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MARKOV SYSTEMS AND THEIR ADDITIVE FUNCTIONALS

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For certain classes of Markov systems (that is, stochastic systems which have Markov representations with transition and cotransition probabilities) considered by the author in previous papers, a correspondence was established between additive functionals of any such system and measures on a certain measurable space. We now prove analogous results for arbitrary Markov systems. Measures corresponding to the additive functionals are defined on a certain σ-algebra in the product space $R \times \Omega$ where R is the real line and Ω is the sample space (we call it the central σ-algebra). The theory is applicable not only to traditional processes but also to a number of generalized stochastic processes introduced by Gelfand and Itô. A situation where the observations are performed over a random time interval and the measure P can be infinite is considered in the concluding section. These generalizations are of special importance for the homogeneous case which will be treated in another publication.

1. Introduction.

1.1. Let a σ-algebra $\mathscr{F}(I)$ in a fixed space Ω be associated with every open interval I of the real line R in such a way that:

1.1.A. If $I_1 \subset I_2$, then $\mathscr{F}(I_1) \subset \mathscr{F}(I_2)$.

1.1.B. If $I_n \uparrow I$, then $\mathscr{F}(I)$ is the minimal σ-algebra which contains the union of the $\mathscr{F}(I_n)$.

Let P be a probability measure on a σ-algebra $\mathscr{F} \supseteq \mathscr{F}(R)$. A collection $(\mathscr{F}(I), P)$ will be called a *stochastic system*. We shall assume that the σ-algebra $\mathscr{F}$ is complete relative to the measure P.

We put for abbreviation $\mathscr{F}_{<t} = \mathscr{F}(-\infty, t)$, $\mathscr{F}_{>t} = \mathscr{F}(t, +\infty)$.

A stochastic system $(\mathscr{F}(I), P)$ is called a *Markov system* if:

1.1.C. For every $s < t$, the σ-algebras $\mathscr{F}_{<s}$ and $\mathscr{F}_{>t}$ are conditionally independent given $\mathscr{F}(s, t)$.

We say that the σ-algebras $\mathscr{A}$ and $\mathscr{B}$ are *conditionally independent given* $\mathscr{C}$ if for any $A \in \mathscr{A}$, $B \in \mathscr{B}$

$$P(AB \mid \mathscr{C}) = P(A \mid \mathscr{C})P(B \mid \mathscr{C}) \quad \text{a.s.} \quad P.$$

The following condition is stronger than 1.1.C.:

1.1.D. For every $s < t$, the σ-algebras $\mathscr{F}_{<t}$ and $\mathscr{F}_{>s}$ are conditionally independent given $\mathscr{F}(s, t)$.

If this is fulfilled, we call $(\mathscr{F}(I), P)$ a *hyper-Markov system*.

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1.2. A number of important σ-algebras are associated with each stochastic system.

First we introduce necessary notations. For each σ-algebra $\mathscr{A}$ in the space Ω, denote by $\mathscr{A}^P$ the minimal σ-algebra which contains $\mathscr{A}$ and all the sets A such that $P(A) = 0$. Denote by $\mathscr{F}(s, t+)$ the intersection of $\mathscr{F}(s, u)$ over all $u > t$ and introduce $\mathscr{F}(s-, t)$, $\mathscr{F}(s-, t+)$, $\mathscr{F}_{<t+}$, etc., in an analogous way. We call a function $\xi_t(\omega)$ *progressive* if its restriction to any interval I is measurable relative to $\mathscr{B}(I) \times \mathscr{F}(I)^P$ where $\mathscr{B}(I)$ is the Borel σ-algebra in I. We say that a function $\xi_t(\omega)$ belongs to the class $\mathscr{S}_+$ ($\mathscr{S}_-$) if it is right continuous (left continuous) in t for almost all ω. A function ξ_t is said to be *adapted* to a family $\mathscr{A}_t$ if ξ_t is $\mathscr{A}_t$-measurable for every t. We write 1_C for the indicator of a set C and use the abbreviation $\xi \in \mathscr{A}$ for the statement ξ is a nonnegative $\mathscr{A}$-measurable function.

We define four fundamental σ-algebras in $R \times \Omega$. These are the σ-algebras generated by the following families of functions:

(i) functions of $\mathscr{S}_+$ adapted to $\mathscr{F}^P_{<t+}$;
(ii) functions of $\mathscr{S}_-$ adapted to $\mathscr{F}^P_{>t-}$;
(iii) functions of $\mathscr{S}_-$ adapted to $\mathscr{F}^P_{<t}$;
(iv) functions of $\mathscr{S}_+$ adapted to $\mathscr{F}^P_{>t}$.

Sets and functions which are measurable with respect to the σ-algebra (i) will be called *right* (Meyer [10] calls them well-measurable or optional). Analogously we shall use the terms *left*, *predictable* and *reconstructable* in the cases of the σ-algebras (ii), (iii), and (iv).

The σ-algebra generated by all predictable and reconstructable functions will be called *solid*. Finally, sets and functions which are simultaneously right, left and solid will be called *central*. The σ-algebra of all the central sets will be denoted by $\mathscr{C}$.

We say that functions ξ and η are *indistinguishable* if $P(\xi_t \neq \eta_t \text{ for some } t) = 0$. Functions which are indistinguishable from 0 are said to be *evanescent*. We apply the same word to sets with evanescent indicators. Denote by $\mathscr{F}_{R\times\Omega}$ the minimal σ-algebra which contains $\mathscr{B}(R) \times \mathscr{F}(R)$ and the evanescent sets. All the σ-algebras introduced in subsection 1.2 are sub-σ-algebras of $\mathscr{F}_{R\times\Omega}$.

Denote by $P\xi$ the integral of a function ξ with respect to a measure P. Put $\xi \in \mathscr{K}$ if there exists a function η such that $P|\eta| < \infty$ and either $\xi \leqq \eta$ for all t, ω or $\xi \geqq \eta$ for all t, ω.

1.3. We need a concept of the right projection (which is identical to Meyer's well-measurable projection) and a dual concept of the left projection.

A function $\tau(\omega)$ taking values in $(-\infty, +\infty]$ is a Markov time relative to $\mathscr{M}_t = \mathscr{F}^P_{<t+}$ if $\{\tau \leqq t\} \in \mathscr{M}_t$ for every $t \in R$. Put $C \in \mathscr{M}_\tau$ if $C \cap \{\tau \leqq t\} \in \mathscr{M}_t$ for every $t \in R$. Let ξ be an $\mathscr{F}_{R\times\Omega}$-measurable function of class $\mathscr{K}$. We say that η is the *right projection* of ξ and write $\eta = \Pi^+\xi$ if η is a right function and

$$P1_C\eta_\tau = P1_C\xi_\tau \tag{1.1}$$

for every Markov time τ and every $C \in \mathscr{M}_\tau$ (by definition $\xi_\tau = \eta_\tau = 0$ for $\tau = +\infty$). For each $\xi \in \mathscr{K}$ the right projection exists and any two right projections are indistinguishable (see Section 3). The *left projection* of ξ is denoted by $\Pi^-\xi$. We prove that $\Pi^+\Pi^-\xi = \Pi^-\Pi^+\xi$ for each solid function ξ. This function is central. It will be called the *central projection* of ξ and denoted by $\Pi\xi$.

1.4. A *finite additive functional* of a Markov system $(\mathscr{F}(I), P)$ is a measure A on the real line R depending on ω in such a way that for each open interval I:

1.4.A. $A(I) \in \mathscr{F}^P(I)$.
1.4.B. $P(A(I) = \infty) = 0$ if I is finite.
1.4.C. $A(I)$ is a solid function of ω.

The condition 1.4.C is equivalent to each of the following conditions:

1.4.C′. $A\{t\}$ is a solid function of t, ω.
1.4.C″. $A\{t\}$ is a central function of t, ω.

(Functionals with the property 1.4.C″ were called normal in [6].)

A function A is said to be a *σ-functional* if it can be represented in the form $A = \sum A_n$ where A_n are finite additive functionals.

For any σ-functional A, a measure μ on $\mathscr{F}_{R\times\Omega}$ is defined by the formula

$$\mu(C) = P \int 1_C(t, \omega)A(dt) \quad (C \in \mathscr{F}_{R\times\Omega}) \,. \tag{1.2}$$

Obviously μ charges no evanescent set. If the functional A is continuous, i.e., if $P(A\{t\} \neq 0$ for some $t) = 0$ then μ charges no *scanty* set (i.e., no set C with the property $P(\omega: (t, \omega) \in C$ for an uncountable set of $t) = 0$).

A remarkable fact is that

$$\mu(\xi) = \mu(\Pi\xi) \tag{1.3}$$

for any solid $\xi \in \mathscr{K}$ (see Section 5). This makes it natural to restrict μ to the central σ-algebra $\mathscr{C}$. The restriction will be called the *spectral measure* of the σ-functional A and will be denoted by μ_A.

We prove that:

(a) The spectral measure of any σ-functional is a σ-measure, i.e., a sum of a countable set of finite measures.

(b) A σ-functional is uniquely determined by its spectral measure. (We do not distinguish functionals which coincide for almost all ω.)

(c) A σ-functional is continuous if and only if its spectral measure vanishes on all the scanty sets.

(d) In the case of a hyper-Markov system $(\mathscr{F}(I), P)$, every σ-measure on $\mathscr{C}$ which charges no evanescent set is a spectral measure of a σ-functional.

Thus for a hyper-Markov system, we have a one-to-one correspondence between σ-functionals and σ-measures on $\mathscr{C}$ which charge no evanescent set.

REMARK. It was stated in [6] that the spectral measure μ_A is σ-finite if the measure A is σ-finite for almost all ω. This follows from Theorem 4.3 there. But the proof given for Theorem 4.3 is false. It remains an open question whether or not the statement of the theorem is true.

1.5. Let us discuss the relationship between Markov systems and traditional Markov processes.

Let a mapping $x_t(\omega)$ of Ω into a measurable space $(E_t, \mathscr{B}_t)$ be associated with any $t \in R$ and let the following conditions be satisfied:

1.5.A. $(\omega: x_t(\omega) \in \Gamma) \in \mathscr{F}(I)$ for all $t \in I$, $\Gamma \in \mathscr{B}_t$.

1.5.B. For each t, the σ-algebras $\mathscr{F}_{<t}$ and $\mathscr{F}_{>t}$ are conditionally independent given x_t.

Then we say that $(x_t, (I), P)$ is a *Markov process* or that x_t is a *Markov representation* of the stochastic system $(\mathscr{F}(I), P)$. We show in Section 2 that a system $(\mathscr{F}(I), P)$ is a Markov if and only if it has a Markov representation.

For any σ-algebra $\mathscr{A}$ in Ω we denote by $\mathscr{A} \vee x_t$ the smallest σ-algebra which contains $\mathscr{A}$ and all the sets $\{x_t \in \Gamma\}$ where $\Gamma \in \mathscr{B}_t$. Let a probability measure $P_{t,x}$ on $\mathscr{F}_{>t}$ be defined for any $x \in E_t$ in such a way that for any $s < t \in R$, and any $\eta \in \mathscr{F}_{>t}$

$$P(\eta \mid \mathscr{F}_{<t} \vee x_t) = P_{t,x_t}\eta \quad \text{a.s.} \quad P$$

and

$$P_{s,x}(\eta \mid \mathscr{F}(s, t) \vee x_t) = P_{t,x_t}\eta \quad \text{a.s.} \quad P_{s,x} .$$

Then we say that $P_{t,x}$ are *forward transition* probabilities of the Markov process $(x_t, \mathscr{F}(I), P)$ or that $(x_t, P_{t,x})$ is a *right Markov representation* of $(\mathscr{F}(I), P)$. The dual concepts of the *backward transition probabilities* $P^{t,x}$ and the *left Markov representation* are introduced similarly. The *two-sided representation* is a collection $(x_t, P_{t,x}, P^{t,x})$ where $(x_t, P_{t,x})$ is a right and $(x_t, P^{t,x})$ is a left representation.

