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ON THE UNIQUENESS AND NONUNIQUENESS OF PROXIMITY PROCESSES

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We discuss the uniqueness of a class of infinite particle systems known as proximity processes, with the aid of certain "dual" Markov chains. By checking whether the dual "explodes," i.e., attempts infinitely many jumps in a finite time, and how it explodes when it does, it is possible in many cases to determine whether or not there is more than one particle system with given flip rates. We then use duality to find an example of a system which is uniquely determined by its flip rates, but whose generator is not the closure of the naive operator formed from these flip rates.

0. Introduction. Let $(\xi_t)_{t\in\mathbb{R}^+}$ be a continuous time spin system (see [10] or [3], for example) with configuration space $\Xi = \{-1, 1\}^V$, V a countable set. Each configuration $\xi = (\xi(x))_{x\in V}$ represents an assignment of $+$ and $-$ "spins" to the *sites* x of V. The dynamics of (ξ_t) are prescribed by means of *flip rates* $c = \{c_x(\xi);\, x \in V,\, \xi \in \Xi\}$: intuitively, $c_x(\xi)$ gives the instantaneous rate of change from $\xi(x)$ to $-\xi(x)$ at site x when the system has value ξ. Certain of these processes may be studied by constructing a "dual" Markov chain $(\hat{\xi}_t)_{t\in\mathbb{R}^+}$, on a suitable denumerable state space $\hat{\Xi}$, such that

$$\mathbb{E}_\xi[g_i(\xi_t)] = \hat{\mathbb{E}}_i[g_{\hat{\xi}_t}(\xi)]\,. \tag{1}$$

Here $\mathbb{E}_\xi$ and $\hat{\mathbb{E}}_i$ are the expectation operators for (ξ_t) starting in ξ and $(\hat{\xi}_t)$ starting in i respectively, and $\{g_i\}$ is a countable collection of continuous functions on Ξ which totally determine the evolution of (ξ_t). The resultant "duality theory" has been developed over the past several years, mainly by Vasershtein and Leontovich [18], Holley and Liggett [9], Harris [6], Holley and Stroock [12], and Holley, Stroock and Williams [13]. The systematic formulation of spin system duality is due to Holley and Stroock. In [9], [6], [12], [5] and [17], assorted special cases of (1) are used to derive *ergodic theorems* for various types of spin systems. Our object in this paper is to discuss the *uniqueness problem* for a special class of spin systems with duals, namely the *proximity processes* introduced by Holley and Liggett in [9]. It was mentioned in [10] that uniqueness of such a system having prescribed rates c is connected with whether or not $(\hat{\xi}_t)$ can "explode," i.e., jump infinitely often in a finite time.[1] With the aid of an exploding dual, the first example of two distinct Feller spin systems with the same rates

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[1] For want of a better term, we say that an "explosion" occurs at the time of the first bad discontinuity. This will not necessarily mean that the total number of particles in the dual has grown without bound.

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was constructed in [3]. More recently, Holley, Stroock and Williams [13] have used duality to discuss nonuniqueness of a class of diffusions with one boundary point on the d-dimensional torus. As an immediate consequence of the methods of [13], we show in Section 1 that a Markovian spin system (ξ_t) satisfying (1) with respect to the *multiplicative* $\alpha = 1$ *basis* $\{f_i\}$ and *branching process with interference* $(\hat{\xi}_t)$ is uniquely determined by its rates c provided that $(\hat{\xi}_t)$ does not explode. Sufficient conditions for uniqueness and ergodicity, due to Holley and Stroock [11], are easy corollaries.

The paper next discusses proximity processes (ξ_t) with exploding duals. It turns out that there are two types of explosion: "weak," and "strong," which must be distinguished. In Section 2 a *continuum* of Feller systems with the same rates is exhibited whenever (ξ_t) has a weakly exploding dual. As an illustration, "voter models" on $V = \mathbb{N} = \{0, 1, \cdots\}$ with rapidly exploding duals are mentioned. We show how to produce a wide range of behavior "at ∞" by exploiting the boundary theory of birth and death processes and its implications for the dual.

Then in Section 3 we show *uniqueness* for rates c which are bounded and bounded away from 0 if $(\hat{\xi}_t)$ has only a strong explosion. The property of spin systems known as the *strong extension property* (s.e.p.) is studied. We note that a process with a dual enjoys this property if the dual's expected total number of jumps is uniformly bounded. We next give a uniqueness example with a strongly exploding dual and *without* s.e.p. This example (Theorem 4), the main result of the paper, settles a problem from [3] and [16]. It shows that there is a pregenerator which uniquely determines a Markov spin system whose true generator is *not* the minimal closure of this pregenerator. Additional consequences of Theorem 4 are discussed at the end of Section 3.

1. Preliminaries. We outline the theory of spin systems, following most closely the notation in [3] and [5]. Write $\Xi = \{-1, 1\}^V$ (discrete product topology), $\xi = (\xi(x))_{x \in V} \in \Xi$, $\mathscr{B}_\Xi =$ the Borel σ-algebra on Ξ. $\mathscr{C}$ denotes the continuous functions on Ξ topologized by the supremum norm $\|\ \|$, and $\mathscr{F}$ consists of those functions in $\mathscr{C}$ which depend on only finitely many sites in V. Note that $\bar{\mathscr{F}} = \mathscr{C}$, i.e., $\mathscr{F}$ is dense in $\mathscr{C}$. Let $c = \{c_x(\xi); x \in V, \xi \in \Xi\}$ satisfy $0 \leqq c_x(\cdot) \in \mathscr{C}$ for each x. Given c, introduce the *pregenerator* $G : \mathscr{F} \to \mathscr{C}$ defined by

$$Gf(\xi) = \sum_{x \in V} c_x(\xi)[f({}_x\xi) - f(\xi)] ,$$

where

$$\begin{aligned} {}_x\xi(y) &= \xi(y) \qquad y \neq x \\ &= -\xi(x) \quad y = x . \end{aligned}$$

Put $\mathbb{D} =$ the right continuous functions with left limits from $\mathbb{R}^+ = [0, \infty)$ to Ξ, and let $\mathscr{B}$ be the usual σ-algebra on $\mathbb{D}$. A *spin system with rates* c is given by the collection $(\mathbb{D}, \mathscr{B}, \{\mathbb{P}_\xi\}_{\xi \in \Xi}, (\xi_t)_{t \in \mathbb{R}^+})$, where (ξ_t) is the canonical coordinate process with path space $\mathbb{D}$ and state space Ξ. The measures $\mathbb{P}_\xi$ should satisfy

$\mathbb{P}_\bullet(B)$ $\mathscr{B}_\Xi$-measurable for each $B \in \mathscr{B}$,

(2) $$\mathbb{P}_\xi(\xi_0 = \xi) = 1$$

and

(3) $$f(\xi_t) - \int_0^t Gf(\xi_s)\,ds \quad \text{is a } \mathbb{P}_\xi\text{-martingale}$$

for all $\xi \in \Xi$, $f \in \mathscr{F}$ (cf. [10]). In this paper we will be largely concerned with *Markov spin systems*, i.e., processes which satisfy the stronger condition:

