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INTERPOLATION OF MARTINGALES

By David Heath

Cornell University

We show that every discrete-time martingale can be interpolated to give a continuous, continuous-time martingale, and provide a necessary and sufficient condition for the existence of an interpolated martingale with no flat spots.

1. Introduction. This note presents two results; they represent slight generalizations of Theorems 11.1 and 11.2 of Chacon [2], but the proofs presented here are quite different from those of [2].

THEOREM 1. *Suppose* $(X_k, \mathscr{F}_k, k = 0, 1, \cdots)$ *is a martingale on* $(\Omega, \mathscr{F}, P)$. *There is then (on a possibly enlarged version of* $(\Omega, \mathscr{F}, P)$*) a martingale* $(Y_t, \mathscr{G}_t, t \geqq 0)$ *such that*:

(i) Y_t *has continuous sample paths*

and

(ii) *For* $k = 0, 1, \cdots, Y_k = X_k$ *and* $\mathscr{F}_k \subseteqq \mathscr{G}_k$.

THEOREM 2. (i) *If* $P\{X_{k+1} \neq X_k \mid \mathscr{F}_k\}$ *is zero with positive probability, then any martingale* $(Y_t, \mathscr{G}_t, t \geqq 0)$ *satisfying the conclusion of Theorem* 1 *must, with positive probability, be constant in t for* $t \in [k, k+1]$.

(ii) *If* $P\{X_{k+1} \neq X_k \mid \mathscr{F}_k\} > 0$ a.e. *for all* $k = 0, 1, \cdots,$ *then almost every path of the martingale* $(Y_t, \mathscr{G}_t, t \geqq 0)$ *constructed in the proof of Theorem* 1 *has no intervals of constancy*.

REMARKS. 1. It is interesting to let $(X_k, \mathscr{F}_k, k = 0, 1, \cdots)$ be a simple random walk, for then the distribution of Y_t, as constructed, is absolutely continuous if t is not an integer and is discrete if t is an integer.

2. The above results, together with the representation of continuous martingales as time-changed Brownian motion (see [3]) yield another construction for the "Skorokhod embedding."

2. Proofs of the theorems. To construct the martingale $(Y_t, \mathscr{G}_t, t \geqq 0)$ for the proof of Theorem 1, we assume (by construction of a product space if necessary which we then rename Ω) that there are countably many standard Brownian motions (B_t^k), $k \geqq 0$ on $(\Omega, \mathscr{F}, P)$, independent of each other and of $\bigvee_{k \geqq 0} \mathscr{F}_k$. Let ϕ be any continuously differentiable function on (0, 1] for which $\phi(0+) = +\infty$ and $\phi(1) = 0$, with $\phi' < 0$ (for example, take $\phi(s) = s^{-1} - 1$), and set

$$\mathscr{G}_k = \mathscr{F}_k \vee \sigma\{B_s^j : 0 \leqq s < \infty, 0 \leqq j < k\} \qquad \text{if } k = 0, 1, 2, \cdots$$

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and

$$\mathscr{G}_t = \mathscr{G}_{[t]} \vee \sigma\{X_{k+1} + B^k_{\phi(s)}, 0 < s \leqq t - [t]\} \qquad \text{if } t \geqq 0, \ t \text{ not an integer.}$$

($[t]$ denotes the greatest integer not exceeding t.)

Let Y_t be a separable version of the process given by $E(X_{[t]+1} \mid \mathscr{G}_t)$. (The subscript on X is unimportant provided it is an integer not less than t.)

Clearly $(Y_t, \mathscr{G}_t, t \geqq 0)$ satisfies condition (ii) of Theorem 1. To check (i) and to investigate "intervals of constancy," we reduce the problem to the following special case: consider the martingale $(X_k, \mathscr{F}_k, k = 0, 1)$ where X_0 is identically 0 and $\mathscr{F}_0$ is trivial. Since we shall use only one standard Brownian motion (assumed independent of $\mathscr{F}_1$), we eliminate the superscript.

In this case,

$$\begin{aligned} Y_t &= 0 && \text{if } \ t = 0 \\ &= E(X_1 \mid X_1 + B_{\phi(s)} : 0 < s \leqq t) && \text{for } \ t \in (0, 1]\,. \end{aligned}$$

Since $\sigma(X_1 + B_{\phi(s)}, 0 < s \leqq t) = \sigma(X_1 + B_{\phi(t)}) \vee \sigma(B_{\phi(s)} - B_{\phi(t)}, 0 < s \leqq t)$ and $\sigma(B_{\phi(s)} - B_{\phi(t)}, 0 < s \leqq t)$ is independent of X_1 and $X_1 + B_{\phi(t)}$, we obtain $Y_t = E(X_1 \mid X_1 + B_{\phi(t)})$ for $t \in (0, 1]$, and thus by an elementary computation $Y_t = H(t, X_1 + B_{\phi(t)})$ for $t \in (0, 1)$, where H is defined by

$$H(t, z) = \frac{\int_{-\infty}^{\infty} x \exp(-(z - x)^2/2\phi(t))\, dF(x)}{\int_{-\infty}^{\infty} \exp(-(z - x)^2/2\phi(t))\, dF(x)} \tag{1}$$

where F is the distribution function of X_1.