The functions

$$\begin{aligned} h(t, x) &= P_{t,x}(A) \qquad \text{for} \quad t < u , \\ &= 0 \qquad\qquad\quad \text{for} \quad t \geqq u \end{aligned}$$

where $u \in R$ and $A \in \mathscr{F}_{>u}$ are called the base functions of the right representation $(x_t, P_{t,x})$. The right representation is called *regular* if $h(t, x_t)$ belongs to $\mathscr{S}^+$ for each base function h.

Assume now that the collection $\mathscr{F}(I)$ has the following property:

1.5.α. Let $I_n \uparrow I$. Let P_n be a probability measure on $\mathscr{F}(I_n)$ and $P_n = P_{n-1}$ on $\mathscr{F}(I_{n-1})$. Then there exists a measure P on $\mathscr{F}(I)$ which coincides with P_n on $\mathscr{F}(I_n)$, $n = 1, 2, \cdots$.

Under this condition every stochastic system which has a right representation has also a regular right representation (see [5]).

The definition of regularity and the existence theorem can be extended to left and two-sided representations.

Let $(x_t, P_{t,x}, P^{t,x})$ be a regular two-sided representation of a stochastic system

$(\mathscr{F}(I), P)$. Introduce into the union $\mathscr{E}$ of the spaces E_t a measurable structure generated by the base functions of $(x_t, P_{t,x})$ and $(x_t, P^{t,x})$. We prove in Section 3 that a function ξ is central if and only if it is indistinguishable from a function of the form $f(t, x_t)$ with a measurable f. This makes it possible to characterize additive functionals of $(\mathscr{F}(I), P)$ by measures on the state space E which charge no set inaccessible for x_t. This is the way we described additive functionals in [6].

1.6. The concept of a stochastic system can be generalized as follows. Consider a collection $\mathscr{F}(I)$ of σ-algebras in Ω satisfying conditions 1.1.A—B and a (possibly infinite) measure P on a σ-algebra $\mathscr{F} \supseteq \mathscr{F}(R)$. Assume that to each $\omega \in \Omega$ there corresponds an open interval or the empty set $\Delta(\omega)$. We say that $(\mathscr{F}(I), P, \Delta)$ is a *stochastic system on time interval* Δ if the following conditions are fulfilled:

1.6.A. For every $t \in R$ $\{\omega: t \in \Delta(\omega)\} \in \mathscr{F}(t-, t+)$ and there exist sets $C_n \in \mathscr{F}(t-, t+)$, $n = 1, 2, \cdots$ such that $P(C_n) < \infty$ and $\{\omega: t \in \Delta(\omega)\}$ is the union of C_n.

1.6.B. For every I, the set $\{\omega: \Delta(\omega) \cap I = \emptyset\}$ is an atom of $\mathscr{F}(I)$.

1.6.C. $P\{\Delta(\omega) = \emptyset\} < \infty$.

The fact that $\{\Delta \cap I = \emptyset\} \in \mathscr{F}(I)$ follows from 1.6.A. Intuitively, 1.6.B means that nothing is observed outside the time interval Δ. Condition 1.6.C does not affect generality essentially since $\{\Delta = \emptyset\}$ is an atom of $\mathscr{F}(R)$ by 1.6.B. Conditions 1.6.A—C imply that P is a σ-finite measure.

Put $\Delta(\omega) = (\alpha(\omega), \beta(\omega))$ if $\Delta(\omega) \neq \emptyset$ and $\alpha(\omega) = +\infty$, $\beta(\omega) = -\infty$ if $\Delta(\omega) = \emptyset$.

For a system on a random time interval, the condition 1.4.B in the definition of an additive functional has to be replaced by the following one:

1.6.α. $P\{\alpha < s < u < \beta, A(s, u) = \infty\} = 0$ for all $s < u \in R$.

In addition, we suppose that:

1.6.β. The measure $A(-, \omega)$ is concentrated on the interval $\Delta(\omega)$.

Obviously, the spectral measure of any additive functional is concentrated on the set $\{(t, \omega): t \in \Delta(\omega)\}$. With this reservation, all the results formulated in subsection 1.4 can be extended to Markov systems on random time intervals. The proofs need some modifications which are described in Section 7.

The concept of a stochastic system on a random time interval is of special importance for investigating Markov processes with stationary forward transition probabilities. Many problems concerning such processes can be reduced to problems concerning the processes $(x_t, \mathscr{F}(I), P)$ with stationary forward and backward transition probabilities. The last property implies that P is invariant up to a constant factor, relative to the shift operators θ_t. However, generally, the measure P is infinite and x_t is defined on a random time interval.

2. Markov systems.

2.1. In this section we study various formulations of Markov and hyper-Markov properties. We describe also an example due to Molčan (cf. [11], page 27) of a Markov system which is not hyper-Markov.

We make use of the following elementary lemma (see e.g., [9], 25.3.A, or [10], Chapter II, T51):

LEMMA 2.1. *Two σ-algebras $\mathscr{A}$ and $\mathscr{B}$ are conditionally independent* (c.i.) *given $\mathscr{C}$ if and only if for every $\xi \in \mathscr{B}$*

$$P(\xi \mid \mathscr{C}) = P(\xi \mid \mathscr{A} \vee \mathscr{C}) \quad \text{a.s.} \tag{2.1}$$

The last condition is equivalent to the following one: for every $\xi \in \mathscr{B}$, there exists an $f \in \mathscr{C}$ such that

$$P(\xi \mid \mathscr{A} \vee \mathscr{C}) = f \quad \text{a.s.} \tag{2.2}$$

2.2. THEOREM 2.1. *Each of the following four conditions is equivalent to the Markov property* 1.1.C:

2.2.A. $\mathscr{F}_{<t}$ *and* $\mathscr{F}_{>t}$ *are* c.i. *given* $\mathscr{F}(t, t+)$.

2.2.B. *There exists a Markov representation* x_t.

2.2.C. $P\{\xi \mid \mathscr{F}_{<t+}\} = P\{\xi \mid \mathscr{F}(t, t+)\}$ *for all* $\xi \in \mathscr{F}_{>t}$.

2.2.D. *For* $s \leqq t$, $\mathscr{F}_{<s}$ *and* $\mathscr{F}_{>t}$ *are* c.i. *given* $\mathscr{F}(s, t+)$.

REMARK. The statement of Theorem 2.1 remains valid if we replace conditions 2.2.A—D by the dual conditions obtained by time reversal. We use asterisks for references to dual statements. For example, 2.2.A* means: "$\mathscr{F}_{>t}$ and $\mathscr{F}_{<t}$ are c.i. given $\mathscr{F}(t-, t)$."

PROOF. Let us agree to denote by r a variable taking rational values and to omit the letters "a.s." in our calculations.

1°. Let 1.1.C be fulfilled. Then $P\{\xi \mid \mathscr{F}_{<t} \vee \mathscr{F}(t, r)\} = P\{\xi \mid \mathscr{F}(t, r)\}$ for $t < r < u$, $\xi \in \mathscr{F}_{>u}$. Denote by $\mathscr{G}$ an intersection of $\mathscr{F}_{<t} \vee \mathscr{F}(t, r)$ over all $r > t$. Letting $r \downarrow t$, we have $P\{\xi \mid \mathscr{G}\} = P\{\xi \mid \mathscr{F}(t, t+)\}$. Since $\mathscr{G} \supseteqq \mathscr{F}_{<t} \vee \mathscr{F}(t, t+) \supseteqq \mathscr{F}(t, t+)$, this implies that $P\{\xi \mid \mathscr{F}_{<t} \vee \mathscr{F}(t, t+)\} = P\{\xi \mid \mathscr{G}\} = P\{\xi \mid \mathscr{F}(t, t+)\}$. In view of 1.1.B the last equality is true for all $\xi \in \mathscr{F}_{>t}$. Thus 2.2.A is fulfilled.

2°. In order to deduce 2.2.B from 2.2.A it is sufficient to put $E_t = \Omega$, $\mathscr{B}_t = \mathscr{F}(t, t+)$ and $x_t(\omega) = \omega$.

3°. If 2.2.B is true, then

$$P\{\xi \mid \mathscr{F}_{<r} \vee x_r\} = P\{\xi \mid x_r\} \qquad \text{for} \quad t < r < u, \quad \xi \in \mathscr{F}_{>u}.$$

Letting $r \downarrow t$, we have $P\{\xi \mid \mathscr{F}_{<t+}\} \in \mathscr{F}(t, t+)^P$. Hence 2.2.C is fulfilled.

4°. Condition 2.2.C implies 2.2.D since the σ-algebra $\mathscr{M} = \mathscr{F}_{<s} \vee \mathscr{F}(s, t+)$ contains $\mathscr{F}(t, t+)$ and is contained in $\mathscr{F}_{<t+}$, and therefore, for $\xi \in \mathscr{F}_{>t}$

$$\begin{aligned} P\{\xi \mid \mathscr{M}\} &= P[P\{\xi \mid \mathscr{F}_{<t+}\} \mid \mathscr{M}] = P[P\{\xi \mid \mathscr{F}(t, t+)\} \mid \mathscr{M}] \\ &= P\{\xi \mid \mathscr{F}(t, t+)\} \in \mathscr{F}(s, t+)^P. \end{aligned}$$

5°. Finally, 2.2.D implies that $P\{\xi \mid \mathscr{F}_{<s} \vee \mathscr{F}(s, r+)\} = P\{\xi \mid \mathscr{F}(s, r+)\}$ for $s < r < t$ and $\xi \in \mathscr{F}_{>t}$. Letting $r \uparrow t$, we have $P\{\xi \mid \mathscr{F}_{<s} \vee \mathscr{F}(s, t)\} \in \mathscr{F}(s, t)^P$. Thus 1.1.C is fulfilled.

2.3. THEOREM 2.2. *Each of the following conditions is equivalent to the hyper-Markov property* 1.1.D:

2.3.A. *For all* $s < t$ *and* $\eta \in \mathscr{F}$

$$P\{\eta \mid \mathscr{F}(s, t)\} = P[P\{\eta \mid \mathscr{F}_{>s}\} \mid \mathscr{F}_{<t}] \quad \text{a.s.}$$

2.3.B. *For all* $s \leqq t$ *and* $\eta \in \mathscr{F}$

$$P\{\eta \mid \mathscr{F}(s, t+)\} = P[P\{\eta \mid \mathscr{F}_{>s}\} \mid \mathscr{F}_{<t+}] \quad \text{a.s.}$$

2.3.C. *For* $s \leqq t$, *the* σ*-algebras* $\mathscr{F}_{>s}$ *and* $\mathscr{F}_{<t+}$ *are* c.i. *given* $\mathscr{F}(s, t+)$.

PROOF. If 2.3.A is fulfilled, then

$$P\{\eta \mid \mathscr{F}(s, t)\} = P\{\eta \mid \mathscr{F}_{<t}\} \qquad \text{for} \quad \eta \in \mathscr{F}_{>s}.$$

By Lemma 2.1 this is equivalent to 1.1.D. On the other hand, $\xi = P\{\eta \mid \mathscr{F}_{>s}\} \in \mathscr{F}_{>s}$ for every $\eta \in \mathscr{F}$. Therefore, if 1.1.D is true, then $P\{\xi \mid \mathscr{F}_{<t}\} = P\{\xi \mid \mathscr{F}(s, t)\}$. The right side is equal to $P\{\eta \mid \mathscr{F}(s, t)\}$. Hence 1.1.D implies 2.3.A.