> G extends to a generator G^e for a conservative Markov semigroup $(P^t)_{t \in \mathbb{R}^+}$,

and *Feller spin systems*, i.e., Markov spin systems such that

(4) $$P^t \quad \text{takes} \quad \mathscr{C} \quad \text{into itself.}$$

As is customary, we often suppress most of the structure and think of (ξ_t) as the spin system. Holley and Stroock proved in [10] that there is always at least one spin system (ξ_t) with (continuous) rates c, and that if there is a unique system (i.e., a unique collection $\{\mathbb{P}_\xi\}_{\xi \in \Xi}$ satisfying (2) and (3)), then (4) is automatic. What is more, a method of Krylov [14] (described in Remark (2.6) of [10]) shows that there is always a *Markov* (ξ_t) with given c, and that if there is only one, then there is only one spin system. A useful condition which guarantees uniqueness is

(5) $$\sup_{x \in V} \sum_{y \in V} \sup_{\xi \in \Xi} |c_x({}_y\xi) - c_x(\xi)| < \infty$$

(cf. [15], [3]).

Let $\mathscr{Y}_0$ denote the collection of finite sets $A \subset V$, the empty set $\varnothing$ included. Throughout this paper the letters A, B and Λ will always represent *finite* subsets of V, even when not explicitly identified as such. Endow $\mathscr{Y}_0$ with the discrete topology, and throw in an additional isolated point Δ to get $\hat{\Xi} = \mathscr{Y}_0 \cup \{\Delta\}$. This will be the canonical state space for the dual process $(\hat{\xi}_t)$. The path space is $(\hat{\mathbb{D}}, \hat{\mathscr{B}})$, defined analogously to $(\mathbb{D}, \mathscr{B})$. Measures for the dual will be $\hat{\mathbb{P}}_\xi$, expectations $\hat{\mathbb{E}}_\xi$. The corresponding function space is $\hat{\mathscr{C}} =$ the bounded functions on $\hat{\Xi}$. Define $f_x(\xi) = (\xi(x) + 1)/2$, $x \in V$, $\xi \in \Xi$. The *multiplicative* ($\alpha = 1$) *basis* $\mathscr{F}_0$ consists of all functions f_A, $A \in \mathscr{Y}_0$, given by $f_A = \prod_{x \in V} f_x$ ($f_\varnothing = 1$) (cf. [12]). Note that f_A is simply the indicator of "all $+1$'s on A." It is easy to see that the finite linear span of $\mathscr{F}_0$ is $\mathscr{F}$. The basic assumptions one makes of the flip rates c which give rise to a *proximity process* ([9], [12]) are that

(6) $$Gf_x(\xi) = \prod_{B \in \mathscr{Y}_0} q_{xB} f_B(\xi) \qquad q_{xB} \in \mathbb{R},\ q_{xB} \in \mathbb{R}^+ \ \text{for} \ B \neq x\,,$$

and that with $q_x = -q_{xx}$,

(7) $$\kappa_x = q_x - \sum_{B \neq x} q_{xB} \geqq 0 \qquad \text{for all} \quad x \in V\,.$$

(Here and below we confound x with $\{x\}$ whenever convenient.) Thus the rates

c have the form

$$c_x(\xi) = \frac{q_x}{2}\left[1 + \xi(x) - 2\xi(x) \textstyle\sum_{B\neq x} \left(\frac{q_{xB}}{q_x}\right) f_B(\xi)\right]. \tag{8}$$

Note that (8) implies $c_x(\cdot) \in \mathscr{C}$ for each fixed x.

Consider a process $(\hat{\xi}_t)$ on the state space $\hat{\Xi}$ with the following transition mechanism. If $\hat{\xi}_t = A$, then each $x \in A$ attempts to "branch" into a set B, $B \neq x$, at rate q_{xB}, and to send the entire system to Δ at rate κ_x. If the first "occupied" site x to attempt such a transition chooses B, then the sites of $B \cap (A - x)^c$ are added to the occupied set, and the sites of $B \cap (A - x)$ remain occupied. If the first x to jump chooses Δ, then the system is absorbed at Δ. Thus the states $\varnothing$ and Δ are traps. Define $f_\Delta = 0$. By imposing conditions on the rates c for a spin system (ξ_t) which ensure (5)—(8) and that the process (ξ_t) just described does not attempt infinitely many jumps in a finite time, it is shown in [9] and [12] that for all $\xi \in \Xi$, $A \in \mathscr{Y}_0$,

$$\mathbb{E}_\xi[f_A(\xi_t)] = \hat{\mathbb{E}}_A[f_{\hat{\xi}_t}(\xi)]. \tag{9}$$

This is the basic form of (1) we wish to consider. The process $(\hat{\xi}_t)$ is a *branching process with interference* (b.p.i.). Let $\hat{G}$ be its "Q-matrix." Then (6) may be rewritten as

$$Gf_x(\xi) = \hat{G}f_x(\xi), \tag{10}$$

where $\hat{G}$ operates in the *subscript* with ξ fixed. To obtain (9) one wants

$$Gf_A(\xi) = \hat{G}f_A(\xi) \qquad \text{for all} \quad A, \xi. \tag{11}$$

But it follows from the definition of G that

$$Gf_A = \textstyle\sum_{x\in A} (\prod_{y\in A;y\neq x} f_y)Gf_x. \tag{12}$$

Thus $\hat{G}f_A(\xi)$ can be computed using (10)—(12), and the matrix elements q_{AB} of $\hat{G}$ can be read off. Moreover, $\kappa_A = q_{A\Delta} = -q_{AA} + \sum_{B\neq A} q_{AB} = \sum_{x\in A} \kappa_x \geqq 0$, so $\hat{G}$ is a proper Q-matrix. In fact, the corresponding chain $(\hat{\xi}_t)$ is exactly the b.p.i. described above. Interference corresponds to the reduction of terms involving $[\xi(x)]^2 = 1$ in (12). For details, and the formulation of more general duality theory, the reader is referred to [12], [6], [5] and [17].

