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INEQUALITIES FOR CONDITIONED NORMAL APPROXIMATIONS

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Let X_n be a sequence of i.i.d. random variables with mean 0 and variance 1. Let $S_n^* = n^{-\frac{1}{2}} \sum_{\nu=1}^n X_\nu$. We investigate in this paper the convergence order in conditioned central limit theorems, that is, the convergence order of $\sup_{t\in\mathbb{R}} |P(S_n^* < t \mid B) - \phi(t)|$.

1. Introduction and notations. Let $(\Omega, \mathscr{A}, P)$ be a probability space and X_n: $\Omega \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of independent and identically distributed (i.i.d.) random variables with mean 0 and variance 1. Let $S_n^* = n^{-\frac{1}{2}} \sum_{k=1}^n X_k$.

The conditioned central limit theorem of Rényi [2] states that

$$\alpha_n(B) \equiv \sup_{t\in\mathbb{R}} |P(S_n^* < t \mid B) - \Phi(t)| \to 0 \quad \text{as} \quad n \to \infty$$

for all $B \in \mathscr{A}$ with $P(B) > 0$.

For $B = \Omega$ the theorem of Berry–Esseen yields that $n^{\frac{1}{2}}\alpha_n(\Omega)$ is bounded. It would be worthwhile to determine a sequence $\delta_n \to \infty$—and if possible the "best"—such that $\delta_n \alpha_n(B)$ is bounded for each $B \in \mathscr{A}$ with $P(B) > 0$. Unfortunately it turns out (see Example 1) that no sequence of i.i.d. random variables admits such a sequence $\delta_n \to \infty$, i.e., each rate of convergence for $\alpha_n(B)$ can be destroyed by a suitable $B \in \mathscr{A}$. Therefore only convergence rates depending on the set B are available. We prove an inequality for conditioned sums which yields the following corollaries:

(i) A uniform inequality:

$$\alpha_n(B) \leqq c_r(P(B))^{-(1/r)} \left(\frac{k}{n}\right)^{\frac{1}{2}}, \qquad B \in \mathscr{F}_k \equiv \sigma(X_1, \cdots, X_k), \quad r \geqq 2$$

which can be applied to obtain general limit theorems as well as convergence rates for $\alpha_n(B)$, even for sets B varying with $n \in \mathbb{N}$. (c_r is an appropriate constant only depending on r.)

(ii) A result on convergence a.e.:

$$\left(\frac{n}{k(n) \log\log k(n)}\right)^{\frac{1}{2}} \sup_{t\in\mathbb{R}} |P(S_n^* < t \mid \mathscr{F}_{k(n)}) - \Phi(t)|$$

is P-a.e. bounded if the sequence $k(n)$ fulfills the condition $k(n)\log\log k(n)/n \to 0$ as $n \to \infty$.

(iii) The conditioned central limit theorem of Rényi.

Denote by $\sigma(X_i, i \in I)$ the σ-field induced by the random variables X_i, $i \in I$.

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Write $P(A, \varphi)$ for $\int_A \varphi(\omega)P(d\omega)$ and denote by $P(\varphi \mid \mathscr{F}_n)$ the conditional expectation of φ given $\mathscr{F}_n = \sigma(X_1, \cdots, X_n)$ with respect to P. Denote the q-norm by $||\varphi||_q = (P|\varphi|^q))^{1/q}$.

2. An inequality for the distribution of conditioned sums with applications. At first we give an example which shows that for each sequence of i.i.d. random variables and each sequence $\varepsilon_n \to 0$ the rate of convergence for

$$\alpha_n(A) \equiv \sup_{t\in\mathbb{R}} |P(S_n^* < t \mid A) - \phi(t)|$$

is worse than $O(\varepsilon_n)$ for a suitable chosen $A \in \mathscr{A}$. The rate of convergence can even be destroyed for a single $t \in \mathbb{R}$.

