sequence chosen which completes the proof of (3).

It is possible that T can be infinite valued. This is equivalent to $\{M(T) = \infty\}$ and

$$P(T = \infty) = \exp(-\textstyle\int_0^\infty (\Phi(z)/\int_{z-a}^z \Phi(u)\, du)\, dz) . \tag{11}$$

4. Derivation of $E(\exp(\alpha M(T) - \beta T))$. The joint Laplace transform of $M(T)$ and T can be calculated by conditioning on $M(T)$ as follows:

$$\begin{aligned} E(\exp(\alpha M(T) - \beta T)) &= E(\exp(\alpha M(T))E(\exp(-\beta T) \mid M(T))) \\ &= \textstyle\int_0^\infty \exp(\alpha x)E(\exp(-\beta T) \mid M(T) = x) f_{M(T)}(x)\, dx \end{aligned}$$

where $f_{M(T)}(x)$ is the density of $M(T)$ derived from (3). We compute $E(\exp(-\beta T) \mid M(T) = x)$ using the discrete approximation technique of Section 3.

We introduce an arbitrary partition sequence $\{s_{ni}, 0 \leqq i \leqq n+1\}$ with $0 = s_{n0} < \cdots < s_{nn} = x < s_{nn+1}$ and $\varepsilon_{nk} = s_{nk} - s_{nk-1}$. We let $m_n = \max_{1 \leqq k \leqq n+1} \varepsilon_{nk}$ and assume $m_n \to 0$ as $n \to \infty$. Define a sequence of stopping times

$$\tau_{nk} = \inf\{t > 0 \mid X(S_{nk-1} + t) - X(S_{nk-1}) = \varepsilon_{nk} \text{ or } -a\}, \quad 1 \leqq k \leqq n+1$$

with $S_{nk} = \sum_{i=1}^{k} \tau_{ni}$. Each τ_{ni} is a.s. finite since $\sigma(x) > 0$.

In this discrete formulation $\{M(T) = x\}$ is approximated by $\{(\bigcap_{k=1}^{n} (X(S_{nk}) - X(S_{nk-1}) = \varepsilon_{nk})) \cap (X(S_{nn+1}) - X(S_{nn}) = -a)\} = B_n$ and $E(\exp(-\beta T) \mid M(T) = x)$ by $E(\exp(-\beta S_{nn+1}) \mid B_n) = E(\prod_{i=1}^{n+1} \exp(-\beta\tau_{ni}) \mid B_n) = E_n$. The stopping time τ_{nk+1} is in the future of S_{nk} for $0 \leqq k \leqq n$, and the X-process is time homogeneous, thus the strong Markov property can be applied to give

$$(12) \qquad E_n = \prod_{i=1}^{n} E(\exp(-\beta\tau_{ni}) \mid X(S_{ni-1}) = s_{ni-1}, X(S_{ni}) = s_{ni}) \times E(\exp(-\beta\tau_{nn+1}) \mid X(S_{nn}) = x, X(S_{nn+1}) = x - a).$$

Each of the $n+1$ conditional expectations in (12) can be computed by using Lemma 2. We find

$$(13) \qquad E_n = \left(\prod_{i=1}^{n} u(s_{ni-1} - a, s_{ni}, s_{ni-1})\right) \Big/ \prod_{i=1}^{n} p(s_{ni-1} - a, s_{ni}, s_{ni-1}) \times (v(x - a, s_{nn+1}, x)/q(x - a, s_{nn+1}, x))$$

where p, q, u, and v are defined in Lemmas 1 and 2.

We now let $n \to \infty$ and show $\lim_{n\to\infty} E_n$ exists and is independent of the partition sequence chosen. We identify this limit as $E(\exp(-\beta T) \mid M(T) = x)$. The limits are taken in the manner outlined in Section 3. We take logs, use $m_n \to 0$, and use the continuity of $g'(x)$ and $h'(x)$ to show

$$(14) \qquad \lim_{n\to\infty} \prod_{i=1}^{n} u(s_{ni-1} - a, s_{ni}, s_{ni-1}) = \exp\left(-\int_0^x b(z)\, dz\right)$$

with

$$b(z) = \frac{g(z-a)h'(z) - h(z-a)g'(z)}{g(z-a)h(z) - h(z-a)g(z)}.$$

Furthermore,

$$(15) \qquad \lim_{n\to\infty} \left(\prod_{i=1}^{n} p(s_{ni-1} - a, s_{ni}, s_{ni-1})\right) q(x - a, s_{nn+1}, x)/\varepsilon_{nn+1} = f_{M(T)}(x)$$

where the limit of the product was calculated in Section 3. Finally,

$$(16) \qquad \lim_{n\to\infty} v(x - a, s_{nn+1}, x)/\varepsilon_{nn+1} = c(x)$$

where $c(x)$ is defined in (4).

Each of the limits (14), (15), and (16) is independent of the particular sequence chosen, hence we have

$$(17) \qquad E(\exp(-\beta\tau) \mid M(T) = x) = \exp\left(-\int_0^x b(z)\, dz\right) c(x)/f_{M(T)}(x).$$

Substitution of (17) into the original expression for $E(\exp(\alpha M(T) - \beta T))$ yields (4) and completes the derivation.