The equivalence of 2.3.B and 2.3.C can be proved in an analogous way. The equivalence of 2.3.A and 2.3.B is proved by a simple passage to the limit using the fact that $\mathscr{F}(s, r) \downarrow \mathscr{F}(s, t+)$ as $r \downarrow t$ and $\mathscr{F}(s, r+) \uparrow \mathscr{F}(s, t)$ as $r \uparrow t$.

2.4. A Markov system is hyper-Markov if the collection $\mathscr{F}(I)$ satisfies the following condition:

2.4.A. For all $s \leqq t$, $\mathscr{F}_{>s} = \mathscr{F}(s, t+) \vee \mathscr{F}_{>t}$.

Indeed if $\xi_1 \in \mathscr{F}(s, t+)$ and $\xi_2 \in \mathscr{F}_{>t}$, then $P\{\xi_1\xi_2 \mid \mathscr{F}_{<t+}\} = \xi_1 P\{\xi_2 \mid \mathscr{F}(t, t+)\} \in \mathscr{F}(s, t+)$. Therefore 2.3.C is true.

In particular, condition 2.4.A is fulfilled for all systems with the following property:

2.4.B. There exists a family x_t of mappings of Ω into measurable spaces $(E_t, \mathscr{B}_t)$ such that any σ-algebra $\mathscr{F}(I)$ is generated by x_t for $t \in I$.

2.5. An important class of stochastic systems can be described as follows. Let S be a linear topological space whose elements are real-valued functions on R, let a continuous linear functional $x_\varphi(\omega)$ of $\varphi \in S$ be given for each $\omega \in \Omega$, and let $\mathscr{F}(I)$ denote the σ-algebra in Ω generated by x_φ with $\varphi = 0$ outside I. The collection $\mathscr{F}(I)$ satisfies conditions 1.1.A and 1.1.B. It also satisfies condition 2.4.B if S is the space of all functions whose supports are finite sets.

Now let S be the Schwartz space of all functions $\varphi \in C^\infty$ with compact support (or tending sufficiently rapidly to 0 at ∞). Then (x_φ, P) is a generalized stochastic process according to Gelfand or a random distribution according to Itô. The most important example is white noise which is characterized by the following properties:

(i) $c_1 x_{\varphi_1} + \cdots + c_n x_{\varphi_n}$ is normally distributed for any $\varphi_1, \cdots, \varphi_n \in S$ and any constants $c_1, \cdots, c_n$;

(ii) $Px_\varphi = 0$ for all $\varphi \in S$;

(iii) $Px_\varphi x_\psi = \int \varphi(u)\psi(u)\, du$ for all $\varphi, \psi \in S$.

It is possible to extend the mapping $\varphi \rightarrow x_\varphi$ to all the functions $\varphi \in L^2(R)$ while still preserving these properties.

The derivative $\dot{x}_\varphi = -x_{\dot{\varphi}}$ *of the white noise generates a Markov system which is not hyper-Markov.* In fact, if $s < t$, $\varphi(u) = 0$ for $u > s$ and $\psi(u) = 0$ for $u < t$, then $\dot{x}_\varphi$ and $\dot{x}_\psi$ are independent, and hence 1.1.C is fulfilled. On the other hand we shall show that for any $\varphi \in L^2(R)$

$$P\{x_\varphi \mid \mathscr{F}(s, t)\} = x_\phi \quad \text{a.s.}\,, \tag{2.3}$$

where

$$\phi(u) = 1_{(s,t)}(u)\{\varphi(u) - \textstyle\int_s^t \varphi(v)\, dv\, (t - s)^{-1}\}\,.$$

By passage to the limit, we obtain

$$P\{x_\varphi \mid \mathscr{F}_{>s}\} = x_{1_{u>s}\varphi}\,, \qquad P\{x_\varphi \mid \mathscr{F}_{<t}\} = x_{1_{u<t}\varphi}\,,$$

and hence

$$P\{P[x_\varphi \mid \mathscr{F}_{>s}] \mid \mathscr{F}_{<t}\} = x_{1_{s<u<t}\varphi} \neq P\{x_\varphi \mid \mathscr{F}(s, t)\}$$

if $\int_s^t \varphi(v)\, dv \neq 0$. Thus 2.3.A is not true. In order to prove (2.3), we have to establish that:

(a) If $\psi \in S$ and $\psi = 0$ outside (s, t), then $P(x_\varphi \dot{x}_\psi) = P(x_\phi \dot{x}_\psi)$;

(b) $x_\phi \in \mathscr{F}(s, t)^P$.

Indeed (a) implies that if $\psi_1, \cdots, \psi_n$ vanish outside (s, t), then an arbitrary linear combination $c_1 x_{\psi_1} + \cdots + c_n x_{\psi_n}$ is independent of $x_\varphi - x_\phi$; therefore $x_\varphi - x_\phi$ is independent of $(x_{\psi_1}, \cdots, x_{\psi_r})$ and hence independent of $\mathscr{F}(s, t)$.

The statement (a) follows from an obvious equality

$$\int \varphi\dot{\psi}\, du = \int \phi\dot{\psi}\, du\,.$$

In order to prove (b), it suffices to remark that if $\phi \in L^2(R)$, $\phi = 0$ outside (s, t), $\int \phi\, du = 0$ and if the restriction of ϕ to (s, t) belongs to C^∞, then ϕ can be approximated in $L^2(R)$ by functions of class C^∞ which vanish outside (s, t) and have integrals equal to 0.

3. Fundamental σ-algebras. Projections.

3.1. In this section the fundamental σ-algebras introduced in 1.2 are investigated and some properties of the right, left and central projections will be established.

First we prove a general lemma.

LEMMA 3.1. *Let $\mathscr{Q}$ be a class of nonnegative functions on a set X such that:*

(i) *if $f, g \in \mathscr{Q}$, then $f + g \in \mathscr{Q}$;*
(ii) *if $f \in \mathscr{Q}$ and $0 \leqq g \leqq f$, then $g \in \mathscr{Q}$.*

Put $\xi \in \mathscr{K}$ if $\xi^+ = \xi \vee 0 \in \mathscr{Q}$ or $\xi^- = (-\xi) \vee 0 \in \mathscr{Q}$.

Let $\mathscr{H}$ denote the class of real-valued functions on X which satisfy the following conditions:

3.1.A. *If $\xi \in \mathscr{H}$, then $c\xi \in \mathscr{H}$ for every constant c.*

3.1.B. *If ξ and η are nonnegative and belong to $\mathscr{H}$, then $\xi + \eta \in \mathscr{H}$. If, in addition, ξ or η belongs to $\mathscr{H}$, then $\xi - \eta \in \mathscr{H}$.*

3.1.C. *If $0 \leqq \xi_n \uparrow \xi$ and $\xi_n \in \mathscr{H}$, then $\xi \in \mathscr{H}$.*

Let $\mathscr{A}$ be a class of bounded nonnegative functions on X with the properties:

3.1.D. *If $f, g \in \mathscr{A}$, then $fg \in \mathscr{A}$.*

3.1.E. *There exist functions $f_n \in \mathscr{A}$ such that $f_n \uparrow 1$.*

3.1.F. $\mathscr{A} \subseteq \mathscr{Q}$.

If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathscr{M}$ contains all the functions of $\mathscr{M}$ which are measurable with respect to the σ-algebra $\sigma(\mathscr{A})$ generated by $\mathscr{A}$.

PROOF. Let H_1 denote the class of bounded functions h such that $fh \in \mathscr{H}_1$ for every $f \in \mathscr{A}$. It is easy to verify that $\mathscr{H}_1$ contains all the constants, is closed with respect to the linear operations and that, if $0 \leqq h_n \uparrow h$, $h_n \in \mathscr{H}_1$ for $n = 1, 2, \cdots$ and h is bounded, then $h \in \mathscr{H}_1$. It is easy to deduce from these properties that $\mathscr{H}_1$ is closed with respect to uniform convergence. Since $\mathscr{A} \subseteq \mathscr{H}_1$, $\mathscr{H}_1$ thus contains all the bounded $\sigma(\mathscr{A})$-measurable functions (see [10], Chapter I, T20).

Let f_n be functions which are defined as in 3.1.E; then for every bounded $\sigma(\mathscr{A})$-measurable function h we have $f_n h \uparrow h$ and $f_n h \in \mathscr{H}$. Hence $\eta \in \mathscr{H}$. Relying on 3.1.B and 3.1.C, it is not hard to conclude that $\mathscr{H}$ contains all $\sigma(\mathscr{A})$-measurable functions of $\mathscr{K}$.

We shall apply Lemma 3.1 to functions ξ on the space $X = R \times \Omega$. Put $\xi \in \mathscr{Q}$ if there exists a function η such that $0 \leqq \xi_t(\omega) \leqq \eta(\omega)$ for all t, ω, and $P\eta < \infty$. Obviously $\mathscr{Q}$ satisfies conditions (i)—(ii) and the corresponding $\mathscr{K}$ coincides with the class introduced in 1.2. From now on $\mathscr{Q}$ and $\mathscr{K}$ designate these specific classes unless otherwise stipulated.

Let $\mathscr{A}^0$ be one of the following four families of functions:

$$\mathscr{L}_0\colon 1_{s<t<u}\varphi \quad \text{where } s < u \in R, \quad \varphi \in \mathscr{F}, \quad P\varphi < \infty;$$
$$\mathscr{L}_-\colon \varphi 1_{s<t} \quad \text{where } s \in R, \quad \varphi \in \mathscr{F}_{<s}, \quad P\varphi < \infty;$$
$$\mathscr{L}_+\colon 1_{t<u}\psi \quad \text{where } u \in R, \quad \psi \in \mathscr{F}_{>u}, \quad P\psi < \infty;$$
$$\mathscr{L}\colon \varphi 1_{s<t<u}\psi \quad \text{where } s < u \in R, \quad \varphi \in \mathscr{F}_{<s}, \quad \psi \in \mathscr{F}_{>u}, \quad P\varphi < \infty, \quad P\psi < \infty.$$

Put $f \in \mathscr{A}$ if f is indistinguishable from some function $f' \in \mathscr{A}^0$ which satisfies the inequality $f' \leqq 1$. Obviously $\mathscr{A}$ possess properties 3.1.D, E, F. If $\mathscr{A}^0 = \mathscr{L}_0$, then $\sigma(\mathscr{A}) = \mathscr{F}_{R\times\Omega}$. We show that, for $\mathscr{A}^0 = \mathscr{L}_-$, $\sigma(\mathscr{A})$is the predictable σ-algebra. It suffices to prove that every function $\xi \in \mathscr{S}_-$ adapted to $\mathscr{F}^P_{<t}$ is measurable with respect to $\sigma(\mathscr{A})$. But it is easy to see that ξ is indistinguishable from the $\sigma(\mathscr{A})$-measurable function

$$\xi_t(\omega) = \lim_{n\to\infty} \sum_k 1_{k/n<t\leqq(k+1)/n}\xi_{k/n} \,.$$

It can be proved analogously that, for $\mathscr{A}^0 = \mathscr{L}_+$, $\sigma(\mathscr{A})$ is the reconstructable σ-algebra. Hence, for $\mathscr{A}^0 = \mathscr{L}$, $\sigma(\mathscr{A})$ is the solid σ-algebra.