EXAMPLE 1. Let $X_1, X_2, \cdots$ be i.i.d. with $P(X_k) = 0$ and $P(X_k^2) = 1$. We construct for each sequence $\varepsilon_n \to 0$ as $n \to \infty$ a set $A \in \sigma(X_k\colon k \in \mathbb{N})$ with

$$(^+) \qquad |P(S_n^* < 0 \mid A) - \phi(0)| \geqq \varepsilon_n$$

for infinitely many n.

PROOF. W.l.o.g. we assume $P(X_1 \geqq 0) \geqq \frac{1}{2}$ and $\frac{1}{8} > \varepsilon_n \downarrow 0$. Now we construct inductively $\delta(n)$, $k(n) \in \mathbb{N}$ and $A_n \in \sigma(X_k\colon k \in \mathbb{N})$ with $k(n) < k(n+1)$, $\delta(n) < \delta(n+1)$ and $k(n) \geqq \delta(n)$, $A_n \subset A_{n+1}$ and

(i) $P(A_n) = \frac{1}{2} - \varepsilon_{\delta(n)}$
(ii) $P(S^*_{k(j)} < 0, A_n) \leqq \frac{1}{4} - \varepsilon_{\delta(j)} - \varepsilon_{\delta(n)}$ for $j \leqq n$.

As $\sigma(X_k\colon k \in \mathbb{N})$ is countably generated and $P(A) = 0$ for all atoms A of $\sigma(X_k\colon k \in \mathbb{N})$, $P \mid \sigma(X_k\colon k \in \mathbb{N})$ is a nonatomic measure. Hence there exists according to the theorem of Ljapunoff a set $A_1 \subset \{S_1^* \geqq 0\} = \{X_1 \geqq 0\}$ with $P(A_1) = \frac{1}{2} - \varepsilon_1$. Take $\delta(1) = k(1) = 1$, then (i) and (ii) are fulfilled.

Now assume that $k(j)$, $\delta(j)$, A_j are defined for $j \leqq n$ with the desired properties. According to the theorem of Rényi

$$P(S_m^* < 0, A_n) \to \tfrac{1}{2}P(A_n) = \tfrac{1}{4} - \tfrac{1}{2}\varepsilon_{\delta(n)} \qquad \text{as} \quad m \to \infty$$

and

$$P(S_m^* \geqq 0, \bar{A}_n) \to \tfrac{1}{2}P(\bar{A}_n) = \tfrac{1}{4} + \tfrac{1}{2}\varepsilon_{\delta(n)} \qquad \text{as} \quad m \to \infty\,.$$

Choose $\delta(n+1) > \delta(n)$ with $2\varepsilon_{\delta(n+1)} < \frac{1}{2}\varepsilon_{\delta(n)}$. We can choose consequently $k(n+1) > \max(k(n), \delta(n))$ with

$$(1) \qquad P(S^*_{k(n+1)} < 0, A_n) \leqq \tfrac{1}{4} - 2\varepsilon_{\delta(n+1)}$$

$$(2) \qquad P(S^*_{k(n+1)} \geqq 0, \bar{A}_n) \geqq \tfrac{1}{4}\,.$$

By (2) there exists according to the theorem of Ljapunoff a set $B_n \in \sigma(X_k\colon k \in \mathbb{N})$ with

$$(3) \qquad B_n \subset \{S^*_{k(n+1)} \geqq 0\} \cap \bar{A}_n$$

$$(4) \qquad P(B_n) = \varepsilon_{\delta(n)} - \varepsilon_{\delta(n+1)}\,.$$

Define $A_{n+1} = A_n + B_n$, then $P(A_{n+1}) = P(A_n) + P(B_n) = \frac{1}{2} - \varepsilon_{\delta(n+1)}$, i.e., (i) is fulfilled for $n + 1$.

From (1) and (3) we obtain

$$P(S^*_{k(n+1)} < 0, A_{n+1}) = P(S^*_{k(n+1)} < 0, A_n) \leqq \tfrac{1}{4} - 2\varepsilon_{\delta(n+1)}$$
$$= \tfrac{1}{4} - \varepsilon_{\delta(n+1)} - \varepsilon_{\delta(n+1)} ,$$

i.e., (ii) is fulfilled for $j = n + 1$.