In this paper, a b.p.i. is *any* chain $(\hat{\xi}_t)$ on the state space $\hat{\Xi}$ and path space $(\hat{\mathbb{D}}, \hat{\mathscr{B}})$, governed by a Markov family $\{\hat{\mathbb{P}}_{\hat{\xi}}\}_{\hat{\xi}\in\hat{\Xi}}$, whose Q-matrix $\hat{G}$ is derived from rates c which satisfy (8) according to (10)—(12). Given a b.p.i. $(\hat{\xi}_t)$, define ζ_n = the time of the nth jump ($= \infty$ if no such jump exists), $n \geqq 1$. Introduce the *first explosion time* $\zeta = \lim_{n\to\infty} \zeta_n$. By elementary Markov chain theory, $\hat{G}$ uniquely determines the evolution of $(\hat{\xi}_t)$ up to time ζ. Let $\{\check{\mathbb{P}}_{\hat{\xi}}\}_{\hat{\xi}\in\hat{\Xi}}$ be the Markov family of measures for the *minimal process* with pregenerator $\hat{G}$, i.e., the process which is absorbed at Δ at time ζ. If $\check{\mathbb{P}}_{\hat{\xi}}(\zeta = \infty) = 1$ for all $\hat{\xi} \in \hat{\Xi}$, then the minimal process is the unique chain with Q-matrix $\hat{G}$

Our first theorem states that the pregenerator G with rates c uniquely determines (ξ_t) whenever the Q-matrix $\hat{G}$ for the b.p.i. uniquely determines $(\hat{\xi}_t)$. It relies on the following lemma, which gives a condition for uniqueness. Except for the setting, the proofs are virtually identical to those given in [13] that (7) and (8) of [13] imply uniqueness for the diffusions considered there. We therefore omit these arguments.

LEMMA 1. *Given flip rates* $c = \{c_x(\xi)\}$ *which satisfy* (7)—(8), *let* $\hat{G}$ *be the* Q*-matrix derived from* G, *and let* $(\hat{\xi}_t)$ *be the minimal* b.p.i. *Introduce* $(\Lambda_N)_{N=1}^{\infty}$, *an increasing sequence of sets in* $\mathscr{V}_0$ *which exhausts* V, *and define* $\sigma_N = \inf\{t\colon \hat{\xi}_t \cap \Lambda_N^c \neq \varnothing\}$. *If for every Markov spin system with rates* c,

$$(13)\qquad \lim_{N\to\infty} \check{\mathbb{E}}_{\hat{\xi}}[\mathbb{E}_{\xi}[f_{\hat{\xi}_{\sigma_N}}(\xi_{t-\sigma_N})], \sigma_N < t] = 0 \qquad \text{for all} \quad \hat{\xi}, \xi, t\,,$$

then there is a unique spin system with rates c, *and it is a Feller process whose generator extends* G.

(Note that it suffices to check (13) for Markov systems by Krylov's result.)

THEOREM 1. *Assume that* c *satisfies* (7)—(8), *and let* $(\hat{\xi}_t)$ *be the* b.p.i. *with* Q*-matrix* $\hat{G}$. *If*

$$\check{\mathbb{P}}_{\hat{\xi}}(\zeta = \infty) = 1 \qquad \text{for all} \quad \hat{\xi} \in \hat{\Xi}\,,$$

then there is a unique spin system with rates c, *and it is Feller.*

(To apply Lemma 1, note that $\lim_{N\to\infty} \sigma_N = \zeta$ a.s. since the jump rates are uniformly bounded on any given Λ_N.)

REMARK 1. Lemma 1 and Theorem 1 generalize in a straightforward manner to the α-duals of [12] and more general dual processes. These generalizations can be used to give probabilistic proofs of uniqueness and ergodic theorems. For example, consider the results in Sections 6 and 7 of [10]. Condition (6.3) there says that the dual jumps to Δ with at least a certain minimal positive probability after a finite holding time in any state which is not a trap. It is therefore trapped somewhere, whence $\zeta = \infty$ a.s. and uniqueness ensues by Theorem 1. If, in addition, $q_x > 0$ for all $x \in V$ (in our setting), then either $\varnothing$ or Δ is reached after a finite number of jumps. This implies ergodicity, as in [12], and corresponds to Theorem (7.4) of [10]. Finally, if we also have $\inf_{x\in V} \kappa_x = \kappa > 0$, then the ergodicity is exponential because the b.p.i. goes to Δ with at least rate κ from any state except $\varnothing$. An analogous argument yields Theorem (7.10) of [10], or the more general Theorem (1.8) of [11]. Another efficient method of obtaining these results may be found in [16].

A more intricate application of Lemma 1 will be given in Theorem 3.

Let us now turn to the question of nonuniqueness. Denote $\#A =$ the cardinality of A. When $\zeta < \infty$, write $\hat{\xi}_{\zeta-} = \lim_{s\to\zeta} \bigcap_{s<r<\zeta} \hat{\xi}_r$. Note that this set may have infinite cardinality, and that it equals $\lim_{r\to\zeta} \hat{\xi}_r$ by left limits of the path. Say that ζ is a *weak explosion time* for $(\hat{\xi}_t)$ if $\#\hat{\xi}_{\zeta-} < \infty$, and a *strong explosion* *time* if $\#\hat{\xi}_{\zeta-} = \infty$. The remainder of the paper is devoted to studying the consequences of these two types of explosion.

2. Weak explosions and nonuniqueness. The main result of this section, Theorem 2, shows that a weakly exploding b.p.i. always leads to nonuniqueness of the associated proximity process. The idea of using duality to get nonuniqueness examples was proposed in [10], and exploited in [3] and [13].

THEOREM 2. *Given rates c for a proximity process, let* $\{\check{\mathbb{P}}_{\hat{\xi}}\}$ *be the minimal Markov family with Q-matrix* $\hat{G}$. *If* $\check{\mathbb{P}}_{\hat{A}_0}(\zeta < \infty, \#\hat{\xi}_{\zeta-} < \infty) > 0$ *for some* $\hat{A}_0 \in \hat{\Xi}$, *then there is a continuum of distinct Feller spin systems with rates c. More precisely, let* Λ *be the maximal set in V such that* $\Lambda \subset \hat{\xi}_{\zeta-}$ $\mathbb{P}_{\hat{A}_0}$-a.s. *Then to each probability measure* π *on* $(\hat{\Xi}, \mathscr{B}_{\hat{\Xi}})$ *there corresponds a Feller system of path measures* $\{\mathbb{P}_\xi{}^\pi\}$ *for a process with rates c, and* $\{\mathbb{P}_\xi{}^\pi\} \neq \{\mathbb{P}_\xi{}^{\pi'}\}$ *if* $\pi_{|V-\Lambda} \neq \pi'_{|V-\Lambda}$.

The idea is to consider various extensions of $(\hat{\xi}_t)$ beyond ζ, and see with the aid of the duality equation (9) which of these give rise to a spin system. For clarity, we isolate the easy steps of the proof as a lemma.

LEMMA 2. *Given rates c for a proximity process, let* $(\hat{P}^t)$ *be the semigroup of a* b.p.i. $(\hat{\xi}_t)$ *with Q-matrix* $\hat{G}$, *where* $\hat{P}^t f_{\hat{\xi}}(\xi) = \hat{\mathbb{E}}_{\hat{\xi}}[f_{\hat{\xi}_t}(\xi)]$. *If*

$$\sum_{A \subset D \subset A \cup B} (-1)^{\#(D-A)} \hat{P}^t f_D(\xi) \geqq 0 \tag{14}$$

for all $A, B \in \mathscr{Y}_0$: $A \cap B = \emptyset$, $t \geqq 0$ *and* $\xi \in \Xi$, *then the duality equation*

$$P^t f_A(\xi) = \hat{P}^t f_A(\xi) \qquad A \in \mathscr{Y}_0,\ \xi \in \Xi\,, \tag{15}$$

gives rise to a well-defined Feller semigroup (P^t) *on* $\mathscr{C}$ *whose generator extends G, and hence to a Feller proximity process* (ξ_t) *with rates c.*

PROOF. There are seven steps; we omit some of the routine details.