Applying Lemma 3.1 to the four classes $\mathscr{A}$ described above, we obtain the following proposition:

LEMMA 3.2. *Let a class $\mathscr{H}$ of functions on $R \times \Omega$ satisfy conditions* 3.1.A—C. *If $\mathscr{H}$ contains all the evanescent functions and the family $\mathscr{L}_0$ ($\mathscr{L}_-$, $\mathscr{L}_+$ or $\mathscr{L}$), then $\mathscr{H}$ contains all the $\mathscr{F}_{R\times\Omega}$-measurable (respectively, predictable, reconstructable or solid) functions of class $\mathscr{K}$.*

As the first application of Lemma 3.2, we prove that all the predictable functions are right. Indeed the function $\xi_t(\omega) = \varphi 1_{s<t} \in \mathscr{L}_-$ is equal to the limit of the right functions

$$\xi_t^n(\omega) = 1_{s+1/n\leqq t}\,\varphi(\omega) \,.$$

Hence it is a right function. According to Lemma 3.2, all the predictable functions of class $\mathscr{K}$ are right functions. Therefore all the predictable functions are right.

3.2. We remark that if $\xi' = \Pi^+\xi$, then every function ξ'' indistinguishable from ξ' is also a right projection of ξ. On the other hand, if ξ' and ξ'' are right projections of ξ, then, for each Markov time τ, the integrals of ξ_τ' and ξ_τ'' over an arbitrary set $C \in \mathscr{M}_\tau$ coincide. Hence $\xi_\tau' = \xi_\tau''$ a.s. Consequently ξ' and ξ'' are indistinguishable (see [1], Chapter IV, T13).

We now list some properties of right projections. (Here we write $\xi = \eta$ if ξ and η are indistinguishable.)

3.2.A. If $\Pi^+\xi = \xi'$, then $\Pi^+(c\xi) = c\xi'$ for every constant c.

3.2.B. If ξ and η are nonnegative and $\xi' = \Pi^+\xi$, $\eta' = \Pi^+\eta$, then $\xi' + \eta' = \Pi^+(\xi + \eta)$. If, in addition ξ or η is majorized by a summable function, then $\xi' - \eta' = \Pi^+(\xi - \eta)$.

3.2.C. If $0 \leqq \xi_n \uparrow \xi$ and $\Pi^+\xi_n = \xi_n'$, $n = 1, 2, \cdots$, then $\xi_n' \uparrow \Pi^+\xi$.

3.2.D. $\Pi^+\xi = 0$ if ξ is evanescent.

3.2.E. If I is an arbitrary interval and φ is a summable function then the function

$$\xi_t'(\omega) = \lim_{r\downarrow t} P\{\varphi \mid \mathscr{F}_{<r+}\} 1_I(t)$$

(where r takes rational values) is a right projection of the function $\xi_t(\omega) = \varphi(\omega)1_I(t)$.

3.2.F. If $\xi \geqq 0$ and $\Pi^+\xi = \xi'$ then for any right nonnegative function η, $\Pi^+(\eta\xi) = \eta\xi'$.

Property 3.2.E can be easily deduced from [10], Chapter VI, T4 (cf. [1], Chapter V, T9). The other properties are obvious.

3.3. THEOREM 3.1. *The right projection exists for every $\mathscr{F}_{R\times\Omega}$-measurable function $\xi \in \mathscr{K}$.*

Let $(\mathscr{F}(I), P)$ be a Markov system. Then:

3.3.A. *If ξ is reconstructable, then $\Pi^+\xi = \xi'$ is reconstructable, central and progressive. There exists a version of ξ' such that, for each t, ξ_t' is $\mathscr{F}(t, t+)$-measurable.*

3.3.B. *If a reconstructable function ξ belongs to $\mathscr{S}_+ \cap \mathscr{Q}$, then $\Pi^+\xi \in \mathscr{S}_+$.*

3.3.C. *Let ξ_1 be predictable, ξ_2 be reconstructable and let $\xi = \xi_1\xi_2$. If $\Pi^-\xi_1 = \eta_1$, $\Pi^+\xi_2 = \eta_2$, then*

$$\Pi^+\Pi^-\xi = \Pi^-\Pi^+\xi = \eta_1\eta_2 .$$

3.3.D. $\Pi^+\Pi^-\xi = \Pi^-\Pi^+\xi$ *for each solid function $\xi \in \mathscr{K}$.*

PROOF. Put $\xi \in \mathscr{H}$ if there exists a right projection of ξ. Properties 3.2.A—C imply 3.1.A—C. By virtue of 3.2.D—F $\mathscr{H}$ contains $\mathscr{L}$ and all the evanescent functions of $\mathscr{K}$. By Lemma 3.2 $\mathscr{H}$ contains all $\mathscr{F}_{R\times\Omega}$-measurable functions of class $\mathscr{K}$.

Now denote by $\mathscr{H}$ the totality of all functions ξ for which 3.3.A is fulfilled. Obviously $\mathscr{K}$ satisfies 3.1.A—C and contains all evanescent functions. According to Lemma 3.2, the statement 3.3.A will be proved if we show that $\mathscr{H} \supseteq \mathscr{L}_+$. But the right projection of the function $\xi_t = 1_{t<u}\varphi \in \mathscr{L}_+$ is given by the formula

$$\xi_t'(\omega) = \lim_{r\downarrow t} P\{\varphi \mid \mathscr{F}(r, r+)\}1_{t<u}$$

in view of 3.2.E and 2.2.C and (2.1). It is evident that $\xi' \in \mathscr{S}_+$ and $\xi_t' \in \mathscr{F}(t, t+)$ for each t. Hence ξ' is right, reconstructable and progressive. But all reconstructable functions are solid. They are also left (the dual statement was proved at the end of 3.1). Thus $\xi \in \mathscr{H}$.

Now let $\xi \in \mathscr{S}_+ \cap \mathscr{Q}$ and $\Pi^+\xi = \xi'$. According to [1] (Chapter IV, T28), 3.3.B will be proved if we show that $P\xi'_{\tau_n} \downarrow P\xi_\tau'$ for arbitrary Markov times $\tau_n \downarrow \tau$. But $P\xi_{\tau_n} \downarrow P\xi_\tau$ and by virtue of (1.1), $P\xi'_{\tau_n} = P\xi_{\tau_n}$, $P\xi_\tau' = P\xi_\tau$.

Let us prove 3.3.C. As we know, any predictable function ξ_1 is right. Hence 3.2.F implies that $\Pi^+\xi = \xi_1\eta_2$. But η_2 is a left function according to 3.3.A. Therefore $\Pi^-\Pi^+\xi = \eta_1\eta_2$. Analogously, $\Pi^+\Pi^-\xi = \eta_1\eta_2$.

3.3.D can be easily deduced from 3.3.C and Lemma 3.2.

Relying on 3.3.D, we define the central projection of a solid function $\xi \in \mathscr{K}$ by the formula

$$\Pi\xi = \Pi^+\Pi^-\xi = \Pi^-\Pi^+\xi .$$

In accordance with 1.2, denote the class of all central sets by $\mathscr{C}$. Let $\mathscr{C}_+$ $(\mathscr{C}_-)$

be the totality of all reconstructable (respectively, predictable) central sets. The classes $\mathscr{C}$, $\mathscr{C}_+$ and $\mathscr{C}_-$ are σ-algebras in $R \times \Omega$.

The following propositions hold:

3.4.A. $\mathscr{C} = \mathscr{C}_+ \vee \mathscr{C}_-$.

3.4.B. All central functions are progressive.

3.4.C. Each function in $\mathscr{C}_+$ is indistinguishable from a function ξ with the property: for each t, $\xi_t \in \mathscr{F}(t, t+)$.

3.4.D. If $\xi \in \mathscr{S}_+$ and $\xi_t \in \mathscr{F}(t, t+)$ for each t, then $\xi \in \mathscr{C}_+$.

In fact, 3.3.C and Lemma 3.2 imply that, for any solid $\xi \in \mathscr{K}$, the function $\Pi\xi$ is measurable with respect to the σ-algebra generated by $\eta_1\eta_2$ where η_1 is predictable and η_2 is reconstructable. But $\xi = \Pi\xi$ for any central ξ. Therefore 3.4.A—B follow from 3.3.A—A*. If $\xi \in \mathscr{C}_+$, then $\xi = \Pi^+\xi$, and 3.3.A implies 3.4.C. Proposition 3.4.D is evident.

3.5. Let ξ be an $\mathscr{F}_{R\times\Omega}$-measurable function of class $\mathscr{K}$ and let $\mathscr{A}$ be a sub-σ-algebra of the σ-algebra $\mathscr{F}$. Then there exists a function η_t such that

$$P(\xi_\varphi \,|\, \mathscr{A}) = \eta_\varphi \quad \text{a.s.}$$

for each $\mathscr{A}$-measurable function φ; η is determined uniquely up to indistinguishability; we denote it by $P_{\mathscr{A}}\xi$ and call it the regular conditional mathematical expectation of ξ. The operators $P_{\mathscr{A}}$ have the following properties:

3.5.A. $P_{\mathscr{A}}(c\xi) = cP_{\mathscr{A}}\xi$ for every constant c.

3.5.B. $P_{\mathscr{A}}(\xi + \eta) = P_{\mathscr{A}}\xi + P_{\mathscr{A}}\eta$ for nonnegative ξ and η. If, in addition, ξ or η belongs to $\mathscr{Q}$, then $P_{\mathscr{A}}(\xi - \eta) = P_{\mathscr{A}}\xi - P_{\mathscr{A}}\eta$.

3.5.C. If $0 \leqq \xi_n \uparrow \xi$, then $P_{\mathscr{A}}\xi_n \uparrow P_{\mathscr{A}}\xi$.

3.5.D. $P_{\mathscr{A}}\xi = 0$ for evanescent ξ.

3.5.E. If ξ and η are nonnegative and $\eta \in \mathscr{B}(R) \times \mathscr{A}$, then $P_{\mathscr{A}}(\xi\eta) = \eta P_{\mathscr{A}}\xi$.

3.5.F. $P_{\mathscr{A}}\xi \in \mathscr{S}_+$ for $\xi \in \mathscr{S}_+ \cap \mathscr{Q}$.

All these assertions follow easily from [8] or they can be deduced from 3.2—3.3. In fact the operator $P_{\mathscr{A}}$ coincides with the operator Π^+ for the stochastic system $(\hat{\mathscr{F}}(I), P)$ where $\hat{\mathscr{F}}(I) = \mathscr{A}$ for all I.

3.6. We associate with every open interval I the operator

$$\phi_I : \xi \to \phi_I\xi = P_{\mathscr{F}(I)}(\xi 1_I)\,, \qquad \xi \in \mathscr{K}.$$

The operators $\phi_{<u}$ and $\phi_{>u}$ correspond to intervals $(-\infty, u)$ and $(u, +\infty)$.

If ξ is predictable, then

$$\phi_{<u}(\xi\eta) = \xi\phi_{<u}\eta\,. \tag{3.1}$$

Indeed the function $\xi_t 1_{t<u}$ is measurable with respect to $\mathscr{B}(R) \times \mathscr{F}^P_{<u}$. Therefore (3.1) follows from 3.5.E.

