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A SQUARE FUNCTION INEQUALITY

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For martingales $f \in L_p$ $(2 \leqq p < \infty)$ the inequality $\|Mf\|_p \leqq (p + 1)\|Sf\|_p$ is proved, where $Mf = \sup_n |f_n|$ is the maximal function and $S^2 = \sum_n |f_n - f_{n-1}|^2$ the martingale square function. For integer p the estimate becomes $\|Mf\|_p \leqq p\|Sf\|_p$.

Let $(F_n,\ n \in Z)$ be a stochastic base over the σ-finite measure space (X, F, μ), where $\bigcup_n F_n$ is dense in F and $\bigcap_n F_n$ is denoted by $F_{-\infty}$. For simplicity we denote by E_t the conditional expectation operator given the field F_t.

Consider a martingale $(f_n)_{n \in Z}$ with $f_{-\infty} = 0$ and $f_\infty \in L_p$, $2 \leqq p < \infty$. Denote by Mf and Sf its respective maximal and square function. The inequality

$$\|Mf\|_p \leqq 2p\|Sf\|_p$$

is well known [1]. But, as is seen in [2], the coefficient can be improved, and this is the purpose of this note.

In the above mentioned paper, C. Herz proves that $\|Mf\|_p \leqq C_p\|Sf\|_p$, where $C_p \leqq 2(p-1)$ and p/C_p is bounded as $p \to \infty$. We will get constant C_p satisfying $C_p/p \to 1$ as $p \to \infty$. The exact relation $C_p = p$ holds for integer p and our proof is strong circumstantial evidence for the following:

CONJECTURE. $\|Mf\|_p \leqq p\|Sf\|_p$, $2 \leqq p < \infty$.

The constant presented here behaves like $p + k/p$. First we have the following algebraic facts:

LEMMA. *Let* $p = l + 2 + r$, $0 < r \leqq 1$, l *nonnegative integer*.

(i) *If* $a, b \geqq 0$ *then*

$$a^p - b^p - p(a-b)b^{p-1} \leqq (a-b)^2 a^{p-2} + (a-b)^2 \textstyle\sum_{i=0}^{l} (p-1-i)a^i b^{p-2-i}\,.$$

(ii) *If* $a, b \in R$ *then*

$$|a|^p - |b|^p - p(a-b)b|b|^{p-2} \leqq |a-b|^2|a|^{p-1} + |a-b|^2 \textstyle\sum_{i=0}^{l} (p-1-i)|a|^i|b|^{p-2-i}\,.$$

(iii) *If* $q = p/(p-1)$ *then*

$$\textstyle\sum_{i=0}^{l} (p-1-i)q^{p-2-i} = q^r \dfrac{p-1-r}{q-1}\,.$$

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These relations carry the proof for real-valued martingales. The proof extends for the Banach-valued case, where one uses $a, b \in X$ (some reflexive Banach space X) and $\theta \in X'$ is the Mazur functional of b to write in (ii) instead of $(a - b)b|b|^{p-2}$ the expression $\operatorname{Re}(\theta(a) - |b|)|b|^{p-1}$. The computation gets too elaborate to be worth while reading through.

PROPOSITION. *In the above conditions*

$$C_p^2 \leqq q^2\left[1 + q^r \frac{p-1-r}{q-1}\right].$$

PROOF. We only have to consider martingales of form

$$f = \sum_{l \leqq n \leqq m} (f_n - f_{n-1}),$$

$$\begin{aligned} A_n &= \int (|f_n|^p - |f_{n-1}|^p)\, d\mu = \int (|f_n|^p - |f_{n-1}|^p - p(f_n - f_{n-1})f_{n-1}|f_{n-1}|^{p-2})\, d\mu \\ &\leqq \int (f_n - f_{n-1})^2 [|f_n|^{p-2} + \textstyle\sum_{i=0}^{l} (p-1-i)|f_n|^i |f_{n-1}|^{p-2-i}]\, d\mu \\ &\leqq \int (S_n^2 - S_{n-1}^2)(|f|^{p-2} + \textstyle\sum_{i=0}^{l} (p-1-i)|f|^i M_{n-1}^{p-2-i})\, d\mu, \\ \|f\|^p &= \textstyle\sum_n A_n \leqq \int S^2(|f|^{p-2} + \sum_{i=0}^{l} (p-1-i)|f|^i M^{p-2-i})\, d\mu \\ &\leqq \|S\|^2(\|f\|^{p-2} + \textstyle\sum_{i=0}^{l} (p-1-i)\|f\|^i \|M\|^{p-2-i}). \end{aligned}$$

Hence

$$\begin{aligned} \left(\frac{\|f\|}{\|S\|}\right)^2 &\leqq 1 + \textstyle\sum_{i=0}^{l} (p-1-i)\left(\frac{\|M\|}{\|f\|}\right)^{p-2-i} \\ &\leqq 1 + q^r \frac{p-1-r}{q-1}. \end{aligned}$$

□

In case of $r = 1$ we get

$$C_p^2 = q^2\left(1 + q\frac{p-2}{q-1}\right) = q^2(1 + p^2 - 2p) = p^2$$

and for $r = 0$

$$C_p^2 = q^2\left(1 + \frac{p-2}{q-1}\right) = p^2 + q^2.$$

Therefore the result is better than $p + 1$ if $p \geqq 3$. For the case $2 < p < 3$ there are various ways to get $C_p < p + 1$, for example, the following Neveu type argument.

Let t be the stopping time $t = \inf\{s, M_s f > a\}$.

$$\begin{aligned} \int \{|f| > a\}(|f| - a)^2\, d\mu &\leqq \int \{|f| > a\}\{t < \infty\}(|f| - |f_{t-}|^2)\, d\mu \\ &\leqq \int\{t < \infty\}|f - f_{t-}|^2\, d\mu \leqq \int \{M > a\} S^2\, d\mu. \end{aligned}$$

We integrate $\int_0^\infty a^{p-3}\, da$ to get

$$\frac{2}{p(p-1)(p-2)} \int |f|^p\, d\mu \leqq \frac{1}{p-2} \int S^2 M^{p-2}\, d\mu$$

or

$$\|f\|^2 \leqq \frac{p(p-1)}{2} \|S\|^2 \left(\frac{\|M\|}{\|f\|}\right)^{p-2}$$

which gives

$$C_p^{\,2} \leqq \frac{p^2}{2} q^{p-1} .$$

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