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A COUNTEREXAMPLE IN THE APPROXIMATION THEORY OF RANDOM SUMMATION

BY DIETER LANDERS AND LOTHAR ROGGE

University of Cologne and University of Konstanz

Let X_n, $n \in \mathbb{N}$, be independent and identically distributed random variables and τ_n be random summation indices such that $\tau_n/n \to \tau > 0$ in probability. It is shown that even if τ_n/n converges to τ as quickly as possible (i.e., $\tau_n/n = \tau$) no general approximation orders for suitably normalized random sums $\sum_{\nu=1}^{\tau_n(\omega)} X_\nu(\omega)$ are available. If, however, the limit function τ is independent of X_n, $n \in \mathbb{N}$, we give a positive approximation result.

1. Introduction. Let X_n, $n \in \mathbb{N}$, be a sequence of independent and identically distributed (i.i.d.) random variables with mean 0 and variance 1. Let $\tau_n : \Omega \to \mathbb{N}$, $n \in \mathbb{N}$, and $\tau : \Omega \to (0, \infty)$ be random variables such that

$$(1) \qquad \lim_{n\to\infty} P\left\{\left|\frac{\tau_n(\omega)}{n\tau(\omega)} - 1\right| > \varepsilon\right\} = 0 \qquad \text{for all} \quad \varepsilon > 0\,,$$

$$\text{i.e.,} \quad \tau_n/n \to \tau \quad \text{in probability.}$$

It is well known that under condition (1):

$$(2) \qquad \sup_{t\in\mathbf{R}} \left|P\left\{\frac{\sum_{\nu=1}^{\tau_n(\omega)} X_\nu(\omega)}{(n\tau(\omega))^{\frac{1}{2}}} \leqq t\right\} - \Phi(t)\right| \to_{n\to\infty} 0$$

and

$$(3) \qquad \sup_{t\in\mathbf{R}} \left|P\left\{\frac{\sum_{\nu=1}^{\tau_n(\omega)} X_\nu(\omega)}{(\tau_n(\omega))^{\frac{1}{2}}} \leqq t\right\} - \Phi(t)\right| \to_{n\to\infty} 0\,,$$

where $\Phi(t)$ is the distribution function of a normally distributed random variable with mean 0 and variance 1.

In [2] it is shown that if τ is a constant limit function and (1) is strengthened to

$$(4) \qquad P\left\{\left|\frac{\tau_n(\omega)}{n\tau} - 1\right| > \varepsilon_n\right\} = O((\varepsilon_n)^{\frac{1}{2}})$$

(where $1/n \leqq \varepsilon_n \to 0$) then in (2) and (3) the order of convergence is $O((\varepsilon_n)^{\frac{1}{2}})$. The question arises whether, under assumptions of type (4), convergence orders are also available for nonconstant limit functions τ.

In this paper it is shown (see our example) that for each sequence of i.i.d. random variables each convergence order for (2) and (3) can be destroyed by a two valued limit function τ even with $\tau_n = n\tau$. Hence the maximal sharpening

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of (4) to $P\{|(\tau_n/n\tau) - 1| > 0\} = 0$ does not guarantee any convergence order for (2) or (3), even if τ is two-valued.

It turns out that the limit function τ in our example depends on the sequence X_n, $n \in \mathbb{N}$. For limit functions τ which are independent of X_n, $n \in \mathbb{N}$, we give a positive result for vector-valued random variables which contains the approximation result given in [2] for a constant limit function τ and real-valued X_n. For references concerning the random central limit theorem, see [2].

2. The results. Let $(\Omega, \mathscr{A}, P)$ be a probability space. Denote by $\mathscr{L}_r(\Omega, \mathscr{A}, P, \mathbb{R}^k)$ the space of all $\mathscr{A}$-measurable $\mathbb{R}^k$-valued functions with $P(|f|^r) < \infty$.

EXAMPLE. Let $X_n \in \mathscr{L}_3(\Omega, \mathscr{A}, P, \mathbb{R})$, $n \in \mathbb{N}$, be a sequence of i.i.d. random variables with mean μ and variance $\sigma^2 > 0$. We construct for each sequence $\delta_n \to_{n\to\infty} 0$ a two-valued measurable function $\tau: \Omega \to \{1, 2\}$ with

$$|P\{\sum_{\nu=1}^{n\tau} (X_\nu - \mu) \leqq 0\} - \Phi(0)| \geqq \delta_n$$

for infinitely many $n \in \mathbb{N}$. Put $\tau_n = n\tau$, then (4) is fulfilled for each sequence $\varepsilon_n \to 0$, but the convergence order in (2) and (3) cannot be better than $O(\delta_n)$.