THEOREM 3.2. *Let $(\mathscr{F}(I), P)$ be a hyper-Markov system and let $\xi \in \mathscr{K}$. If ξ is* *reconstructable, then* $\eta = \phi_{<u}\xi$ *is also reconstructable and*

$$\Pi^+\eta = \Pi^+(\xi 1_{t<u}) . \tag{3.2}$$

If ξ *is solid, then* $\eta = \phi_I \xi$ *is solid too and*

$$\Pi\eta = \Pi(\xi 1_I) . \tag{3.3}$$

PROOF. In view of Lemma 3.2 and 3.4.A—D, the equality (3.2) is true for all reconstructable $\xi \in \mathscr{K}$ if it is true for $\xi \in \mathscr{S}_+ \cap \mathscr{Q}$ (since $\mathscr{L}_+ \subseteqq \mathscr{S}_+ \cap \mathscr{Q}$). By 3.5.F, $\eta = \phi_{<u}\xi \in \mathscr{S}_+$. By virtue of 1.1.D and (2.1),

$$\eta_t = P\{\xi_t \mid \mathscr{F}_{<u}\} = P\{\xi_t \mid \mathscr{F}(t, u)\} \in \mathscr{F}_{>t}$$

for $t < u$. Since $\eta_t = 0$ for $t \geqq u$, η is reconstructable.

By virtue of 3.3.B, the functions $\xi' = \Pi^+(\xi 1_{t<u})$ and $\eta' = \Pi^+\eta$ belong to $\mathscr{S}_+$. Both functions vanish for $t \geqq u$. Therefore (3.2) is true if $\xi_t' = \eta_t'$ a.s. for each $t < u$. But (1.1) and 2.2.A imply that for $t < u$

$$\begin{aligned}\eta_t' &= P\{\eta_t \mid \mathscr{F}_{<t+}\} = P\{P\{\xi_t \mid \mathscr{F}(t, u)\} \mid \mathscr{F}(t, t+)\} = P\{\xi_t \mid \mathscr{F}(t, t+)\} \\ &= P\{\xi_t \mid \mathscr{F}_{<t+}\} = \xi_t' \quad \text{a.s.}\end{aligned}$$

In order to prove (3.3), it is sufficient to verify it for $\xi = \xi_1\xi_2$ where ξ_1 and ξ_2 are nonnegative, ξ_1 is predictable and ξ_2 is reconstructable. Let $I = (s, u)$. By 2.3.A, $\eta = \phi_{>s}\phi_{<u}\xi$. By (3.1) $\phi_{<u}\xi = \xi_1\phi_{<u}\xi_2$ and the second factor is reconstructable. It follows from (3.1)* that $\eta = \phi_{>s}\xi_1\phi_{<u}\xi_2$ and the first factor is predictable. Now 3.3.C implies the equalities

$$\begin{aligned}\Pi\eta &= \Pi^-(\phi_{>s}\xi_1)\Pi^+(\phi_{<u}\xi_2) , \\ \Pi(\xi 1_I) &= \Pi^-(\xi_1 1_{t>s})\Pi^+(\xi_2 1_{t<u})\end{aligned}$$

and the right sides are equal in view of (3.2) and (3.2)*.

3.7. We formulate a simple criterion of solidity. Put $\mathscr{F}_{\neq t} = \mathscr{F}_{<t} \vee \mathscr{F}_{>t}$.

THEOREM 3.3. *If* ξ *is solid, then* ξ *is* $\mathscr{F}_{R\times\Omega}$*-measurable and* ξ_t *is* $\mathscr{F}^P_{\neq t}$*-measurable for each* t*. The converse is true if each* σ*-algebra* $\mathscr{F}(I)$ *is generated by* $\mathscr{F}(t-, t+)$, $t \in I$.

The first assertion is deduced easily with the help of Lemma 3.2. The second assertion is proved in [6] (Section 2).

3.8. Now we justify the description of the central σ-algebra formulated in 1.5.

Denote by $\mathscr{H}$ the class of all functions f on the state space E for which $f(t, x_t)$ is central. Evidently, $\mathscr{H}$ contains all the constants and is closed under addition, multiplication and passage to the limit. According to 3.4.D—D* and the definition of regular representation, $\mathscr{H}$ contains all the base functions. Therefore $\mathscr{H}$ contains all measurable functions.

Now we prove that each central function is indistinguishable from a function $f(t, x_t)$ with f measurable. It suffices to establish this for functions $\Pi\xi$ with solid ξ. By virtue of Lemma 3.2 it suffices to consider only functions $\xi \in \mathscr{L}$, i.e., functions $\xi_t = \varphi 1_{s<t<u} \psi$ where $\varphi \in \mathscr{F}_{<s}$, $\psi \in \mathscr{F}_{>u}$, $P\varphi < \infty$ and $P\psi < \infty$. In view of 3.3.C

$$\Pi\xi = (\Pi^- \varphi 1_{s<t})(\Pi^+ \psi 1_{t>u}) .$$

The strong Markov property and the dual property imply that

$$\Pi^+ \psi 1_{t>u} = 1_{t>u} P_{t,x_t} \psi ,$$
$$\Pi^- \varphi 1_{s<t} = 1_{s<t} P^{t,x_t} \varphi$$

(see [3], Theorem 3.1). Thus $\Pi\xi$ is indistinguishable from $f(t, x_t)$, where $f(t, x) = 1_{s<t<u} P_{t,x} \varphi P^{t,x} \varphi$.

4. Finite additive functionals.

4.1. We proceed to investigate additive functionals.

LEMMA 4.1. *Under conditions* 1.4.A—B, *the properties of* 1.4.C, C′, C″ *are equivalent.*

PROOF. For any $s \in R$ the function $\xi_s(t) = 1_{s<t} A(s, t)$ is predictable and the function $\eta_s(t) = 1_{t<s} A(t, s)$ is reconstructable. We have

$$1_{s<t<u} A(s, u) = 1_{s<t<u} A\{t\} + 1_{t<u} \xi_s(t) + 1_{s<t} \eta_u(t)$$

for every $s < u \in R$. Therefore under conditions 1.4.B—C the function $1_{s<t<u} A\{t\}$ is solid. Letting $s \to -\infty$ and $t \to +\infty$, we prove that $A\{t\}$ is also solid. Thus 1.4.C′ is satisfied. On the other hand if 1.4.C′ is true, then $1_{s<t<u} A(s, u)$ is solid, and 1.4.C follows from the relation

$$1_{t \leq s} A(s, u) = \lim_{r \downarrow s} 1_{t<r} A(r, u)$$

and the corresponding dual relation.

The formula

$$A\{t\} = \lim_{s \to -\infty} \lim_{\varepsilon \downarrow 0} 1_{s<t} A(s, t] - 1_{s<t-\varepsilon} A(s, t - \varepsilon]$$

shows that $A\{t\}$ is a right function. By symmetry it is also left. Hence 1.4.C′ and 1.4.C″ are equivalent.

LEMMA 4.2. *Let A be a σ-functional and let f be a nonnegative progressive function. Then*

$$B(\omega, \Gamma) = \int_\Gamma f_t(\omega) A(\omega, dt) , \qquad \Gamma \in \mathscr{B}(R)$$

satisfies 1.4.A; *if f is central and finite, then B is a σ-functional.*

PROOF. The restriction of a progressive function f to the set I is $\mathscr{B}(I) \times \mathscr{F}(I)^P$-measurable and for every $\Gamma \in \mathscr{B}(I)$ the function $A(\omega, \Gamma)$ is $\mathscr{F}(I)^P$-measurable. Therefore (see, e.g., [2], Lemma 1.7), the function

$$F(\omega_1, \omega_2) = \int_I f_t(\omega_1) A(\omega_2, dt)$$

is $\mathscr{F}(I)^P \times \mathscr{F}(I)^P$-measurable and $B(I) = F(\omega, \omega)$ is $\mathscr{F}(I)^P$-measurable. The first assertion of the lemma is proved.

Obviously B satisfies condition 1.4.C'' if f is central. For bounded f, the validity of 1.4.B for A implies its validity for B. Therefore the second assertion is also true.

4.2. The main result of this section is:

THEOREM 4.1. *If A is a finite additive functional, then there exists a positive central function φ such that*

$$P \int \varphi_t A(dt) < \infty . \tag{4.1}$$

In other words, the spectral measure of any finite additive functional is σ-finite.

This result was first proved by Šur [12] in a slightly different form. We adapt his proof as follows:

PROOF. Without loss of generality we can assume that

$$A\{t\} \leqq \rho \qquad \text{for all} \quad t$$

where ρ is a constant. Indeed in view of Lemma 4.2 the formula

$$A_\rho(dt) = 1_{A\{t\} \leqq \rho} A(dt)$$

defines a finite additive functional. Let $\varphi_t{}^\rho$ be a positive central function such that

$$P \int \varphi_t{}^\rho A_\rho(dt) = c_\rho < \infty .$$

Then the function

$$\varphi_t = \sum_{k=1} c_k{}^{-1} 2^{-k} \varphi_t{}^k 1_{A\{t\} \leqq k}$$

is positive, central and satisfies (4.1).

Consider a finite set $\Lambda = \{t_0 < t_1 < \cdots < t_n\}$ and put

$$\xi_t = A(t, t_k] \qquad \text{for} \quad t_{k-1} \leqq t < t_k ,$$
$$\eta_t = F(\xi_t) \qquad \text{where} \quad F(u) = 1 - e^{-u} ,$$
$$\zeta_t = 1_{\xi_t \geqq 1} = 1_{\eta_t \geqq F(1)} .$$

These functions are reconstructable. Let $\bar\eta = \Pi^+ \eta$, $q = t_{i-1}$, $r = t_i$, $\gamma = F(1)/2$. Put

$$B(dt) = 1_{\bar\eta_t \leqq \gamma} A(dt)$$

and consider the Markov times

$$\tau_0 = q ; \qquad \tau_{k+1} = \inf\{t : t > \tau_k, B(\tau_k, t] > 1\} \quad \text{for} \quad k = 0, 1, 2, \cdots .$$

We prove that for each $k > 0$

$$\bar\eta_{\tau_k} \leqq \gamma \quad \text{a.s.} \quad \text{on the set} \quad \{\tau_k < \infty\} . \tag{4.2}$$

Indeed $\eta \in \mathscr{S}_+$. Hence $\bar\eta \in \mathscr{S}_+$. If $\tau_k < \infty$, then $B(\tau_{k-1}, t] \leqq 1$ for $t < \tau_k$ and $B(\tau_{k-1}, t] > 1$ for $t > \tau_k$. Therefore $B(\tau_{k-1}, \tau_k) \leqq 1$ and $B[\tau_k, t] > 0$ for $t > \tau_k$. Consequently, each segment $[\tau_k, t]$ contains t' for which $\bar\eta_{t'} \leqq \gamma$.

We have

$$P\{\tau_{k+1} < r\} \leqq P\{\tau_k < r, B(\tau_k, r] > 1\} \leqq P\{\tau_k < r, \xi_{\tau_k} \geqq 1\} = P 1_{\tau_k < r} \zeta_{\tau_k} .$$

But $\zeta \leqq F(1)^{-1}\eta$ and by virtue of (1.1) and (4.2)

$$P\{\tau_{k+1} < r\} \leqq F(1)^{-1}P1_{\tau_k<r}\eta_{\tau_k} = F(1)^{-1}P1_{\tau_k<r}\bar{\eta}_{\tau_k} \leqq 2^{-1}P\{\tau_k < r\}\,.$$

As we know $B(\tau_{k-1}, \tau_k) \leqq 1$. Hence $B(\tau_{k-1}, \tau_k] \leqq \rho + 1$ and $B(\tau_0, \tau_n] \leqq n(\rho + 1)$. Therefore

$$P\{B(q, r] > n(\rho + 1)\} \leqq P\{\tau_n < r\} \leqq 2^{-n}$$

and $PB(q, r] < \infty$. We have thus proved that (4.1) is fulfilled for $\varphi_t = 1_{t_0<t\leqq t_n, \bar{\eta}_t \leqq \gamma}$. Let Λ_m be an increasing sequence of finite sets with everywhere dense union and consider the corresponding sequences η^m and φ^m. Observe that $\eta_t^m \downarrow 0$ for all t a.s. Put $c_m = \int \varphi_t^m A(dt)$. The function

(4.3) $$\varphi = \sum_{m=1} 2^{-m} c_m^{-1} \varphi^m$$

is positive, central and satisfies (4.1).