(i) Define P^t on $\mathscr{F}_0$ by (15), and note that $P^t f_\emptyset(\xi) = \hat{P}^t f_\emptyset(\xi) = 1$ because $\emptyset$ is a trap for $(\hat{\xi}_t)$. Thus P^t is conservative.

(ii) Extend P^t linearly from $\mathscr{F}_0$ to $\mathscr{F}$.

(iii) Check that P^t is positive on $\mathscr{F}$. To do this, note that any $f \geqq 0$ in $\mathscr{F}$ is a positive linear combination of functions $\chi_{AB}(\xi)$ = the indicator of "all 1's on A, all -1's on B," $A, B \in \mathscr{Y}_0$, $A \cap B = \emptyset$. By Möbius inversion,

$$\chi_{AB} = \sum_{A \subset D \subset A \cup B} (-1)^{\#(D-A)} f_D\,.$$

so (14) ensures that $P^t \chi_{AB} \geqq 0$. The claim follows.

(iv) Extend P^t uniquely to $\mathscr{C}$ by approximation. This is possible because P^t is a contraction on $\mathscr{F}$ by (i)—(iii).

(v) Verify that (P^t) has the semigroup property by computing

$$P^{s+t} f_A(\xi) = \hat{P}^{s+t} f_A(\xi) = \hat{P}^t \hat{P}^s f_A(\xi) = \hat{P}^t P^s f_A(\xi) = P^s \hat{P}^t f_A(\xi) = P^s P^t f_A(\xi)\,,$$

and then extending from $\mathscr{F}_0$ to $\mathscr{F}$ to $\mathscr{C}$ by approximation.

(vi) Show that the generator for (P^t) extends G, i.e., that the Markov spin system (ξ_t) with semigroup (P^t) has flip rates c. It suffices to check that

$$\left|\frac{\hat{P}^h f_A(\xi) - f_A(\xi)}{h} - \hat{G}f_A(\xi)\right| \downarrow 0 \quad \text{as} \quad h \downarrow 0 \quad \text{uniformly in} \quad \xi \quad \text{for each} \quad A\,.$$

For fixed ξ and A the limit is 0 because $(\hat{P}^t)$ has Q-matrix $\hat{G}$. The uniformity in ξ is easily verified.

(vii) Note that (P^t) is Feller. To see this, let ξ^B be the modification of $\xi \in \Xi$ given by

$$\begin{aligned} \xi^B(x) &= \xi(x) \quad & x \in B \\ &= 1 & \text{else}\,, \end{aligned}$$

and observe that

$$|P^t f_A(\xi) - P^t f_A(\xi^B)| = \hat{\mathbb{E}}_A[f_{\hat{\xi}_t}(\xi) - f_{\hat{\xi}_t}(\xi^B)] \to 0$$

as $B \uparrow V$ by bounded convergence. Therefore P^t takes $\mathscr{F}_0$ into $\mathscr{C}$, and hence (4) holds. The construction of $\{\mathbb{P}_\xi\}$ from (P^t) is standard.

Proof of Theorem 2. Given a probability measure π on $(\hat{\Xi}, \hat{\mathscr{B}}_{\hat{\Xi}})$, define $\mathbb{P}_\xi{}^\pi$ as follows. Use the Q-matrix $\hat{G}$ to determine the evolution of $(\hat{\xi}_t)$ up to time ζ. If there is a strong explosion at ζ put $\hat{\xi}_\zeta = \Delta$. In case of a weak explosion, let $\hat{\xi}_\zeta = \hat{\xi}_{\zeta-} \cup \hat{\eta}_1$, where $\hat{\eta}_1 \in \hat{\Xi}$ is a random element chosen independently according to π. The possibility $\hat{\eta}_1 = \Delta$ is allowed; define $\Delta \cup \hat{\xi} = \Delta$ for all $\hat{\xi} \in \hat{\Xi}$. The dual now proceeds from $\hat{\xi}_\zeta$ according to $\hat{G}$ until another explosion occurs. A new independent $\hat{\eta}_2$ with the same distribution π is used to define the state at the second explosion time as before, and so on. If the explosion times have a finite limit point, then send the system to Δ at that time. The resulting family of canonical measures $\{\hat{\mathbb{P}}_{\hat{\xi}}^\pi\}$ is surely Markov with Q-matrix $\hat{G}$. According to Lemma 2, we need only check (14) in order to get a corresponding Feller spin system $\{\mathbb{P}_\xi{}^\pi\}$ with rates c. To this end, fix A, B, and construct a coupling of $\#(A \cup B)$ copies of $(\hat{\xi}_t)$ on a common probability space $(\tilde{\Omega}, \tilde{\mathscr{B}}, \tilde{\mathbb{P}})$, one copy starting from each $x \in A \cup B$. The mechanism of the coupling is quite simple: all copies of the dual which are occupied at a given site x use the same "exponential alarm clock" to decide when to attempt to jump. At explosion times, all copies with simultaneous weak explosions use the same π-distributed random state $\hat{\eta}$. Let $(\hat{\xi}_t{}^x)$ denote the copy starting from x, and define

$$\hat{\xi}_t{}^D = \bigcup_{x \in D} \hat{\xi}_t{}^x \quad D \in \mathscr{Y}_0\,. \tag{16}$$

The key observation is that $(\hat{\xi}_t{}^D)$ obeys the transition law of the b.p.i. starting from D. Letting $\tilde{\mathbb{E}}$ be the expectation operator for $\tilde{\mathbb{P}}$, the sum in (14) may therefore be rewritten as

$$\tilde{\mathbb{E}}[\textstyle\sum_{A \subset D \subset A \cup B} (-1)^{\#(D-A)} f_{\cup_{x \in D} \hat{\xi}_t^x}(\xi)] = \tilde{\mathbb{E}}[\textstyle\sum_{A \subset D \subset (A \cup B) \cap S_\Delta^c} (-1)^{\#(D-A)} f_{\cup_{x \in D} \hat{\xi}_t^x}(\xi)]\,,$$

where $S_\Delta = \{x : \hat{\xi}_t{}^x = \Delta\}$. If $A \cap S_\Delta \neq \varnothing$ the last term in brackets is 0. Otherwise, again by Möbius inversion, it is an indicator. Namely, it equals 1 on