Put $\delta_n' := \delta_n + \rho/\sigma^3(2n)^{\frac{1}{2}}$, where $\rho = P(|X_1 - \mu|^3)$. Define $S_n := \sum_{\nu=1}^{n} [X_\nu - \mu]$. From the central limit theorem we immediately obtain

(i) $\lim_{n\to\infty} P(S_n \leqq 0, S_{2n} > 0) =: c > 0$.

Now construct a subsequence $k(n) \in \mathbb{N}$ such that

(ii) $\sum_{j=n+1}^{\infty} \delta'_{k(j)} \leqq \delta'_{k(n)}/4$;
(iii) $\sum_{n=1}^{\infty} \delta'_{k(n)} < c/4$.

We shall show that we can define inductively an increasing sequence $j(n)$ with

(iv) $j(n) \geqq k(n)$, $\delta'_{j(n)} \leqq \delta'_{k(n)}$

and sets A_n with

(v) $A_n \subset \overline{A_1 + \cdots + A_{n-1}} \cap \{S_{j(n)} \leqq 0, S_{2j(n)} > 0\}$
(vi) $P(A_n) = 2\delta'_{k(n)}$
(vii) $P(S_{j(n)} \leqq 0, A_1 + \cdots + A_n) - P(S_{2j(n)} \leqq 0, A_1 + \cdots + A_n) \geqq \frac{3}{2}\delta'_{k(n)}$.

According to (i) there exists $j(1) \geqq k(1)$ with $\delta'_{j(1)} \leqq \delta'_{k(1)}$ and $P\{S_n \leqq 0, S_{2n} > 0\} \geqq c/2$ for all $n \geqq j(1)$. As $\sigma(X_n : n \in \mathbb{N})$ is countably generated and $P|\sigma(X_n : n \in \mathbb{N})$ has measure zero for each atom, $P|\sigma(X_n : n \in \mathbb{N})$ is a nonatomic measure.

According to a theorem of Ljapunoff (see [1], page 26) the range of a nonatomic measure is connected, hence there exists a set

$$A_1 \subset \{S_{j(1)} \leqq 0, S_{2j(1)} > 0\}$$

with $P(A_1) = 2\delta'_{k(1)}$, whence (iv)–(vii) are fulfilled for $n = 1$. Let (iv)–(vii) be fulfilled for $l \leqq n$. According to the conditional central limit theorem of Rényi (see [4]) there exists $j(n+1) > \max(k(n+1), j(n))$ such that

$$(^+) \qquad |P(S_{j(n+1)} \leqq 0, A_1 + \cdots + A_n) - P(S_{2j(n+1)} \leqq 0, A_1 + \cdots + A_n)| < \tfrac{1}{2}\delta'_{k(n+1)} .$$

W.l.o.g. we may assume that $\delta'_{j(n+1)} \leqq \delta'_{k(n+1)}$.

As by (vi) and (iii)

$$P(\overline{A_1 + \cdots + A_n} \cap \{S_{j(n+1)} \leqq 0, S_{2j(n+1)} > 0\}) \geqq \frac{c}{2} - 2 \textstyle\sum_{j=1}^{n} \delta'_{k(j)} > 2\delta'_{k(n+1)}$$

there exists according to the theorem of Ljapunoff a set

$$A_{n+1} \subset \overline{A_1 + \cdots + A_n} \cap \{S_{j(n+1)} \leqq 0, S_{2j(n+1)} > 0\}$$

with

$$P(A_{n+1}) = 2\delta'_{k(n+1)} .$$

Hence (iv)–(vi) are fulfilled for $n+1$. As furthermore, using $(^+)$

$$\begin{aligned} P(S_{j(n+1)} \leqq 0, A_1 + \cdots + A_{n+1}) &- P(S_{2j(n+1)} \leqq 0, A_1 + \cdots + A_{n+1}) \\ &= P(S_{j(n+1)} \leqq 0, A_1 + \cdots + A_n) \\ &\quad - P(S_{2j(n+1)} \leqq 0, A_1 + \cdots + A_n) + P(A_{n+1}) \\ &\geqq 2\delta'_{k(n+1)} - \tfrac{1}{2}\delta'_{k(n+1)} = \tfrac{3}{2}\delta'_{k(n+1)} , \end{aligned}$$

this concludes the induction.