Furthermore we obtain for $j \leqq n$ from (4) and the inductive assumption

$$P(S^*_{k(j)} < 0, A_{n+1}) \leqq P(S^*_{k(j)} < 0, A_n) + P(B_n)$$
$$\leqq \tfrac{1}{4} - \varepsilon_{\delta(j)} - \varepsilon_{\delta(n)} + \varepsilon_{\delta(n)} - \varepsilon_{\delta(n+1)}$$
$$= \tfrac{1}{4} - \varepsilon_{\delta(j)} - \varepsilon_{\delta(n+1)} ,$$

i.e., (ii) is fulfilled for $j \leqq n$.

This concludes the inductive construction.

Let $A = \bigcup_{n=1}^{\infty} A_n \in \sigma(X_k : k \in \mathbb{N})$. Then according to (i) we have $P(A) = \frac{1}{2}$. According to (ii) we have for all $j \in \mathbb{N}$:

$$P(S^*_{k(j)} < 0, A) \leqq \tfrac{1}{4} - \varepsilon_{\delta(j)}$$

and hence

$$\phi(0) - P(S^*_{k(j)} < 0 \mid A) \geqq 2\varepsilon_{\delta(j)} \geqq 2\varepsilon_{k(j)} .$$

This proves $(^+)$.

From the following theorem we get our corollaries. Especially we get an inequality for $\alpha_n(B)$.

THEOREM 1. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *with* $P(X_k) = 0$ *and* $P(X_k^2) = 1$. *Define* $\mathscr{F}_k = \sigma(X_1, X_2, \cdots, X_k)$, $S_n^* = n^{-\frac{1}{2}} \sum_{k=1}^{n} X_k$ *and* $F_n(t) = P\{S_n^* < t\}$. *Then for* $k < n$ *we have* P-a.e.:

$$\sup_{t \in \mathbb{R}} |P(S_n^* < t \mid \mathscr{F}_k) - \phi(t)|$$
$$\leqq \sup_{t \in \mathbb{R}} |F_{n-k}(t) - \phi(t)| + (2\pi)^{-\frac{1}{2}} \left(\frac{k}{n-k}\right)^{\frac{1}{2}} |S_k^*|$$
$$+ (8\pi e)^{-\frac{1}{2}} \frac{k}{n-k} .$$

PROOF. (i) Since $X_1, \cdots, X_n$ are i.i.d. the function

$$\omega \to F_{n-k}\left(\left(\frac{n}{n-k}\right)^{\frac{1}{2}} t - \left(\frac{k}{n-k}\right)^{\frac{1}{2}} S_k^*(\omega)\right)$$

is a version of the conditional expectation $P(S_n^* < t \mid \mathscr{F}_k)$.

(ii) We have

$$\sup_t \left| F_{n-k}\left(\left(\frac{n}{n-k}\right)^{\frac{1}{2}} t - \left(\frac{k}{n-k}\right)^{\frac{1}{2}} S_k^*\right) - \phi(t)\right|$$
$$\leqq \sup_t \left| F_{n-k}\left(\left(\frac{n}{n-k}\right)^{\frac{1}{2}} t - \left(\frac{k}{n-k}\right)^{\frac{1}{2}} S_k^*\right) - \phi\left(\left(\frac{n}{n-k}\right)^{\frac{1}{2}} t - \left(\frac{k}{n-k}\right)^{\frac{1}{2}} S_k^*\right)\right|$$
$$+ \sup_t \left| \phi\left(\left(\frac{n}{n-k}\right)^{\frac{1}{2}} t - \left(\frac{k}{n-k}\right)^{\frac{1}{2}} S_k^*\right) - \phi(t)\right|$$