4.3. Theorem 4.1 implies that each σ-functional can be represented as a sum of functionals A_n for which $PA_n(R) < \infty$. Consequently, the spectral measure of each σ-functional is a σ-measure.

It is sufficient to prove this assertion for finite functionals A. Put $A^c(dt) = 1_{\varphi_t \geqq c} A(dt)$ where φ satisfies condition (4.1). We have

$$PA^c(R) \leqq P \int_R c^{-1}\varphi_t A(dt) < \infty\,.$$

Let $c_n = 2^n$, $n = \cdots, -2, -1, 0, 1, 2, \cdots$. Then A is a sum of the functionals $A_n = A^{c_n} - A^{c_{n-1}}$.

5. The fundamental identity and its implications.

5.1. Let μ be the measure defined by formula (1.2). Our aim is to prove the fundamental identity (1.3). In view of 4.3 we can assume that μ is finite. We wish to apply Lemma 3.2 to the set $\mathscr{H}$ of all solid functions ξ satisfying (1.3). Conditions 3.1.A—C are fulfilled for $\mathscr{H}$ by virtue of 3.2.A—C and 3.2.A*—C*. In view of 3.2.D—D*, $\mathscr{H}$ contains all evanescent functions. It remains to prove the inclusion $\mathscr{H} \supseteq \mathscr{L}$.

Let

$$\xi_t = \varphi 1_{s<t<u}\psi\,, \qquad \text{where} \quad \varphi \in \mathscr{F}_{<s}\,, \quad \psi \in \mathscr{F}_{>u}\,, \quad P\psi < \infty\,, \quad P\psi < \infty\,.$$

By 3.3.C

$$\Pi\xi = \eta\zeta \qquad \text{where} \quad \eta_t = \Pi^-(\varphi 1_{s<t})\,, \quad \zeta_t = \Pi^+(1_{t<u}\psi)\,.$$

By (1.1) and (1.1)* for every $s < q < r < u$

$$P\{\psi \mid \mathscr{F}_{<r+}\} = \zeta_r\,, \qquad P\{\varphi \mid \mathscr{F}_{>q-}\} = \eta_q \quad \text{a.s.}$$

Therefore

(5.1) $$P\varphi A(q, r)\psi = P\{\varphi A(q, r) P\{\psi \mid \mathscr{F}_{<r+}\}\} = P\varphi A(q, r)\zeta_r = P\eta_q A(q, r)\zeta_r\,.$$

Let $\Lambda = \{t_1 < t_2 < \cdots < t_m\}$ be a finite subset of the interval (s, u) such that

$PA(\Lambda) = 0$. Put $t_0 = s$, $t_{m+1} = u$ and introduce the functions

$$\gamma(t) = t_{k-1}, \qquad \delta(t) = t_k \quad \text{for} \quad t \in [t_{k-1}, t_k) .$$

By (5.1)

$$(5.2) \qquad \mu(\xi) = P \int \xi_t A(dt) = \sum P\varphi A(t_{k-1}, t_k)\psi = \sum P\eta_{t_{k-1}} A(t_{k-1}, t_k)\zeta_{t_k}$$
$$= P \int_s^u \eta_{\gamma(t)} \zeta_{\delta(t)} A(dt) .$$

Let Δ_m be an expanding sequence of finite sets such that $\bigcup \Lambda_m$ is everywhere dense in (s, u). Consider the corresponding functions γ_m and δ_m and observe that $\gamma_m(t) \uparrow t$ and $\delta_m(t) \downarrow t$. According to 3.3.B—B*, $\eta \in \mathscr{S}_-$ and $\zeta \in \mathscr{S}_+$. Therefore $\eta_{\gamma_m(t)} \zeta_{\delta_m(t)} \to \eta_t \zeta_t$ and the right side of (5.2) converges to $\mu(\Pi\xi)$. Hence $\mathscr{L} \subseteq \mathscr{H}$. By Lemma 3.2, $\mathscr{H}$ contains all solid functions of class $\mathscr{K}$.

5.2. THEOREM 5.1. *Two σ-functionals with identical measures are indistinguishable.*

PROOF. 1°. Let the measure $\mu_A = \mu_B = \mu$ be finite. The fundamental identity (1.3) implies that if ξ is solid and independent of t, then

$$P\xi A(I) = P\xi B(I) = \mu(\xi 1_I) \qquad \text{for all} \quad I .$$

The function $\xi = A(I) - B(I)$ is solid by 1.4.C. Hence $P\xi^2 = P\xi[A(I) - B(I)] = 0$ and $A(I) = B(I)$ a.s. But as two finite measures coincide if they coincide on all the intervals (r_1, r_2) with rational r_1 and r_2, we must have $A = B$. (The equality sign $=$ means here and later indistinguishability of functionals.)

2°. If the spectral measure μ_A is finite and if

$$d\mu_{\tilde{A}} = f\, d\mu_A$$

with a central function f, then

$$\tilde{A}(dt) = f_t A(dt) .$$

In fact the right side of the last formula defines an additive functional with the spectral measure $\mu_{\tilde{A}}$ (cf. Lemma 4.2).

3°. Every σ-functional A can be represented in the form

$$(5.3) \qquad A(dt) = f_t \tilde{A}(dt) ,$$

where $\tilde{A}$ is an additive functional with a finite spectral measure and f is a central function. The measures μ_A and $\mu_{\tilde{A}}$ charge the same sets (i.e., assign positive measure to the same sets).

Indeed according to 4.3, $A = \sum A_n$ where $PA_n(R) = c_n < \infty$. Put

$$\tilde{A} = \sum c_n^{-1} 2^{-n} A_n , \qquad \mu_A = \mu , \quad \mu_{\tilde{A}} = \tilde{\mu} , \qquad \mu_{A_n} = \mu_n .$$

The measures μ and $\tilde{\mu}$ charge the same sets, μ_n and $\tilde{\mu}$ are finite and the μ_n are absolutely continuous with respect to $\tilde{\mu}$. By the Radon–Nikodym theorem $d\mu_n = f^n\, d\tilde{\mu}$ where f^n is a central function. By 2°, $A_n(dt) = f_t^n \tilde{A}(dt)$. Hence (5.3) is fulfilled with $f = \sum f^n$.

4°. Now let A and B be arbitrary σ-functionals with identical spectral measures $\mu_A = \mu_B = \mu$. By 3°

$$A(dt) = f_t \bar{A}(dt), \qquad B(dt) = g_t \bar{B}(dt),$$

where f and g are central functions and $\bar{A}$ and $\bar{B}$ are additive functionals with finite spectral measures $\mu_{\bar{A}}$ and $\mu_{\bar{B}}$. The measures μ, $\mu_{\bar{A}}$, and $\mu_{\bar{B}}$ charge the same sets. Hence $d\mu_{\bar{A}} = h\, d\mu_{\bar{B}}$ and, according to 2°, $\bar{A}(dt) = h_t \bar{B}(dt)$. Hence $A(dt) = f_t g_t \bar{B}(dt)$.

We thus have

$$d\mu = fh\, d\mu_{\bar{B}} = g\, d\mu_{\bar{B}}.$$

Now put $C = \{fh = g\}$ and $D = \{fh \neq g\}$. Obviously,

$$1_C A(dt) = 1_C fh\bar{B}(dt) = 1_C g\bar{B}(dt) = 1_C B(dt).$$

On the other hand, $\mu_{\bar{B}}(D) = 0$. Hence

$$P \int 1_D(t) A(dt) = \mu_A(D) = 0, \qquad P \int 1_D(t) B(dt) = \mu_B(D) = 0$$

and

$$1_D(t) A(dt) = 1_D(t) B(dt) = 0.$$

Thus $A = B$.

5.3. THEOREM 5.2. *A σ-functional A is continuous if and only if its spectral measure μ_A charges no scanty set.*

PROOF. The set $C = \{(t, \omega): A\{t\} > 0\}$ is scanty and central. Therefore $\mu_A(C) = 0$ if μ_A charges no scanty sets. But

$$\mu_A(C) = P \int 1_{A\{t\}>0} A(dt) = P \sum A\{t\}.$$

Hence $P\{A\{t\} > 0 \text{ for some } t\} = 0$.

The remaining part of the theorem is trivial.

6. Construction of additive functionals.

6.1. The aim of this section is to prove that every σ-measure on the central σ-algebra which does not charge any evanescent set is a spectral measure of a σ-functional. This result will be established for hyper-Markov systems.

In fact, a stronger proposition will be proved. Let us say that $(\mathscr{F}(I), \tilde{P})$ is subordinate to $(\mathscr{F}(I), P)$ if, for any finite interval I and any $C \in \mathscr{F}(I)$ the equality $P(C) = 0$ implies the equality $\tilde{P}(C) = 0$. Starting from a measure μ, we construct A which is a σ-functional not only of $(\mathscr{F}(I), P)$ but also of all subordinate stochastic systems.

6.2. Put $C \in \mathscr{S}(I)$ if $C \in \mathscr{F}(I)$ and the set $R \times C$ is solid. Let us prove that $\xi \in \mathscr{F}(s, u)$ belongs to $\mathscr{S}(s, u)$ if and only if $\xi 1_{s<t<u}$ is solid. It suffices to prove that the functions $\xi 1_{t \leq s}$ and $\xi 1_{u \geq t}$ are solid if $\xi \in \mathscr{F}(s, u)$. By virtue of 1.1.B we can assume that $\xi \in \mathscr{F}(q, r)$ where $s < q < r < u$, and in this case $\xi 1_{t \leq s} = \xi 1_{t<q} 1_{t \leq s}$ is reconstructable and $\xi 1_{t \geq u} = \xi 1_{t>r} 1_{t \geq u}$ is predictable.

6.3. LEMMA 6.1. *Let $(\mathscr{F}(I), P)$ be a hyper-Markov system and let μ be a finite measure on the central σ-algebra $\mathscr{C}$ which does not charge any evanescent set. Then for each open interval I, there exists a function $a(I) \in \mathscr{G}(I)$ such that*

$$Pa(I)\xi = \int \Pi(\xi 1_I)\, d\mu \tag{6.1}$$

if $\xi(\omega)1_I(t)$ is a solid function.

PROOF. The integrals of indistinguishable functions with respect to the measure μ coincide. Hence

$$Q_I(C) = \int \Pi[1_C(\omega)1_I(t)]\, d\mu \tag{6.2}$$

is uniquely determined for every $C \in \mathscr{F}(I)$. In view of 3.2.A—C and 3.2.A*—C*, Q_I is a finite measure on $\mathscr{G}(I)$. By 3.2.D—D*, $Q_I(C) = 0$ if $P(C) = 0$. By the Radon–Nikodym theorem, there exists a function $a(I) \in G(I)$ such that

$$Q_I(\xi) = Pa(I)\xi \tag{6.3}$$

for each $\xi \in \mathscr{G}(I)$. Formula (6.1) now follows from (6.2) and (6.3).