Let $A := \sum_{n=1}^{\infty} A_n$ and define

$$\begin{aligned} \tau(\omega) &= 1 \qquad \text{for} \quad \omega \in A \\ &= 2 \qquad \text{for} \quad \omega \notin A . \end{aligned}$$

According to the theorem of Berry-Esseen

$$|P\{S_n \leqq 0\} - \tfrac{1}{2}| \leqq \frac{\rho}{\sigma^3} n^{-\frac{1}{2}} , \qquad n \in \mathbb{N} .$$

Hence it follows from (ii), (iv), and (vii) that

$$\begin{aligned} P\{\textstyle\sum_{\nu=1}^{j(n)\tau} (X_\nu - \mu) \leqq 0\} - \Phi(0) \\ &= P(S_{j(n)} \leqq 0, A) + P(S_{2j(n)} \leqq 0, \bar{A}) - \tfrac{1}{2} \\ &= P(S_{j(n)} \leqq 0, A) - P(S_{2j(n)} \leqq 0, A) + P(S_{2j(n)} \leqq 0) - \tfrac{1}{2} \\ &\geqq P(S_{j(n)} \leqq 0, A_1 + \cdots + A_n) - P(S_{2j(n)} \leqq 0, A_1 + \cdots + A_n) \\ &\qquad - P(\textstyle\sum_{j=n+1}^{\infty} A_j) + P(S_{2j(n)} \leqq 0) - \tfrac{1}{2} \\ &\geqq \tfrac{3}{2}\delta'_{k(n)} - 2 \textstyle\sum_{j=n+1}^{\infty} \delta'_{k(j)} + P(S_{2j(n)} \leqq 0) - \tfrac{1}{2} \\ &\geqq \delta'_{k(n)} + P(S_{2j(n)} \leqq 0) - \tfrac{1}{2} \\ &\geqq \delta'_{j(n)} - \frac{\rho}{\sigma^3} (2j(n))^{-\frac{1}{2}} = \delta_{j(n)} . \end{aligned}$$

This completes the proof.

Thus, for an arbitrary limit function τ no general approximation order in the random central limit theorem is available. We give now a positive result for a limit function τ, which is independent of X_n, $n \in \mathbb{N}$. If $x > 0$, let $[x] := \min\{l \in \mathbb{N} : x \leqq l\}$ · $(x^1, x^2, \cdots, x^k) \leqq (y^1, y^2, \cdots, y^k)$ means $x^i \leqq y^i$ for $i = 1, \cdots, k$.

THEOREM. *Let $X_n \in \mathscr{L}_3(\Omega, \mathscr{A}, P, \mathbb{R}^k)$, $n \in \mathbb{N}$, be a sequence of* i.i.d. *random variables with positive-definite covariance matrix C. Let ε_n be a sequence with $n^{-1} \leqq \varepsilon_n \to_{n \in \mathbb{N}} 0$. Let $\tau_n : \Omega \to \mathbb{N}$, $n \in \mathbb{N}$, and $\tau : \Omega \to (0, \infty)$ be $\mathscr{A}$-measurable functions. Assume that there exist constants $c_1, c_2 > 0$ such that*

$$(\alpha) \qquad P\left\{\omega : \left|\frac{\tau_n(\omega)}{[n\tau(\omega)]} - 1\right| > c_1\varepsilon_n\right\} = O((\varepsilon_n)^{\frac{1}{2}})$$

$$(\beta) \qquad P\left\{\omega : \tau(\omega) < \frac{c_2}{n\varepsilon_n}\right\} = O((\varepsilon_n)^{\frac{1}{2}})$$

$$(\gamma) \qquad \tau \text{ is independent from } X_n,\ n \in \mathbb{N}.$$

Then we obtain

$$\text{(i)} \qquad \sup_{t \in \mathbb{R}^k}\left|P\left\{\omega : C^{-\frac{1}{2}}\frac{\sum_{\nu=1}^{\tau_n(\omega)}(X_\nu(\omega) - P(X_\nu))}{(n\tau(\omega))^{\frac{1}{2}}} \leqq t\right\} - \Phi(t)\right| = O((\varepsilon_n)^{\frac{1}{2}})$$

and

$$\text{(ii)} \qquad \sup_{t \in \mathbb{R}^k}\left|P\left\{\omega : C^{-\frac{1}{2}}\frac{\sum_{\nu=1}^{\tau_n(\omega)}(X_\nu(\omega) - P(X_\nu))}{(\tau_n(\omega))^{\frac{1}{2}}} \leqq t\right\} - \Phi(t)\right| = O((\varepsilon_n)^{\frac{1}{2}})$$

where $\Phi(t)$ is the distribution function of a normally distributed random vector with mean 0 and covariance matrix I.