This formula shows that Y_t is continuous for all $t \in (0, 1)$. Clearly $Y_1 = X_1$; we must show that $\lim_{t\uparrow 1} Y_t = X_1$. Since (Y_t) is a uniformly integrable martingale it has a limit as $t \uparrow 1$; this limit is $E(X \mid \bigvee_{t<1} \sigma(X_1 + B_{\phi(s)} : 0 < s \leqq t)$; from the continuity of Brownian motion at 0, we conclude that $X_1 + B_{\phi(1)} \equiv X_1$ is measurable with respect to the conditioning σ-field, so $\lim_{t\uparrow 1} Y_t = X_1$.

A similar argument, using the fact that Y_t is a reverse-martingale as $t \downarrow 0$ and that $\bigcap_{t>0} \sigma(X + B_{\phi(s)} : 0 < s \leqq t)$ is trivial gives $\lim_{t\downarrow 0} Y_t = 0$.

To reduce the general case to this special case, it is clear that we need consider only the time-interval $[0, 1]$. Then $Y_t = X_0 + E(X_1 - X_0 \mid \mathscr{F}_0 \vee \sigma\{B_{s-1-1} : 0 < s \leqq t\})$. Call the second term on the right Z_t. We now construct a martingale with the same finite-dimensional distributions as Z_t which is continuous. This implies that Z_t is also (since we, of course, choose a separable version).

Let B_t be a Brownian motion on $(\Omega', \mathscr{F}', P')$ and set

$$\Omega'' = \Omega \times \mathbb{R} \times \Omega'$$
$$\mathscr{F}'' = \mathscr{F}_0 \times \mathscr{B} \times \mathscr{F}'\,,$$

where $\mathscr{B}$ denotes the Borel sets in $\mathbb{R}$. Construct P'' by:

$$P''(\Lambda_1 \times \Lambda_2 \times \Lambda_3) = \left(\int_{\Lambda_1} F(\omega, \Lambda_2)\, dP(\omega)\right) P'(\Lambda_3)$$

where $\Lambda_1 \in \mathscr{F}_0$, $\Lambda_2 \in \mathscr{B}$, $\Lambda_3 \in \mathscr{F}'$ and $F(\cdot, \cdot)$ is a regular conditional probability for $X_1 - X_0$ given $\mathscr{F}_0$ (see [1], page 264). For $\omega \in \Omega''$ we write $\omega = (\omega_1, \omega_2, \omega_3)$.

For each fixed ω_1 we have a probability measure P_{ω_1} on $(\mathbb{R} \times \Omega', \mathscr{B} \times \mathscr{F}')$ given by

$$P_{\omega_1}(\Lambda_2 \times \Lambda_3) = F(\omega_1, \Lambda_2)P'(\Lambda_3) .$$

Keeping ω_1 fixed, we have a random variable X_1 defined on $\mathbb{R} \times \Omega'$ by $X_1(\omega_2, \omega_3) = \omega_2$. Moreover $E(X_1) = 0$. We can therefore construct a continuous martingale Z_t' (for each fixed ω_1) as in the special case. It is then trivial to verify that as a function of $\omega = (\omega_1, \omega_2, \omega_3)$ Z_t' is a martingale, has continuous paths, and has the same finite-dimensional distributions as Z_t. Thus Z_t and hence Y_t have the desired properties. This completes the proof of Theorem 1.

To prove Theorem 2, set $\Lambda = \{\omega : P\{X_{k+1} \neq X_k \mid \mathscr{F}_k\}(\omega) = 0\}$; suppose that $P(\Lambda) > 0$. Let T be the stopping time defined by $T = k$ on $\Omega \backslash \Lambda$ and $T = k + 1$ on Λ. Set $Z_t = Y_{t \wedge T} - Y_k$ for $k \leqq t \leqq k + 1$, undefined elsewhere. Since $Z_{k+1} \equiv 0$, Z must always be zero.

Suppose now that $P\{X_{k+1} \neq X_k \mid \mathscr{F}_k\} > 0$ a.e. for $k = 0, 1, \cdots$. We must then show that Y_t has no intervals of constancy. As in the proof of Theorem 1, we need only consider the special case $(X_k, \mathscr{F}_k, k = 0, 1)$ where $X_0 \equiv 0$ and $\mathscr{F}_0$ is trivial; the hypothesis then reduces to the assumption that F is not concentrated at 0. Suppose now that for a particular ω, $Y_t(\omega) \equiv c$ on $(a, b) \subset (0, 1)$. This means that $H(t, z) - c \equiv 0$ for $t \in (a, b)$; we wish to use the implicit function theorem of [5], page 241 to show that z can then be written as a differentiable function of t (for some sub-interval of (a, b)), for then the Brownian path must have been differentiable, and the probability of this is 0 (see [4]). The only hypothesis of the implicit function theorem not obviously satisfied is that $\partial H/\partial z \neq 0$. Elementary calculations yield that $\partial H/\partial z = 0$ implies

$$\frac{\int_{-\infty}^{\infty} x^2 \exp(-(z - x)^2/2\phi(t))\, dF(x)}{\int_{-\infty}^{\infty} \exp(-(z - x)^2/2\phi(t))\, dF(x)} = \left(\frac{\int_{-\infty}^{\infty} x \exp(-(z - x)^2/2\phi(t))\, dF(x)}{\int_{-\infty}^{\infty} \exp(-(z - x)^2/2\phi(t))\, dF(x)}\right)^2$$

which is easily seen to imply that F is concentrated at a point—in contradiction to our assumption.

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DEPARTMENT OF OPERATIONS RESEARCH
CORNELL UNIVERSITY
COLLEGE OF ENGINEERING
UPSON HALL
ITHACA, NEW YORK 14853