$$\leqq \sup_t |F_{n-k}(t) - \phi(t)|$$
$$+ \sup_t \left|\phi\left(\left(\frac{n}{n-k}\right)^{\frac{1}{2}} t - \left(\frac{k}{n-k}\right)^{\frac{1}{2}} S_k^*\right) - \phi\left(\left(\frac{n}{n-k}\right)^{\frac{1}{2}} t\right)\right|$$
$$+ \sup_t \left|\phi\left(\left(\frac{n}{n-k}\right)^{\frac{1}{2}} t\right) - \phi(t)\right|$$
$$\leqq \sup_t |F_{n-k}(t) - \phi(t)| + (2\pi)^{-\frac{1}{2}}\left(\frac{k}{n-k}\right)^{\frac{1}{2}} |S_k^*| + (8\pi e)^{-\frac{1}{2}} \frac{k}{n-k}$$

where the last inequality follows, since

$$|\phi(u - v) - \phi(u)| \leqq (2\pi)^{-\frac{1}{2}}|v| , \quad u, v \in \mathbb{R}$$

and

$$\left|\phi\left(\left(\frac{n}{n-k}\right)^{\frac{1}{2}} t\right) - \phi(t)\right| \leqq (2\pi)^{-\frac{1}{2}}|t|e^{-t^2/2}\left(\left(\frac{n}{n-k}\right)^{\frac{1}{2}} - 1\right)$$
$$\leqq (2\pi e)^{-\frac{1}{2}}\left(\left(\frac{n}{n-k}\right)^{\frac{1}{2}} - 1\right) \leqq (8\pi e)^{-\frac{1}{2}} \frac{k}{n-k} .$$

Now (i) and (ii) imply the assertion.

COROLLARY 1. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *with* $P(X_k) = 0$, $P(X_k^2) = 1$ *and* $P(|X_k|^q) < \infty$ *for some* $q \geqq 3$. *Then for each* r *with* $2 \leqq r \leqq q$ *there exists a constant* c_r *such that for all* $B \in \mathscr{F}_k \equiv \sigma(X_1, \cdots, X_k)$ *with* $P(B) > 0$

$$\sup_{t \in \mathbb{R}} |P(S_n^* < t \mid B) - \Phi(t)| \leqq c_r(P(B))^{-1/r}\left(\frac{k}{n}\right)^{\frac{1}{2}} .$$

PROOF. Let w.l.o.g. $k \leqq (n/2)$. We have according to the Hölder inequality

$$\sup_t |P(S_n^* < t, B) - \Phi(t)P(B)|$$
$$= \sup_t |P([P(S_n^* < t \mid \mathscr{F}_k) - \Phi(t)]1_B)|$$
$$\leqq P(B)^{1-1/r} \sup_t P(|P(S_n^* < t \mid \mathscr{F}_k) - \Phi(t)|^r)^{1/r} .$$

Hence it suffices to prove

$$(^{++}) \qquad ||\sup_t |P(S_n^* < t \mid \mathscr{F}_k) - \phi(t)| ||_r \leqq c_r \left(\frac{k}{n}\right)^{\frac{1}{2}} .$$

Since $\sup_{k \in \mathbb{N}} ||S_k^*||_r < \infty$ according to Doob [1], page 225, $(^+)$ follows from Theorem 1 using the triangle inequality and the theorem of Berry–Esseen.

REMARK. It is not possible to obtain in Corollary 1 an inequality of the form

$$(*) \qquad \sup_t |P(S_n^* < t \mid B) - \Phi(t)| \leqq d\left(\frac{k}{n}\right)^{\frac{1}{2}}$$

where d is a constant not depending on $B \in \mathscr{F}_k$: If for instance $P(X_1 < t) < 1$ for all t, then $\lim_{s \to \infty} P(S_n^* > 0 \mid X_1 > s) = 1$ which contradicts $(*)$.