PROOF. W.l.o.g. we may assume that $P(X_\nu) = 0$ and $C = I$, the identity-matrix. First we prove that

$$(1) \qquad \sup_{t \in \mathbb{R}^k}|P\{\textstyle\sum_{\nu=1}^{[n\tau(\omega)]} X_\nu(\omega) \leqq ([n\tau(\omega)])^{\frac{1}{2}}t\} - \Phi(t)| = O((\varepsilon_n)^{\frac{1}{2}}).$$

Using the fact that τ is independent of X_n, $n \in \mathbb{N}$, we obtain from (β) and the theorem of Berry-Esseen that

$$\begin{aligned}
&\sup_{t \in \mathbb{R}^k}|P\{\textstyle\sum_{\nu=1}^{[n\tau(\omega)]} X_\nu(\omega) \leqq ([n\tau(\omega)])^{\frac{1}{2}}t\} - \Phi(t)| \\
&\quad = \sup_{t \in \mathbb{R}^k}|\textstyle\sum_{l=1}^{\infty} P\{\sum_{\nu=1}^{l} X_\nu(\omega) \leqq l^{\frac{1}{2}}t,\ [n\tau(\omega)] = l\} - \Phi(t)| \\
&\quad = \sup_{t \in \mathbb{R}^k}|\textstyle\sum_{l=1}^{\infty} P\{[n\tau(\omega)] = l\}(P\{\sum_{\nu=1}^{l} X_\nu(\omega) \leqq l^{\frac{1}{2}}t\} - \Phi(t))| \\
&\quad \leqq \textstyle\sum_{l=[c_2/\varepsilon_n]}^{\infty} P\{[n\tau(\omega)] = l\}\sup_{t \in \mathbb{R}^k}|P\{\sum_{\nu=1}^{l} X_\nu(\omega) \leqq l^{\frac{1}{2}}t - \Phi(t)|) \\
&\qquad + P\left\{n\tau(\omega) < \frac{c_2}{\varepsilon_n}\right\} \\
&\quad \leqq \textstyle\sum_{l=[c_2/\varepsilon_n]}^{\infty} P\{[n\tau(\omega)] = l\}\dfrac{c}{l^{\frac{1}{2}}} + O((\varepsilon_n)^{\frac{1}{2}}) = O((\varepsilon_n)^{\frac{1}{2}}).
\end{aligned}$$

Let

$$I_n(\omega) := \{j \in \mathbb{N} : [n\tau(\omega)](1 - c_1\varepsilon_n) \leqq j \leqq [n\tau(\omega)](1 + c_1\varepsilon_n)\}.$$

Let

$$t = (t^1, \cdots, t^k) \quad \text{and} \quad X_\nu(\omega) = (X_\nu^1(\omega), \cdots, X_\nu^k(\omega)),$$

$$A_n(t) := \{\omega : \max_{j \in I_n(\omega)} \textstyle\sum_{\nu=1}^{j} X_\nu^i(\omega) \leqq t^i([n\tau(\omega)])^{\frac{1}{2}} \text{ for } i = 1, \cdots, k\},$$

and

$$B_n(t) := \{\omega : \min_{j\in I_n(\omega)} \sum_{\nu=1}^{j} X_\nu^i(\omega) \leq t^i([n\tau(\omega)])^{\frac{1}{2}} \text{ for } i = 1, \cdots, k\}.$$

In the second step we prove

$$(2) \qquad \sup_{t\in\mathbf{R}^k} |P(B_n(t)) - P(A_n(t))| = O((\varepsilon_n)^{\frac{1}{2}}).$$