Consider next the case when ξ is not $\mathscr{F}(I)$-measurable but $\xi 1_I$ is solid, and let

$$\xi' = P\{\xi \mid \mathscr{F}(I)\}\,. \tag{6.4}$$

It is clear that $\phi_I \xi = \xi' 1_I$. By Theorem 3.2, $\xi' 1_I$ is solid and

$$\Pi(\xi' 1_I) = \Pi(\xi 1_I)\,. \tag{6.5}$$

Therefore, according to 6.2, ξ' is solid. Hence $\xi' \in \mathscr{G}(I)$ and (6.1) is fulfilled for ξ'. By virtue of (6.4) and (6.5) this formula is true also for ξ.

6.4. THEOREM 6.1. *Let $(\mathscr{F}(I), P)$ be a hyper-Markov system. Suppose that a finite measure μ on the central σ-algebra does not charge any evanescent set. Then μ is a spectral measure of a finite additive functional A. This functional can be constructed in such a way that there exists a set Ω_0 with the properties:*

(i) $P(\Omega_0) = 0$;

(ii) *Ω_0 is the union of a countable family of sets which belong to the σ-algebras $\mathscr{F}(I)$ for the finite I;*

(iii) *$A(I)$ coincides with a $\mathscr{F}(I)$-measurable function outside Ω_0.*

PROOF. Put $\omega \in B_I$ if

$$a(r_1, r_2) \leqq a(q_1, q_2) \tag{6.6}$$

for all rational $q_1 < r_1 < r_2 < q_2 \in I$ and

$$\lim_{r \downarrow t} a(s, r) + a(t, u) = a(s, u) \tag{6.7}$$

for all rational $s < t < u \in I$. (Passage to the limit in (6.7) as well as in subsequent formulas (6.8)—(6.11) is performed over the set of rational numbers.) Consider the complement C_I of B_I and the union Ω_0 of C_I over all finite intervals I with rational endpoints. It is obvious that B_I and C_I belong to $\mathscr{F}(I)$ and that $P(\Omega_0) = 0$.

Now suppose $\omega \notin \Omega_0$. Then (6.6)—(6.7) are fulfilled for all rational $q_1 < r_1 < r_2 < q_2$ and $s < t < u$. For any rational s consider the function on the half-line $[s, +\infty)$ defined by the formula

(6.8) $$F_s(u) = \lim_{r \downarrow u} a(s, r) .$$

The function F_s is nondecreasing and right continuous and $F_s(s) = 0$. Hence there exists a measure A_s on $(s, +\infty)$ such that $F_s(u) = A_s(s, u]$ for all $u > s$. By (6.7),

$$A_s(s, t] + A_t(t, u] = A_s(s, u]$$

for all rational $s < t < u$. Therefore the measures A_s and A_t coincide on $(t, +\infty)$ and there exists a measure A on R such that

(6.9) $$A(s, t] = \lim_{r \downarrow t} a(s, r)$$

for all rational $s < t$. It follows from (6.9) that

(6.10) $$A(s, t) = \lim_{r_1 \downarrow s, r_2 \uparrow t} a(r_1, r_2)$$

for all $s < t \in R$.

Put $A(\omega, \Gamma) = 0$ for $\omega \in \Omega_0$. It is clear that A satisfies conditions 1.4.A and C. The function

(6.11) $$\bar{A}(s, t) = \lim_{r_1 \downarrow s, r_2 \uparrow t} a(r_1, r_2)$$

belongs to $\mathscr{F}(s, t)$ and $A = \bar{A}$ outside Ω_0. The equality

(6.12) $$\mu(\Pi\xi) = P \int \xi_t A(dt)$$

is satisfied for $\xi \in \mathscr{L}$ by virtue of (6.1) and (6.10). By Lemma 3.2, it is also true for all solid functions $\xi \in \mathscr{K}$. In particular, if ξ is central, then $\Pi\xi = \xi$ and (6.12) implies that A satisfies 1.4.B and that μ is the spectral measure of A.

6.5. Theorem 6.2. *Let $(\mathscr{F}(I), P)$ be a hyper-Markov system and let μ be a σ-measure on the central σ-algebra which charges no evanescent set. Then a σ-functional A of $(\mathscr{F}(I), P)$ can be constructed with the spectral measure μ such that A is also a σ-functional of every system $(\mathscr{F}(I), \tilde{P})$ which is subordinate to $(\mathscr{F}(I), P)$.*

Suppose that the density $d\tilde{P}/dP$ on $\mathscr{F}(s, t)$ is equal to $g_s(\omega)h^t(\omega)$, where g and h are bounded, $g \in \mathscr{S}_-$, and $h \in \mathscr{S}_+$. Then

(6.13) $$\tilde{P} \int \xi_t A(dt) = P \int \xi_t g_t h^t A(dt)$$

for every function ξ which is progressive relative to $(\mathscr{F}(I), P)$.

Proof. The general case can be easily reduced to the case of finite μ and bounded ξ. Consider the finite additive functional A with the spectral measure μ described in Theorem 6.1. The equality $P(\Gamma) = 0$ implies that $\tilde{P}(\Gamma) = 0$. Hence $A(I) \in \mathscr{F}(I)^{\tilde{P}}$ and 1.4.A is fulfilled for the measure $\tilde{P}$. Obviously, 1.4.B—C are fulfilled too.

In order to prove (6.13) we put $B(dt) = \xi_t A(dt)$. According to Lemma 4.2 $B(I) \in \mathscr{F}(I)^P$. Hence

$$\tilde{P}B(s, t) = Pg_s B(s, t)h^t$$

for every $s < t$. It suffices to prove (6.13) for ξ vanishing outside a finite interval (s, u). Consider the set Λ and the functions γ and δ introduced in 5.1. We have

$$\tilde{P} \int \xi_t A(dt) = P \int g_{\gamma(t)} h^{\delta(t)} \xi_t A(dt) . \tag{6.14}$$

Formula (6.13) can be deduced from (6.14) in the same way as (1.3) was deduced from (5.2) in 5.1.

7. Markov systems on a random time interval.

7.1. We now extend the results of the preceding sections to systems on a random time interval introduced in 1.6.

First of all we establish some properties of such systems.

7.1.A. For each $t \in R$, $\{\alpha \geqq t\}$ is an atom of $\mathscr{F}_{<t}$, $\{\alpha > t\}$ is an atom of $\mathscr{F}_{<t+}$ and $\{t \bar{\in} [\alpha, \beta)\}$ is an atom of $\mathscr{F}(t, t+)$.

The first assertion follows from 1.6.B with $I = (-\infty, t)$. Now let $t_n \downarrow t$. Then $\{\alpha \geqq t_n\} \uparrow \{\alpha > t\}$. Hence $\{\alpha > t\} \in \mathscr{F}_{>t+}$. If $\xi \in \mathscr{F}_{<t+}$, then ξ is constant on each set $\{\alpha \geqq t_n\}$. Therefore ξ is constant on $\{\alpha > t\}$. This proves the second assertion. To prove the third, let B_n denote the intersection of $\{r \in \Delta(\omega)\}$ over all rational $r \in (t, t_n)$. Clearly $B_n \in \mathscr{F}(t, t_n)$ and $B_n = \{\alpha \leqq t < t_n \leqq \beta\}$. Hence $\{t \in [\alpha, \beta)\} \in \mathscr{F}(t, t+)$. If $\xi \in \mathscr{F}(t, t+)$, then, by 1.6.B, ξ is constant on each set $C_n = \{(\alpha, \beta) \cap (t, t_n) = \emptyset\}$. But $C_n \uparrow \{t \bar{\in} [\alpha, \beta)\}$.

7.1.B. $\{\omega: I \subseteq \Delta(\omega)\} \in \mathscr{F}(I)$ for every open interval I.

Indeed Δ contains I if and only if Δ contains all rational points of the interval I. Therefore 7.1.B follows from 1.6.A.

7.1.C. The function $1_{\alpha<t}$ is predictable. Every predictable function ξ_t is indistinguishable from the sum $\eta + \zeta$, where $\eta_t = 0$ for $t \leqq \alpha$ and ζ_t is measurable in t and does not depend on ω.

The first statement follows from the fact that $\{\alpha < t\} \in \mathscr{F}_{<t}$. The second statement will be proved if we prove it for $\xi \in \mathscr{S}_-$. Since $\{\alpha \geqq t\}$ is an atom of $\mathscr{F}_{<t}$, there exists a constant c_t such that $\xi_t 1_{\alpha \geqq t} = c_t 1_{\alpha \geqq t}$. Put $\zeta_t = \lim_{r \uparrow t} c_r$ (r irrational). The functions $\zeta_t 1_{\alpha \geqq t}$ and $\xi_t 1_{\alpha \geqq t}$ are indistinguishable. Hence ξ_t is indistinguishable from $\eta_t + \zeta_t$, where $\eta_t = (\xi_t - \zeta_t) 1_{\alpha<t}$.

7.1.D. The function $1_{\alpha \leqq t}$ is a right function. Each right function ξ_t is indistinguishable from $\eta_t + \zeta_t$, where $\eta_t = 0$ for $t < \alpha$ and ζ_t is measurable in t and does not depend on ω.

The proof is analogous to the proof of 7.1.C.

7.1.E. The function $1_\Delta(t)$ is a central function. Each central function is indistinguishable from $\eta_t + \zeta_t$, where $\eta_t = 0$ for $t \bar{\in} [\alpha, \beta]$ and ζ_t is measurable in t and does not depend on ω.

In fact 1_Δ is solid according to 7.1.C—C*. Denote by φ_t^n the indicator of the set $\{\alpha \leqq t - n^{-1} < t < \beta\}$. The function φ^n belongs to $\mathscr{S}_+$. By 7.1.A the sets $\{\alpha \leqq t - n^{-1} < \beta\}$ and $\{\alpha \leqq t < \beta\}$ belong to $\mathscr{F}_{<t+}$. Hence $\varphi^n \in \mathscr{F}_{<t+}$ and φ^n

is a right function. The function 1_Δ is also right since $\varphi^n \uparrow 1_\Delta$. By symmetry, 1_Δ is also left. The second part of 7.1.E can be easily deduced from 7.1.D—D*.

7.2. We say that the σ-algebras $\mathscr{A}$ and $\mathscr{B}$ are c.i. on Ω_0 given $\mathscr{C}$ if $\Omega_0 \in \mathscr{C}$ and, for every $A \in \mathscr{A}$ and $B \in \mathscr{B}$,

$$P(AB \mid \mathscr{C}) = P(A \mid \mathscr{C})P(B \mid \mathscr{C}) \quad \text{a.s.} \quad P, \Omega_0 .$$

(The abbreviation a.s. P, Ω_0 means for all $\omega \in \Omega_0$ except a set of P-measure 0.) In Sections 1 and 2 we considered c.i. relative to the σ-algebras $\mathscr{F}(I)$, $\mathscr{F}(t, t+)$, $\mathscr{F}(s, t+)$. For each of them we now fix an element Ω_0 according to the following list:

$$\begin{array}{llll} \mathscr{C}: & \mathscr{F}(I) & \mathscr{F}(t, t+) & \mathscr{F}(s, t+) \\ \Omega_0: & \{I \subseteq \Delta\} & \{t \in [\alpha, \beta)\} & \{(s, t] \subseteq [\alpha, \beta)\} . \end{array}$$

(In view of 1.6.A and 7.1.A, B, $\Omega_0 \in \mathscr{C}$ in all three cases.) We now modify the conditions 1.1.B, C, 2.2.A, C, D and 2.3.C by replacing c.i. with c.i. on Ω_0 and use the subscript r when referring to the modified formulations. For example, the modified version of 1.1.C is:

1.1.C_r. For any $s < t$, σ-algebras $\mathscr{F}_{<s}$ and $\mathscr{F}_{>t}$ are c.i. on $\{(s, t) \subseteq \Delta\}$ given $\mathscr{F}(s, t)$.