COROLLARY 2. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *with* $P(X_k) = 0$ *and* $P(X_k^2) = 1$. *Let* $k(n)$ *be a sequence of integers with* $k(n) \log\log k(n)/n \to 0$, *then*

$$\text{(i)} \qquad \lim_{n \to \infty} \sup_{t \in \mathbb{R}} |P(S_n^* < t \mid \mathscr{F}_{k(n)}) - \Phi(t)| = 0 \quad \text{a.s.}$$

and if $P(|X_k|^3) < \infty$, *then*

(ii) $$\left(\frac{n}{k(n)\log\log k(n)}\right)^{\frac{1}{2}} \sup_{t\in\mathbb{R}} |P(S_n^* < t \mid \mathscr{F}_{k(n)}) - \Phi(t)|$$

is a.s. *bounded.*

PROOF. (i) Since $k(n)/n \to 0$, the central limit theorem implies

$$\lim_{n\to\infty} \sup_t |F_{n-k(n)}(t) - \phi(t)| = 0 .$$

Since $k(n)\log\log k(n)/n \to 0$, the law of the iterated logarithm implies

$$(2\pi)^{-\frac{1}{2}}\left(\frac{k(n)}{n-k(n)}\right)^{\frac{1}{2}} |S_{k(n)}^*| \to 0 \quad \text{a.s.}$$

The assertion follows now from Theorem 1.

(ii) Since $k(n)/n \to 0$, the theorem of Berry–Esseen implies

$$\left(\frac{n}{k(n)\log\log k(n)}\right)^{\frac{1}{2}} \sup_t |F_{n-k(n)}(t) - \phi(t)| \leqq c\left(\frac{n}{k(n)\log\log k(n)}\right)^{\frac{1}{2}} n^{-\frac{1}{2}} .$$

Since $k(n)\log\log k(n)/n \to 0$, the law of the iterated logarithm implies that

$$\left(\frac{n}{k(n)\log\log k(n)}\right)^{\frac{1}{2}} (2\pi)^{-\frac{1}{2}}\left(\frac{k(n)}{n-k(n)}\right)^{\frac{1}{2}} |S_k^*| \leqq c(\log\log k(n))^{-\frac{1}{2}}|S_k^*|$$

is a.s. bounded.

The assertion follows now from Theorem 1.

We also obtain as a corollary the conditioned central limit theorem of Rényi (see [2]).

COROLLARY 3. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *with* $P(X_k) = 0$ *and* $P(X_k^2) = 1$. *Let* $B \in \mathscr{A}$ *be a set with* $P(B) > 0$, *then*

$$\lim_{n\to\infty} P(S_n^* < t \mid B) = \phi(t) .$$

PROOF. Let $\mathscr{F}_\infty = \sigma(X_n : n \in \mathbb{N})$. There exist $\mathscr{F}_n$-measurable functions φ_n with $0 \leqq \varphi_n \leqq 1$ and

$$\lim_{n\to\infty} P(|P(B \mid \mathscr{F}_\infty) - \varphi_n|) = 0 .$$

Let $\varepsilon > 0$ be given there exists $k \in \mathbb{N}$ with

$$P(|P(B \mid \mathscr{F}_\infty) - \varphi_k|) < \frac{\varepsilon}{4} .$$

Using Theorem 1 we obtain therefore

$$\begin{aligned} |P(S_n^* < t, B) - \phi(t)P(B)| &\leqq |P(S_n^* < t, P(B \mid \mathscr{F}_\infty)) - P(S_n^* < t, \varphi_k)| \\ &\quad + |P(S_n^* < t, \varphi_k) - \phi(t)P(\varphi_k)| + |\phi(t)(P(\varphi_k) - P(B))| \\ &\leqq \frac{\varepsilon}{2} + P(|P(S_n^* < t \mid \mathscr{F}_k) - \phi(t)|\varphi_k) \end{aligned}$$

$$\leqq \frac{\varepsilon}{2} + \|P(S_n^* < t \mid \mathscr{F}_k) - \phi(t)\|_2$$

$$\leqq \frac{\varepsilon}{2} + \sup_t |F_{n-k}(t) - \phi(t)| + (2\pi)^{-\frac{1}{2}} \left(\frac{k}{n-k}\right)^{\frac{1}{2}} + (8\pi e)^{-\frac{1}{2}} \frac{k}{n-k} \leqq \varepsilon$$

for sufficiently large n, using the central limit theorem.

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