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TIMID PLAY WHEN LARGE BETS ARE PROFITABLE

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The total variation of a simple, symmetric random walk with absorbing barrier at zero, is stochastically larger than the total variation of any other nonnegative, integer-valued supermartingale with the same initial position. This strengthens a result of David Freedman on the optimality of *timid play* for maximizing the time to bankruptcy in certain gambling situations.

Let $S = (S_0, S_1, \cdots)$ be a sequence of random variables and let $X_i = S_i - S_{i-1}$ for $i \geqq 1$. Define the total variation of S by

$$V(S) = \sum_{i=1}^{\infty} |X_i| .$$

To state the main result of this note, associate with every positive integer k the collection $M(k)$ of all nonnegative, integer-valued supermartingales S which start at $S_0 = k$. Let $T(k)$ be that member of $M(k)$ which corresponds to a simple, symmetric random walk with absorbing barrier at 0. Recall that a random variable X is stochastically larger than a random variable Y if $P[X \geqq t] \geqq P[Y \geqq t]$ for every real number t.

THEOREM 1. *When S varies over $M(k)$, the total variation $V(S)$ is stochastically maximized at $S = T(k)$.*

To relate Theorem 1 to a result of Freedman [2], let S be in $M(k)$ and denote by $\tau(S)$ the first time, if ever, at which S reaches 0. The supermartingale S is said to have *no pauses* if all its increments are almost surely nonzero prior to the absorption time $\tau(S)$. For such an S, $\tau(S)$ is known to be almost surely finite.

COROLLARY (Freedman [2]). *Let S be in $M(k)$. If S has no pauses, then $\tau(T(k))$ is stochastically larger than $\tau(S)$.*

PROOF. Observe that $\tau(S) \leqq V(S)$ and conclude from Theorem 1 that $V(T(k))$ is stochastically larger than $\tau(S)$. However, $V(T(k)) = \tau(T(k))$. □

In gambling terms, Freedman's result says that for a subfair gambling situation on the nonnegative integers, timid play stochastically maximizes playing time. Theorem 1 strengthens this result by asserting that even though large

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bets yield a bigger immediate reward in terms of variation, timid play is still optimal.

PROOF OF THEOREM 1. It is convenient to introduce, for each $v \geqq 0$, the following gambling problem.

On the fortune space $F = \{(k, c): k = 0, 1, \cdots; c \geqq 0\}$, define the utility function u by

$$u(k, c) = \begin{cases} 0 & c < v \\ 1 & c \geqq v, \end{cases}$$

and the set $\Gamma(k, c)$ of available gambles at (k, c) in F, as the set of all distributions of $(k + X, c + |X|)$, where X is any integer-valued random variable with nonpositive mean and support in $[-k, \infty)$. Heuristically, a gambler in state k with cash c, when choosing the lottery X, moves to the new state $k + X$ and his cash becomes $c + |X|$. The gambler's goal, as reflected by his utility function u, is to obtain cash of at least v. How should he play so as to maximize the probability of reaching his goal? One available strategy is *timid play*, which selects at each stage before absorption at 0, the lottery corresponding to the random variable X which assumes the values ± 1 with equal probabilities. The process $T(k)$ in $M(k)$ can thus be viewed as the sequence of states of a gambler with initial state k who plays timidly, while $V(T(k))$ corresponds to the additional cash accumulated under the timid strategy. Furthermore, every S in $M(k)$ can be generated, in a similar way, by a strategy available in this gambling problem, and $V(S)$ will then correspond to the additional cash resulting from this strategy. Thus Theorem 1 is equivalent to the optimality of timid play for the gambling problem. To prove this optimality, let

$$Q(k, c) = P[c + V(T(k)) \geqq v] \tag{1}$$

be the probability of reaching the goal by playing timidly from the initial fortune (k, c) in F.

Another strategy available to the gambler at the fortune (k, c) consists of selecting some X at the first stage, and then, from the new position $k + X$, with the new cash $c + |X|$, playing timidly. The probability of attaining the goal (cash $\geqq v$) with such a strategy is $EQ(k + X, c + |X|)$. If timid play is optimal, then, clearly,

$$EQ(k + X, c + |X|) \leqq Q(k, c)\,. \tag{2}$$

Conversely, if (2) holds then, since obviously $Q \geqq u$, Theorem 2.12.1 of Dubins and Savage [1] applies to show that timid play is indeed optimal. To prove Theorem 1, it thus suffices to establish (2) for all (k, c) in F and all integer-valued random variables $X \geqq -k$ with $EX \leqq 0$.