$\{\xi: \xi(x) = 1$ for all $x \in \bigcup_{y \in A} \hat{\xi}_t^y\} \cap \{\xi: \xi(x) = -1$ for some $x \in \hat{\xi}_t^y$ whenever $y \in B \cap S_\Delta^c\}$, and equals 0 otherwise. Hence the expectation is nonnegative, as desired. To finish the proof, let A_0 and Λ be as in the statement of the theorem. Then $\hat{\mathbb{P}}^\pi_{A_0}$ uniquely determines $\pi_{|V-\Lambda}$. So if $\pi_{|V-\Lambda} \neq \pi'_{|V-\Lambda}$, we must have $\hat{\mathbb{P}}^\pi_{A_0}(A \subset \hat{\xi}_t) \neq \hat{\mathbb{P}}^{\pi'}_{A_0}(A \subset \hat{\xi}_t)$ for some $t \geqq 0$, $A \in \mathscr{V}_0$. Let $\xi_0 \in \Xi$ be the configuration with 1's on A and -1's on $V - A$ to deduce from (15) that

$$\mathbb{E}^\pi_{\xi_0}[f_{A_0}(\xi_t)] = \hat{\mathbb{E}}^\pi_{A_0}[f_{\hat{\xi}_t}(\xi_0)] \neq \hat{\mathbb{E}}^{\pi'}_{A_0}[f_{\hat{\xi}_t}(\xi_0)] = \mathbb{E}^{\pi'}_{\xi_0}[f_{A_0}(\xi_t)] \,.$$

Thus $\mathbb{P}^\pi_{\xi_0} \neq \mathbb{P}^{\pi'}_{\xi_0}$.

REMARK 2. If $V = \{1, 2, \cdots\}$, $q_{xx+1} = -q_{xx} = n^2$, then the nonuniqueness example in [3] consists of the Feller families $\{\mathbb{P}^{\pi^+}_\xi\}$ and $\{\mathbb{P}^{\pi^-}_\xi\}$, where $\pi^+(\varnothing) = \pi^-(\Delta) = 1$. The fact that the duality equation is positivity-preserving was treated much too lightly there.

When $(\hat{\xi}_t)$ has weak explosions, the continuations of Theorem 2 are by no means the only ones which give rise to spin systems with rates c, at least in general. We illustrate this point by considering flip rates for certain rapidly exploding voter models. The ergodic theory of more well-behaved voter models has been studied in some detail (e.g., in [9], [12], [5], [17]). Let $V = \mathbb{N} = \{0, 1, \cdots\}$, and let c be of the form

$$c_0(\xi) = \frac{r_0}{2}[1 - \xi(0)\xi(1)]\,;$$

$$c_n(\xi) = \frac{r_n}{2}[1 - p_n\xi(n)\xi(n+1) - q_n\xi(n)\xi(n-1)] \qquad n \geqq 1\,,$$

for some $r_n > 0$, $0 < p_n < 1$, with $q_n = 1 - p$. There is a minimal b.p.i. for these rates, which consists of birth and death processes (b.d.p.'s) on $\mathbb{N}$, all with jump rates r_n and probabilities p_n of a jump to the right, q_n of a jump to the left, from site n ($p_0 = 1$). These processes are independent except for the usual collision rule. If there is an explosion the entire system goes to Δ. The cardinality of the dual is deterministically nonincreasing, so any possible explosion is weak. Explosion occurs a.s. if and only if the one-particle dual, i.e., the (r_n, p_n)-b.d.p. explodes a.s. To find additional ways of continuing $(\hat{\xi}_t)$ beyond ζ we can take advantage of the bounday theory for b.d.p.'s (cf. [1], [2]). In fact, to each (r_n, p_n)-b.d.p. whose absorption coefficient equals 0, there corresponds a distinct Feller spin system with rates c. In simple cases, such spin systems have "waves" of $+1$'s and -1's "coming in from ∞." But in general (ξ_t) must have very complicated behavior at ∞, since the one-particle dual at ∞ reflects *and* makes infinitely many jumps in time ε, then goes to Δ after an exponential amount of Brownian local time. We sketch the construction of (ξ_t), since the argument leading to Theorem 2 must be modified somewhat. Let $\bar{V} = \mathbb{N} \cup \{\infty\}$, $\bar{\mathscr{V}}_0 =$ finite subsets of $\bar{V}$, and temporarily redefine $\hat{\Xi} = \bar{\mathscr{V}}_0 \cup \{\Delta\}$. Let $\hat{\xi}_t^x$ be a copy, starting at x, of the given (r_n, p_n)-b.d.p. Construct $\hat{\xi}_t^A$, the dual starting from A, according to the usual collision rule: two particles coalesce when they occupy the same site.

Now let $\{\hat{\mathbb{P}}_{\hat{\xi}}\}_{\hat{\xi} \in \hat{\Xi}}$ be the induced family of measures on the canonical path space, and introduce the corresponding operators $\hat{P}^t$. If the absorption coefficient is 0, then the duality equation (15) makes sense without defining $f_{\hat{\xi}}$ when $\infty \in \hat{\xi}$. The proof of Lemma 2 goes through with virtually no change, and the coupling used to get Theorem 2 can be altered to ensure that (16) defines the desired copies of the b.p.i. The rest of the argument is the same.

The general problem of when (15) gives rise to a Feller semigroup, and the more ambitious task of finding all systems with rates c, are undoubtedly difficult even in this simple voter model case. A more complete analysis is available for the class of Feller diffusions on the 1-dimensional torus considered in [13], where a one-parameter family of chains (with reflection and extinction at ∞) are the only possible duals.

REMARK 3 (added in revision). Two very recent papers treat a large class of "additive" interacting particle systems which enjoy property (16) when constructed on an appropriate joint probability space. For this general context, Bertein and Galves [0] give more details of the coupling in our Theorem 2. Harris [7] constructs infinite additive systems directly with the aid of Poisson flows. Both papers focus on infinite systems whose duals do not explode, but the nonuniqueness construction used here extends in a straightforward manner to any additive process with a weakly exploding dual. We suspect that the "graphical representation" in [7] makes sense even for our ill-mannered voter models.

3. Strong explosions, uniqueness and s.e.p. The construction behind Theorem 2 shows that the minimal b.p.i. $\{\check{\mathbb{P}}_{\hat{\xi}}\}$ *always* gives rise to a Feller proximity process via (15). It also suggests that it may be difficult to continue the b.p.i. beyond a strong explosion time without sending it to Δ, since $\hat{\xi}_{\zeta-} \cup A$ is not in $\hat{\Xi}$ for any $A \in \mathscr{Y}_0$. This intuition is essentially correct: if $\zeta < \infty$ with positive probability but the first explosion is strong with probability one, *and* if the flip rates satisfy certain uniformity conditions, then it turns out that there is only one system with rates c. Some estimates similar to the ones in the proof of our next result have been used in [18], [6] and [12] to derive ergodic theorems.

THEOREM 3. *Let c be rates for a proximity process, such that*

$$\inf_{x \in V} \inf_{\xi \in \Xi : \xi(x) = 1} c_x(\xi) > 0 \qquad \text{and} \qquad \sup_{x \in V} \sup_{\xi \in \Xi : \xi(x) = -1} c_x(\xi) < \infty .$$

Assume that $\check{\mathbb{P}}_A(\zeta = \infty$ or ζ a strong explosion$) = 1$ for all $A \in \mathscr{Y}_0$. Then the process (ξ_t) derived from $\{\check{\mathbb{P}}_\xi\}$ by way of (15) is the unique proximity process with rates c.