Conditions 2.3.A—B represent $P\{\eta \mid \mathscr{C}\}$ for the two σ-algebras $\mathscr{C}$ by iterated conditional expectations. We now suppose that this representation holds only on the Ω_0 specified in the list (7.1) and arrive in this way at the conditions 2.3.A_r—B_r.

A Markov representation of a system $(\mathscr{F}(I), P, \Delta)$ is a function $x_t(\omega)$ defined for $t \in \Delta(\omega)$ and satisfying 1.5.A and the following condition:

1.5.B_r. $\mathscr{F}_{<t}$ and $\mathscr{F}_{>t}$ are c.i. on $\{t \in \Delta\}$ given x_t.

Apart from this modification, Theorems 2.1 and 2.2 remain true for systems on random time intervals. The proofs rely on the criterion of c.i. on Ω_0 (equalities (2.1) and (2.2) in Lemma 2.1 have to hold only on Ω_0).

7.3. Section 3 needs the following comments.

In 3.1 we associated a class $\mathscr{A}$ with each of the families $\mathscr{L}_0$, $\mathscr{L}_-$, $\mathscr{L}_+$ and $\mathscr{L}$. For systems on random time intervals, $\mathscr{A}$ satisfies, just as before, the conditions 3.1.D, F but not necessarily 3.1.E. We say that $X \subseteq R \times \Omega$ belongs to the class $\hat{\mathscr{A}}$ if there exists $f_n \in \mathscr{A}$ such that $f_n \uparrow 1_X$. In the case $\mathscr{A}^0 = \mathscr{L}_0$, $R \times \{\omega: r \in \Delta(\omega)\} \in \hat{\mathscr{A}}$ for all r by 1.6.A and $R \times \{\Delta = \varnothing\} \in \hat{\mathscr{A}}$ by 1.6.C. Hence $R \times \Omega \in \hat{\mathscr{A}}$ and Lemma 3.2 remains valid. Analogously $\{\alpha < t\} \in \hat{\mathscr{A}}$ for $\mathscr{A}^0 = \mathscr{L}_-$, $\{\alpha > t\} \in \hat{\mathscr{A}}$ for $\mathscr{A}^0 = \mathscr{L}_+$, and $\{t \in \Delta\} \in \hat{\mathscr{A}}$ for $\mathscr{A}^0 = \mathscr{L}$. Condition 3.1.E is fulfilled if we set X equal to $\{\alpha < t\}$, $\{\alpha > t\}$, and $\{t \in \Delta\}$ respectively. Taking into account 7.1.C—C* we arrive at the following version of Lemma 3.2.

LEMMA 3.2_r. *Let $\mathscr{H}$ be a class of functions on $R \times \Omega$ which satisfies the conditions* 3.1.A—C *and which contains all the evanescent functions. If all the functions* *of t which are independent of ω belong to $\mathscr{H}$, and if $\mathscr{L}_- \subset \mathscr{H}$ ($\mathscr{L}_+ \subseteqq \mathscr{H}$), then $\mathscr{H}$ contains all the predictable (respectively, all the reconstructable) functions of $\mathscr{K}$. If all the functions which vanish on Δ belong to $\mathscr{H}$ and $\mathscr{L} \subseteqq \mathscr{H}$, then $\mathscr{H}$ contains all the solid functions of $\mathscr{K}$.*

All the results of 3.2 and 3.3 remain valid. The proofs are based on the general propositions established in [1] and [10] for a finite measure but these propositions hold also for σ-finite measures. Lemma 3.2 has to be replaced by Lemma 3.2_r.

The operators ϕ_I introduced in 3.6 have now to be defined by the formula

$$\phi_I \xi = P_{\mathscr{F}(I)}(\xi 1_I \Omega_I) ,$$

where $\Omega_I = \{\omega : \Delta(\omega) \cap I \neq \varnothing\}$. In particular,

$$\phi_{<u}\xi = P_{\mathscr{F}_{<u}}(\xi 1_{\alpha<u,t<u}) \qquad \text{and} \qquad \phi_{>s}\xi = P_{\mathscr{F}_{<s}}(\xi 1_{\beta>s,t>s}) .$$

Instead of (3.1) and (3.3) we have the following formulas:

$$(3.1_r) \qquad \Pi^+\eta = \Pi^+(\xi 1_{t<u,\alpha<u}) ,$$

$$(3.3_r) \qquad \Pi\eta = \Pi(\xi 1_I \Omega_I) .$$

The results of 3.7 do not change. The characterization of the central σ-algebra established in 3.8 has to be reformulated as follows:

The class of central functions vanishing outside Δ coincides with the class of functions which are indistinguishable from

$$(7.1) \qquad \begin{aligned} \xi_t &= f(t, x_t) && \text{for } t \in (-\infty, u) \cap \Delta , \\ &= 0 && \text{for } t \,\bar{\in}\, (-\infty, u) \cap \Delta , \end{aligned}$$

where f is a measurable function on $\mathscr{E}$.

7.4. In the case of a random time interval, the proof of Lemma 4.1 is more complicated. Associate with every finite set $\Lambda = \{t_1 < t_2 < \cdots < t_m\}$ two random variables

$$(7.2) \qquad \alpha_\Lambda = t_k \quad \text{for} \quad t_{k-1} \leqq \alpha < t_k ; \qquad \beta_\Lambda = t_k \quad \text{for} \quad t_k < \beta \leqq t_{k+1}$$

(here $t_0 = -\infty$, $t_{m+1} = +\infty$). The functions $\varphi_t = 1_{\alpha_\Lambda < t}$ and $\xi_t = \varphi_t A(\alpha_\Lambda, t)$ are predictable and the functions $\psi_t = 1_{\beta_\Lambda > t}$ and $\eta_t = \psi_t A(t, \beta_\Lambda)$ are reconstructable. We have

$$1_{\alpha_\Lambda < t < \beta_\Lambda} A(\alpha_\Lambda, \beta_\Lambda) = 1_{\alpha_\Lambda < t < \beta_\Lambda} A\{t\} + \varphi_t \eta_t + \psi_t \xi_t .$$

Therefore conditions $1.6.\alpha$ and 1.4.C imply that $1_{\alpha_\Lambda < t < \beta_\Lambda} A\{t\}$ is solid. Condition 1.4.C′ is fulfilled because $1_{\alpha_\Lambda < t < \beta_\Lambda} \uparrow 1$ as Λ runs over an expanding sequence of finite sets with everywhere dense union. On the other hand, 1.4.C′ implies the solidity of $1_{\alpha_\Lambda < t < \beta_\Lambda} A(\alpha_\Lambda, \beta_\Lambda)$. Let Λ run over an expanding sequence of subsets of (s, u) whose union is dense in (s, u). Then $1_{\alpha_\Lambda < t < \beta_\Lambda} A(\alpha_\Lambda, \beta_\Lambda) \uparrow 1_{s<t<u} A(s, u)$, and condition 1.4.C can be checked just as in n° 4.1.

Let us show that 1.4.C′ implies 1.4.C″. For every $\varepsilon > 0$ and every finite Λ

$$1_{\alpha_\Lambda < t-\varepsilon, t<\beta_\Lambda} A(t - \varepsilon, t] = 1_{\alpha_\Lambda < t-\varepsilon, t<\beta_\Lambda}\{A(\alpha_\Lambda, t] - A(\alpha_\Lambda, t - \varepsilon]\}$$

is a right function. Passing to the limit first over ε and then over Λ, we see that $1_{\alpha<t<\beta}A\{t\} = A\{t\}$ is a right function. By symmetry it is also a left function.

The conclusions of Lemma 4.2 and Theorem 4.1 remain true. The most essential change in the proofs is caused by the fact that $A(t, t']$ can be infinite if $t = \alpha$ and therefore it does not necessarily tend to 0 as $t' \downarrow t$. However, $1_\Delta \eta_t^m \downarrow 0$ for all t where η^m are the functions introduced at the end of 4.2. Therefore we can define φ by (4.3) for $t \in \Delta$ and put $\varphi = 1$ for $t \bar{\in} \Delta$.

7.5. Section 5 needs no comments. Section 6 has to be changed as follows. Put $C \in \hat{\mathscr{F}}(s, t)$ if $C \in \mathscr{F}(s, t)$ and $C \subseteq \{\alpha < s, t < \beta\}$. Denote by $\hat{\mathscr{G}}(I)$ the class of sets $C \in \mathscr{F}(I)$ for which $R \times \{C, \Delta \cap I \neq \varnothing\}$ is solid. We substitute:

(i) $\hat{\mathscr{F}}(I)$ for $\mathscr{F}(I)$ in the definition of subordination;
(ii) $1_I \Omega_I$ for 1_I in the statement of Lemma 6.1 and formulas (6.2) and (6.5);
(iii) $\hat{\mathscr{G}}(I)$ for $\mathscr{G}(I)$ in the proof of Lemma 6.1.

We include in the statement of Theorem 6.1 the condition that μ is concentrated on $\{(t, \omega) : t \in \Delta(\omega)\}$. (By virtue of 1.6.$\beta$ this condition is fulfilled for all spectral measures.)

To prove Theorem 6.1, we introduce a class $\mathscr{H}(I)$ of functions ξ_t for which $\xi_t 1_{\alpha<s<t<u<\beta}$ is solid for all $s < u \in I$. Consider α_Λ, β_Λ defined by (7.2) and put $a_\Lambda = a(\alpha_\Lambda, \beta_\Lambda)$ if $\alpha_\Lambda < \beta_\Lambda$ and $a_\Lambda = 0$ for the other Λ. It is easy to check that:

(a) $a_\Lambda \in \mathscr{H}(t_1, t_m)$;
(b) a_Λ is $\mathscr{F}(t_1, t_m)$-measurable;
(c) $Pa_\Lambda \xi = \int \Pi(\xi 1_{\alpha_\Lambda<t<\beta_\Lambda})\, d\mu$ for $\xi \in \mathscr{H}(t_1, t_m)$.

Consider an expanding sequence $\Lambda_n \subseteq I$ whose union is dense in I. It follows from (a) and (c) that a_{Λ_n} increases. Hence it tends to a limit $b(I)$. It follows from (a) and (b) that $b(I) \in \mathscr{H}(I)$ and we can assume that $b(I) \in \mathscr{F}(I)$. Passing to the limit in (c) and taking into account that $(\alpha_\Lambda, \beta_\Lambda) \uparrow I \cap \Delta$, 1_Δ is central and μ is concentrated on $\{t \in \Delta\}$, we get

$$Pb(I)\xi = \int \Pi(\xi 1_I)\, d\mu \tag{7.3}$$

for $\xi \in \mathscr{H}(I)$. Using $b(I)$ instead of $a(I)$ and relying on (7.3) instead of (6.1), we prove the theorem in the same way as in 6.4. (Ω_0 is defined as the union of sets $\{\alpha < q_1, C_{(q_1, q_2)}, q_2 < \beta\}$ over all rational $q_1 < q_2$.)

In Theorem 6.2 we have to replace $\mathscr{F}(s, t)$ by $\hat{\mathscr{F}}(s, t)$.

Some of the results of this paper were announced in [7].

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