The proof of (2) proceeds in two stages. First (2) is reduced to two-valued X and then it is proved for such X. For the reduction stage, recall that every distribution with finite mean is an average of two-point distributions with the same mean. In fact, it follows easily from Freedman [3, Lemma (108), page 68], that given a random variable X with finite mean and support in $K = \{-k, -k+1, -k+2, \cdots\}$, there is a sequence, $\{X_t;\, t = 1, 2, \cdots\}$, of two-valued random variables with support in K and the same mean as X, and a probability measure μ on $\{1, 2, \cdots\}$ such that,

$$Eg(X) = \int Eg(X_t)\, d\mu(t) \tag{3}$$

for all bounded, real-valued functions g defined on K. In particular, (3) holds for all the functions $g = g_{k,c}$, $(k, c) \in F$, defined by $g(x) = Q(k + x, c + |x|)$, $x \in K$. Thus it suffices to prove (2) for two-valued X.

Let, therefore, X be a two-valued random variable with support in K and $EX \leqq 0$. Suppose $X = -a$ with probability α and $X = b$ with the complementary probability $\beta = 1 - \alpha$, where a, b are integers, $0 \leqq a \leqq k$ and $0 \leqq b$. Since $EX \leqq 0$, $\alpha = b/(a+b) + \delta$ and $\beta = a/(a+b) - \delta$ for some $\delta \geqq 0$. Consequently,

$$\begin{aligned} EQ(k + X, c + |X|) &= \alpha Q(k-a, c+a) + \beta Q(k+b, c+b) \\ &= \frac{b}{a+b} Q(k-a, c+a) + \frac{a}{a+b} Q(k+b, c+b) \\ &\quad - \delta(Q(k+b, c+b) - Q(k-a, c+a))\,. \end{aligned} \tag{4}$$

For $k, l \geqq 0$, almost every path of $T(k+l)$ visits k. Also, the time to reach k from $k + l$ is at least l. It follows that $\tau(T(k+l)) \equiv V(T(k+l))$ stochastically majorizes $\tau(T(k)) + l \equiv V(T(k)) + l$. In particular, $V(T(k+b))$ is stochastically larger than $V(T(k-a)) + (a+b)$. Hence, $Q(k+b, c+b) \geqq Q(k-a, c+a)$ and so, by (4),

$$\begin{aligned} &\frac{b}{a+b} Q(k-a, c+a) + \frac{a}{a+b} Q(k+b, c+b) \\ &\qquad \geqq EQ(k+X, c+|X|)\,. \end{aligned} \tag{5}$$

Let τ be the first time for which the process $T(k)$ reaches either $k - a$ or $k + b$ and denote by s_τ the state of the process at time τ. Let E_a be the event that $s_\tau = k - a$ and let E_b be the event that $s_\tau = k + b$. Now calculate

$$\begin{aligned} Q(k, c) &= P[V(T(k)) \geqq v - c] \\ &= P[V(T(s_\tau)) \geqq v - c - \tau] \\ &= \frac{b}{a+b} P[V(T(s_\tau)) \geqq v - c - \tau \mid E_a] \\ &\quad + \frac{a}{a+b} P[V(T(s_\tau)) \geqq v - c - \tau \mid E_b] \\ &\geqq \frac{b}{a+b} P[V(T(k-a)) \geqq v - c - a] \\ &\quad + \frac{a}{a+b} P[V(T(k+b)) \geqq v - c - b] \\ &= \frac{b}{a+b} Q(k-a, c+a) + \frac{a}{a+b} Q(k+b, c+b)\,. \end{aligned} \tag{6}$$

The first equality is by (1); the second holds because $V(T(k))$ has the same distribution as $\tau + V(T(s_\tau))$; the third is true because s_τ has the same distribution as $k + Y$, where Y is the two-valued random variable with mean zero and values $-a$ and b; the inequality uses the strong Markov property of the process $T(k)$ together with the facts that $\tau \geqq a$ on E_a and $\tau \geqq b$ on E_b; the final equality is again by (1). Inequality (2) now follows from (6) and (5). This completes the proof of Theorem 1.

REMARK. For nonnegative integers $k \leqq n$, let $M_n(k)$ be the collection of all *martingales* which start at k and have their values in $\{0, 1, \cdots, n\}$. Theorem 1 and its corollary remain valid if in their statements $M(k)$ is replaced by $M_n(k)$ while at the same time $T(k)$ is modified so as to be absorbed at n and at 0. In particular, for S in $M_n(k)$, $EV(S) \leqq EV(T(k))$. This assertion would be false if supermartingales on the grid $\{0, 1, \cdots, n\}$ were included in $M_n(k)$.

REFERENCES

[1] DUBINS, L. E. and SAVAGE, L. J. (1965, 1976). *How to Gamble if You Must: Inequalities for Stochastic Processes.* Dover, New York.

[2] FREEDMAN, DAVID A. (1967). Timid play is optimal. *Ann. Math. Statist.* **38** 1281–1283.

[3] FREEDMAN, DAVID A. (1971). *Brownian Motion and Diffusion.* Holden-Day, San Francisco.

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