PROOF. By Lemma 1 it suffices to prove that

$$\check{\mathbb{E}}_A[\mathbb{P}_\xi(\xi_{t-\sigma_N}(x) = 1 \ \text{ for all } \ x \in \hat{\xi}_{\sigma_N}), \sigma_N \leqq t] \tag{17}$$

has limit 0 as $N \to \infty$ for all A, ξ, $t > 0$. Majorize (17) by

$$(A_N) \qquad \check{\mathbb{P}}_A(\sigma_N \in [t - \delta, t])$$

$$(B_N) \qquad + \check{\mathbb{P}}_A(\sigma_N < t - \delta, \#\hat{\xi}_{\sigma_N} \geqq M) \sup_{\#B \geqq M} \sup_{s \geqq \delta} \mathbb{P}_\xi(\xi_s(x) = 1 \text{ for all } x \in B)$$

$$(C_N) \qquad + \check{\mathbb{P}}_A(\sigma_N \leqq t, \#\hat{\xi}_{\sigma_N} < M)$$

for arbitrary $\delta \in (0, t)$, $M \geqq 0$. Choose $\varepsilon > 0$. Note first that

$$\limsup_{N\to\infty} \check{\mathbb{P}}_A(\sigma_N \in [t-\delta, t]) \leqq \check{\mathbb{P}}_A(\sigma_N \in [t-\delta, t] \text{ for infinitely many } N) = \check{\mathbb{P}}_A(\zeta \in [t-\delta, t]) .$$

Since ζ is the independent sum of the exponential variable ζ_1 and the remaining time $\zeta - \zeta_1$, the $\check{\mathbb{P}}_A$-distribution of ζ is absolutely continuous on $(0, \infty)$. Thus we can choose $\delta > 0$ so that $\limsup_{N\to\infty} A_N \leqq \varepsilon/2$. Next we take M large enough that $B_N \leqq \varepsilon/2$ for all N. The claim here is that $B_N = B_N(M, \delta) \to 0$ as $M \to \infty$. This is intuitively clear, since by hypothesis there is a positive lower bound l for the rate of flipping from 1 to -1 and a finite upper bound L on the rate of flipping from -1 to 1, *independent* of x and ξ. Here is a rigorous proof based on a rough estimate. Let $f = f_B$ be the indicator of "all 1's on B," $\#B = M$. Apply (3) to f, then differentiate to get

$$\begin{aligned} \frac{dP^s f(\xi)}{ds} &= \mathbb{E}_\xi[Gf(\xi_s)] \\ &= \textstyle\sum_{x\in B} \mathbb{E}_\xi[-c_x(\xi_s), f(\xi_s) = 1] + \sum_{x\in B} \mathbb{E}_\xi[c_x(\xi_s), f({}_x\xi_s) = 1] \\ &\leqq -MlP^s f(\xi) + L \textstyle\sum_{x\in B} P^s f({}_x\xi_s) \\ &\leqq -MlP^s f(\xi) + L[1 - P^s f(\xi)] . \end{aligned}$$

By Gronwall's inequality,

$$P^s f(\xi) \leqq \frac{L}{Ml + L} + \frac{Ml}{Ml + L} e^{-(Ml+L)s} \to 0 \qquad \text{as} \quad M \to \infty ,$$

uniformly in $s \geqq \delta$. (With a little more care one can show that $P^s f(\xi)$ goes to 0 exponentially in M.) This controls the second term of (B_N); bound the first by 1 to establish the claim. Finally, with δ and M as chosen, note that $\lim_{N\to\infty} C_N = \check{\mathbb{P}}_A(\zeta \leqq t, \#\xi_\zeta < M) = 0$, again by hypothesis. Thus the limit in N of (17) is less than ε, and uniqueness is proved.

REMARK 4. Theorem 3 generalizes easily to spin systems with α-duals,[2] additive processes etc. Note that the existence of a "minimal" spin system (ξ_t) corresponding to $\{\check{\mathbb{P}}_\xi\}$ implies that convergence of (17) to 0 is *necessary* for uniqueness. Moreover, if $\mathbf{1} \in \Xi$ is the configuration "all 1's," if $c_x(\mathbf{1}) = 0$ for all x, and if $\{\mathbb{P}_\xi'\}$ is the Markov family obtained by stopping the minimal proximity process when it hits $\mathbf{1}$, then quantity (17) computed with respect to $\mathbb{P}_\mathbf{1}'$ does not tend to 0 if an explosion occurs with positive probability. Evidently the positivity condition cannot be dropped in Theorem 3.

[2] When $\alpha = 0$ one must assume that all the rates are uniformly bounded and bounded away from 0. When $\alpha < 0$, reverse the roles of $+1$ and -1 in the hypotheses.

The pregenerator G with flip rates c is said to satisfy the *strong extension property* (s.e.p.) if

(18) the closure $\bar{G}$ of G is the generator of a Feller semigroup $(P^t)_{t\in\mathbb{R}^+}$.

The remainder of the paper centers around a discussion of this property. Recall that $\bar{G}$ satisfies graph $(\bar{G}) = \overline{\text{graph}\,(G)}$ in $\mathscr{C} \times \mathscr{C}$. Thus if $\bar{G}$ has domain $\mathscr{D}(\bar{G})$ and $h \in \mathscr{D}(\bar{G})$, then there are functions $f_n \in \mathscr{F}$ such that $\|h - f_n\| \to 0$ and $\|\bar{G}h - Gf_n\| \to 0$. If G has s.e.p. then there is a unique spin system with rates c. To see this, use Krylov's theorem, the minimality of $\bar{G}$ among all generators extending G, and the fact that a generator cannot have a proper extension which is also a generator. The converse question—whether uniqueness implies s.e.p.—is a problem which has been mentioned in [3] and [16], for example. We prove in Theorem 4 that the answer is no, by constructing rates c for which there is a unique proximity process (ξ_t), but such that G does not have s.e.p. First, though, as motivation we mention a sufficient condition for a proximity process to have s.e.p. Namely, by mimicking the argument leading to Theorem 1.3.6 in [16], we can show that G has s.e.p. whenever

(19) $$\sup_{A\in\mathscr{Y}_0} \sum_{n=1}^{\infty} \check{\mathbb{P}}_A(\zeta_n < \infty) < \infty ,$$

i.e., when the expected total number of jumps by the minimal b.p.i. is uniformly bounded. Thus the processes described in Remark 1 have s.e.p. This was shown already in [10] for $\alpha = 0$ duals. One can construct additional examples, though they tend to be artificial. The proof makes use of the Hille–Yosida theorem, which yields (18) whenever

(20) $$\text{Range}\,(\lambda - G) \supset \mathscr{F} \quad \text{for all sufficiently large} \quad \lambda$$

(see [14], [3]). The truncated generators introduced in [14] correspond to Q-matrices for b.p.i.'s which are sent to Δ upon leaving Λ_n. Condition (19) says that if we define an operator L on probability densities on $\hat{\Xi}$ by $L\pi = \check{\mathbb{E}}_\pi(\hat{\xi}_n \in \cdot)$, and on $l^1(\hat{\Xi})$ by linearity, then $(1 - L)^{-1}$ is invertible in $l^1(\hat{\Xi})$. This is the key tool for checking that the truncated generators approximate G well enough that (20) can be checked.