We comment that conditionally on $\{M(T) = x\}$, T is a.s. finite. Nevertheless T and $M(T)$ can be infinite with the same positive probability, and $P(T < \infty)$

can be calculated from (4) by setting $\alpha = 0$ and letting $\beta \to 0$. The result agrees with (11).

5. Further results. Given the distribution function for $M(T)$ defined by (3) one can derive many results including moments and asymptotic distributions. Since $X(T) = M(T) - a$, these results can also be derived for $X(T)$. In the case $\gamma(x) = \gamma$, which includes Brownian motion, $M(T)$ is exponential.

The marginal Laplace transform of T can be obtained from (4) merely by setting $\alpha = 0$. The resulting expression for $E(\exp(-\beta T))$ is still, however, formidable. The parameter β does not appear explicitly, but instead appears in g and h which are actually functions of β. In the Brownian motion case, g and h are exponential and the dependence on β is easy to deduce. Unfortunately, the differential equation (6) has, except in the Brownian motion case, non-constant coefficients, hence except in very special cases there will be no closed form solution. In the special case $\sigma^2(x) = \sigma^2$ and $a(x) = -\rho x$ (the Ornstein–Uhlenbeck process), the solution of (6) involves Weber or Kummer functions ([4], Section 3).

It is theoretically possible to use the Laplace transform for T to derive the moments of T. This can be done by differentiation with respect to β and letting $\beta \to 0$. To carry out this program one must be able to evaluate $g(x) = g(x, \beta)$ and $h(x, \beta)$ at $\beta = 0$ as well as $\partial^k g(x, \beta)/\partial\beta^k|_{\beta=0}$ and $\partial^k h(x, \beta)/\partial\beta^k|_{\beta=0}$. Letting $\beta \to 0$ in (6) $g(x, 0)$ and $h(x, 0)$ become solutions of

$$\tfrac{1}{2}\sigma^2(x)f''(x) + a(x)f'(x) = 0\,. \tag{18}$$

One solution is exponential, the other is constant.

Next, differentiate (6) with respect to β and let $\beta \to 0$, to find

$$\tfrac{1}{2}\sigma^2(x)f_{xx\beta}(x, \beta) + a(x)f_{x\beta}(x, \beta) = f(x, 0)\,. \tag{19}$$

This equation can be solved for $f_\beta(x, \beta)$ which gives the first derivative with respect to β of g and h. Repetitions of the argument give higher derivatives. The reader is referred to [5] for information about T in the Brownian motion case.

The derivation used in Sections 3 and 4 to find the distribution of $M(T)$ (or $X(T)$) and the joint Laplace transform of $M(T)$ and T can be generalized in two important ways.

First, one need not assume $X(0) = 0$ a.s. but can allow $X(0) = x$ a.s. or even allow $X(0)$ to be a random variable which is nonanticipative with respect to the process $\{X(t), t \geqq 0\}$. It is only necessary to condition on the initial value and later average over its distribution.

A much more important generalization is to allow other kinds of stopping times. Instead of stopping when the process falls a units below the current maximum, one can stop when the process falls $u(M(t))$ below the current maximum $M(t)$, thus

$$T_u = \inf\{t \mid M(t) - X(t) \geqq u(M(t))\}\,.$$

Assume $u(x) > 0$ for $x \geqq 0$ and, merely for convenience, $u(x)$ is continuous. The special case $u(x) = a$ was studied in Sections 3 and 4, but all of those results carry over to T_u. We do not repeat the proof but merely state

$$P(M(T_u) \geqq x) = \exp(-\textstyle\int_0^x (\Phi(z)/\int_{z-u(z)}^z \Phi(u)\,du)\,dz) \tag{20}$$

$$E(\exp(\alpha M(T_u) - \beta T_u)) = \textstyle\int_0^\infty \exp(\alpha x - \int_0^x b_u(z)\,dz) c_u(x)\,dx \tag{21}$$

with

$$b_u(z) = \frac{g(z - u(z))h'(z) - h(z - u(z))g'(z)}{g(z - u(z))h(z) - h(z - u(z))g(z)}$$

and

$$c_u(x) = \frac{g(x)h'(x) - g'(x)h(x)}{g(x - u(x))h(x) - g(x)h(x - u(x))}\,.$$

$P(T_u = \infty) = P(M(T_u) = \infty)$ can be calculated from (20).

In the special case $\gamma(x) = \gamma$ which includes Brownian motion, (20) becomes

$$\begin{aligned} P(M(T_u) \geqq x) &= \exp[-\textstyle\int_0^x 2\gamma/(\exp(2\gamma u(z)) - 1)\,dz] \qquad & \gamma \neq 0 \\ &= \exp[-\textstyle\int_0^x dz/u(z)] & \gamma = 0\,. \end{aligned} \tag{22}$$

This distribution is no longer exponential if $u(z)$ is not constant.

Markov times T_u are of interest in studying stock market rules mentioned in [5]. Such rules allow one to consider strategies where $u(x) = ax + b$, that is one for which a sales decision is made when the price falls a fixed fraction below the maximum to date rather than a fixed number of dollars below. The reader is referred to [5] for other possible applications.

Finally we mention that the methods used in Sections 3 and 4 apply to random walk processes as well as diffusion processes. In this case, $M(T)$ will be a geometric random variable rather than an exponential random variable when the transition probabilities are spatially homogeneous.

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