The fact that a condition as strong as (19) seems necessary to prove s.e.p. suggested looking for a system with uniqueness but without s.e.p. among processes with exploding duals. We therefore considered the simplest possible proximity processes to which Theorem 3 is applicable, and which have very strong explosions. This led to the example which follows. We do not know (a) whether there is always s.e.p. when the dual does not explode, or (b) whether there can be s.e.p. when the dual explodes strongly. Neither can we offer any additional insights into the proof of Theorem 4, which is based on only the grossest of estimates.

THEOREM 4. *Let $V = \mathbb{N}$, and define flip rates c by*

$$\begin{aligned} c_n(\xi) &= 0 && \text{if } \ \xi(n) = -1 \\ &= 1 && \text{if } \ \xi(m) = 1, \quad 0 \leqq m \leqq n+1 \\ &= 100^{n^2} && \text{else}, \end{aligned}$$

$n \in \mathscr{V}$. *Then there is a unique spin system with rates c, but the pregenerator G for these rates does not have* s.e.p.

PROOF. Observe that

$$c_n(\xi) = \frac{100^{n^2}}{2}[1 + \xi(n) - 2\xi(n)(1 - 100^{-n^2})f_{\{0,1,\cdots,n+1\}}(\xi)],$$

so the rates c admit a b.p.i. with a very simple transition mechanism. A particle at site n jumps at rate 100^{n^2}, choosing the set $\{0, 1, \cdots, n+1\}$ with probability $1 - 100^{-n^2}$, and sending $(\hat{\xi}_t)$ to Δ with the remaining probability. Govern $(\hat{\xi}_t)$ by $\{\check{\mathbb{P}}_{\hat{\xi}}\}$. Then $\check{\mathbb{P}}_A(\zeta < \infty) > 0$ whenever $A \neq \varnothing$, but the explosion must be strong. In fact, $\hat{\xi}_{\zeta-} = V$ whenever $\zeta < \infty$. The uniformity conditions of Theorem 3 also hold, so uniqueness follows from that result. Let (P^t) be the unique (Feller) semigroup for (ξ_t), $G^e: \mathbb{D}(G^e) \to \mathscr{C}$ its generator. Define $\varphi \in \mathscr{C}$ by

$$\begin{aligned} \varphi(\xi) &= 100^{-n_\xi^2} && n_\xi = \min\{m : \xi(m) = -1\} < \infty \\ &= 0 && \xi = \mathbf{1} = \text{“all 1's.”} \end{aligned}$$

To show that G does not have s.e.p. we will establish two claims:

(i) $\varphi \in \mathbb{D}(G^e)$;
(ii) If $f \in \mathscr{F}$ and $||\varphi - f|| \leqq \frac{1}{100}$, then $||G^e\varphi - Gf|| > \frac{1}{100}$.

Together, (i) and (ii) show that $\varphi \in \mathbb{D}(G^e) - \mathbb{D}(\bar{G})$, so that $\bar{G}$ is not a generator. To check (i), put $\Lambda_n = \{0, 1, \cdots, n\}$, then define $G_n : \mathscr{F}_0 \to \mathscr{C}$ by

$$G_n f_A(\xi) = \textstyle\sum_{B \subset \Lambda_n} q_{AB} f_B(\xi),$$

where q_{AB} are the jump rates for our b.p.i. Note that G_n is bounded, extend to $\mathscr{C}$, and write $P_n{}^t = e^{tG_n}$. These are simply the truncated generators and semigroups mentioned earlier. Let $\mathscr{F}_n$ = functions on Ξ which depend only on sites in Λ_n. Consider $\psi \in \mathscr{C}$ given by

$$\begin{aligned} \psi(\xi) &= (1 - G_n)\varphi(\xi) && \text{if } \ n_\xi = n+1 \\ &= -(1 + \textstyle\sum_{i=0}^\infty 100^{-i^2}) && \text{if } \ \xi = \mathbf{1}. \end{aligned}$$

To see that ψ is continuous at $\mathbf{1}$, let ξ_n be the configuration with 1's on $\{0, 1, \cdots, n\}$ and -1's on $\{n+1, n+2, \cdots\}$. Then a simple computation yields

$$\begin{aligned} \lim_{\xi \to \mathbf{1}} \psi(\xi) &= \lim_{n\to\infty} \psi(\xi_n) = \lim_{n\to\infty} [\varphi(\xi_n) - G_n\varphi(\xi_n)] \\ &= \lim_{n\to\infty} [100^{-(n+1)^2} - \textstyle\sum_{i=0}^{n-1} (100^{-i^2} - 100^{-(n+1)^2}) \\ &\qquad - 100^{n^2}(100^{-n^2} - 100^{-(n+1)^2})] \\ &= \psi(\mathbf{1}). \end{aligned}$$

Now $P_n{}^t\psi(\xi) = P^t\psi(\xi)$ if $n_\xi \leqq n + 1$, so $P_n{}^t\psi(\mathbf{1}) = P_n{}^t\psi(\xi_n) = P^t\psi(\xi_n) \to P^t\psi(\mathbf{1})$, by continuity of $P^t\psi$. Thus $P_n{}^t\psi(\xi) \to P^t\psi(\xi)$ for all $\xi \in \Xi$. Hence

$$(1 - G^e)^{-1}\psi = \int_0^\infty e^{-t}P^t\psi(\xi)\,dt = \int_0^\infty e^{-t}\lim_{n\to\infty} P_n{}^t\psi(\xi)\,dt$$
$$= \lim_{n\to\infty}\int_0^\infty e^{-t}P_n{}^t\psi(\xi)\,dt = \lim_{n\to\infty}(1 - G_n)^{-1}\psi(\xi) = \varphi(\xi)\,,$$

$\xi \in \Xi$. Thus $\varphi \in \mathbb{D}(G^e)$, and $G^e\varphi = \varphi - \psi$. In particular,

$$\|G^e\varphi\| = G^e\varphi(\mathbf{1}) = 1 + \sum_{i=0}^\infty 100^{-i^2}\,. \tag{20}$$

Claim (i) is proved, so we proceed to claim (ii). Fix $f \in \mathscr{F}_n$ for some $n \geqq 1$, $\|\varphi - f\| \leqq \frac{1}{100}$, and assume $\|G^e\varphi - Gf\| \leqq \frac{1}{100}$. Write $\Delta_i h(\xi) = h({}_i\xi) - h(\xi)$, $i \geqq 0$, $h \in \mathscr{C}$, $\xi \in \Xi$. We estimate $|\sum| = |\sum_{i=0}^{n-1}\Delta_i f(\xi_n)|$ in two different ways to get a contradiction. First,

$$|\textstyle\sum| = |Gf(\xi_n) - 100^{n^2}\Delta_n f(\xi_n)| \geqq |Gf(\xi_n)| - 100^{n^2}|\Delta_n f(\xi_n)|\,.$$

Now $|Gf(\xi_n)| \geqq |G^e\varphi(\xi_n)| - \frac{1}{100} = G_n\varphi(\xi_n) - \frac{1}{100} > 1.98$ for all $n \geqq 1$. Moreover,

$$|Gf(\xi_n) - Gf(\xi_{n+1})| = |\textstyle\sum_{i=0}^{n-1}\Delta_i f(\xi_n) + 100^{n^2}\Delta_n f(\xi_n)$$
$$- \textstyle\sum_{i=0}^{n}\Delta_i f(\xi_{n+1}) - 100^{(n+1)^2}\Delta_{n+1}f(\xi_{n+1})|$$
$$= (100^{n^2} - 1)|\Delta_n f(\xi_n)|$$

because $f \in \mathscr{F}_n$, and at the same time

$$|Gf(\xi_n) - Gf(\xi_{n+1})| \leqq 2\|G^e\varphi - Gf\| + |G^e\varphi(\xi_n) - G^e\varphi(\xi_{n+1})| < .03\,.$$

We conclude that $|\sum| \geqq 1.98 - \frac{100}{99}(.03) > 1.94$. On the other hand,

$$|\textstyle\sum| \leqq \sum_{i=0}^{n-1}|\Delta_i f(\xi_i) - \Delta_i f(\xi_n)| + \sum_{i=0}^{n-1}|\Delta_i f(\xi_i)|$$
$$\leqq \textstyle\sum_{i=0}^{n-1}\sum_{j=i}^{n-1}|\Delta_i f(\xi_{j+1}) - \Delta_i f(\xi_j)| + \sum_{i=0}^{n-1}|\Delta_i f(\xi_i)|$$
$$= \textstyle\sum_{i=0}^{n-1}\sum_{j=i}^{n-1}|\Delta_{j+1}f(\xi_{j+1}) - \Delta_{j+1}f({}_i\xi_{j+1})| + \sum_{i=0}^{n-1}|\Delta_i f(\xi_i)|\,.$$

Note that for $0 \leqq i < j < \infty$, $|\Delta_j f({}_i\xi_j)| \leqq 2^{-(j+3)}$; if not, then

$$|Gf({}_i\xi_j)| = |\textstyle\sum_{k=0}^{i-2}\Delta_k f({}_i\xi_j) + 100^{(i-1)^2}\Delta_{i-1}f({}_i\xi_j) + \sum_{k=i+1}^{j}100^{k^2}\Delta_k f({}_i\xi_j)|$$
$$\geqq 100^{j^2}2^{-(j+3)} - 2j100^{(j-1)^2}\|f\|$$
$$\geqq 100^{j^2}2^{-(j+3)} - 2j100^{(j-1)^2}(\|\varphi\| + \tfrac{1}{100})$$
$$> 3 > \|G^e\varphi\| + \tfrac{1}{100} \quad \text{by (20)}\,.$$

The same bound holds for $|\Delta_j f(\xi_j)|$ when $j \geqq 1$, so

$$|\textstyle\sum| \leqq 2\sum_{i=0}^{n-1}\sum_{j=i}^{n-1}2^{-(j+4)} + \sum_{i=1}^{n-1}2^{-(i+3)} + 2\|f - \varphi\| + \|\varphi\| < 1.65\,.$$

This contradiction yields claim (ii), completing the proof of the theorem.

REMARK 5 (added in revision). Once uniqueness is established for rates c, interest centers on the ergodic theory for the corresponding spin system, and in particular on its invariant measures. Recall that μ is *invariant* for (ξ_t) if $\mu P_*{}^t = \mu$

for all $t \in \mathbb{R}^+$, where $P_*{}^t$ is the adjoint to P^t. Equivalently, μ is invariant if and only if

$$(21) \qquad \int G^e f \, d\mu = 0 \qquad \text{for all} \quad f \in \mathbb{D}(G^e)\,.$$

In practice, one wants invariance of μ to be implied by the naive condition

$$(22) \qquad \int Gf \, d\mu = 0 \qquad \text{for all} \quad f \in \mathscr{F}\,.$$

It turns out that if μ is a *probability measure*, then (22) implies (21) whenever (ξ_t) is uniquely determined by c. The proof of this follows easily from the construction in the proof of Theorem 2.4 of Higuchi and Shiga [8]. For each finite $A \subset V$ they produce rates $c_{x,A}$ (using the positivity of μ and (22)) such that $\|c_{x,A} - c_x\| \to 0$ as $A \uparrow V$, $c_{x,A} \equiv 0$ for $x \in A^c$, and such that $\mu_A =$ (projection of μ onto $\{-1, 1\}^A) \times \nu_A$ is invariant for the unique spin system $\{\mathbb{P}_{\xi,A}\}$ with rates $c_{x,A}$ where ν_A is any fixed probability measure on $\{-1, 1\}^{V-A}$. It follows from Theorem 2.3 of [10] and uniqueness that for all $f \in \mathscr{F}$, $\|P_A{}^t f - P^t f\| \to 0$ as $A \uparrow V$, so that $\int P^t f \, d\mu = \lim_{A \uparrow V} \int P^t f \, d\mu_A = \lim_{A \uparrow V} \int P_A{}^t f \, d\mu_A = \lim_{A \uparrow V} \int f \, d\mu_A = \int f \, d\mu$. However, this proof fails when μ is a signed measure. In that case, one must check a more general condition on G^e to get (21), namely

$$(23) \qquad \overline{\text{Range}\,(G)} = \mathscr{C}\,.$$

In this connection, Stroock has pointed out that one can use the spin system in Thoerem 4 to construct a system (no longer a *spin* system) where (23) fails and yet the martingale problem is well-posed. Simply take the system in Theorem 4 and send it to a "cemetery" with exponential rate 1. The new pregenerator $\tilde{G}$ uniquely determines this system, and Higuchi and Shiga's argument can be extended to show that (22) is sufficient for a probability measure to be invariant. At the same time, there is a signed measure μ satisfying (22) which is not invariant. Presumably G and $\tilde{G}$ both have more than one extension which generates a signed semigroup, even though they have unique extensions generating a positive semigroup. Therefore, positivity must play an essential role in the proof of Theorem 3. It is used only in the derivation of the estimate for B_N, in the step just before Gronwall's inequality.

The interest in Theorem 4 is that it demonstrates that in certain delicate situations a pregenerator may determine only one semigroup of probabilistic interest (i.e., positive) and yet it may not determine one semigroup among all semigroups. The martingale approach can see this, but the analytic theory cannot.

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