Let $A_n(t^i) := \{\omega : \max_{j\in I_n(\omega)} \sum_{\nu=1}^{j} X_\nu^i(\omega) \leq t^i([n\tau(\omega)])^{\frac{1}{2}}\}$ and $B_n(t^i) := \{\omega : \min_{j\in I_n(\omega)} \sum_{\nu=1}^{j} X_\nu^i(\omega) \leq t^i([n\tau(\omega)])^{\frac{1}{2}}\}$. Then $B_n(t) - A_n(t) \subset \bigcup_{i=1}^{k} \{B_n(t^i) - A_n(t^i)\}$. Hence it suffices to prove that

$$(*) \qquad \sup_{t^i} |P(B_n(t^i) - A_n(t^i))| = O((\varepsilon_n)^{\frac{1}{2}}).$$

Using the fact that τ is independent of X_n, $n \in \mathbb{N}$, as in (1) and arguing similarly as is in the proof of Theorem 1 of [2], we obtain (*) and hence (2).

According to (α)

$$P\{\omega : \tau_n(\omega) \notin I_n(\omega)\} = O((\varepsilon_n)^{\frac{1}{2}})$$

and therefore

$$(3) \quad P(A_n(t)) - O((\varepsilon_n)^{\frac{1}{2}}) \leq P\{\textstyle\sum_{\nu=1}^{\tau_n(\omega)} X_\nu(\omega) \leq ([n\tau(\omega)])^{\frac{1}{2}} t\} \leq P(B_n(t)) + O((\varepsilon_n)^{\frac{1}{2}})$$

and

$$(4) \qquad P(A_n(t)) \leq P\{\textstyle\sum_{\nu=1}^{[n\tau(\omega)]} X_\nu(\omega) \leq ([n\tau(\omega)])^{\frac{1}{2}} t\} \leq P(B_n(t)).$$

Now (1), (2), (3), and (4) imply

$$(5) \qquad \sup_{t\in\mathbf{R}^k} |P\{\textstyle\sum_{\nu=1}^{\tau_n(\omega)} X_\nu(\omega) \leq ([n\tau(\omega)])^{\frac{1}{2}} t\} - \Phi(t)| = O((\varepsilon_n)^{\frac{1}{2}}).$$

For (i) it therefore suffices to show according to a lemma of [3]—which also holds true for vector-valued random variables—that

$$P\left\{\left|\left(\frac{n\tau(\omega)}{[n\tau(\omega)]}\right)^{\frac{1}{2}} - 1\right| > (c_2^{-1}\varepsilon_n)^{\frac{1}{2}}\right\} = O((\varepsilon_n)^{\frac{1}{2}}).$$

This follows from (β) as

$$\begin{aligned}\left\{\left|\left(\frac{n\tau(\omega)}{[n\tau(\omega)]}\right)^{\frac{1}{2}} - 1\right| > (c_2^{-1}\varepsilon_n)^{\frac{1}{2}}\right\} &\subset \left\{\left|\frac{n\tau(\omega)}{[n\tau(\omega)]} - 1\right| > c_2^{-1}\varepsilon_n\right\} \\ &\subset \{[n\tau(\omega)] - n\tau(\omega) > c_2^{-1}\varepsilon_n[n\tau(\omega)]\} \subset \{1 > c_2^{-1}\varepsilon_n[n\tau(\omega)]\} \\ &\subset \{1 > c_2^{-1}\varepsilon_n n\tau(\omega)\}.\end{aligned}$$

Thus we have proved (i).

Assertion (ii) follows from (5) and a lemma of [3], since according to (α)

$$P\left\{\left|\left(\frac{\tau_n(\omega)}{[n\tau(\omega)]}\right)^{\frac{1}{2}} - 1\right| > (c_1\varepsilon_n)^{\frac{1}{2}}\right\} \leq P\left\{\left|\frac{\tau_n(\omega)}{[n\tau(\omega)]} - 1\right| > c_1\varepsilon_n\right\} = O((\varepsilon_n)^{\frac{1}{2}}).$$

We mention that condition (β) is always fulfilled if τ is bounded away from zero.

REMARK. (i) Our theorem contains Theorem 1 of [2] as a special case, since for a constant τ conditions (β) and (γ) are always fulfilled and condition (*) in Theorem 1 of [2] implies condition (α).

(ii) Example 3 of [2] shows that a "maximal" sharpening of (α), (β) and (γ), namely $\tau \equiv 1$

$$P\left\{\left|\frac{\tau_n(\omega)}{n\tau} - 1\right| > \varepsilon_n\right\} = 0\,, \qquad n \in \mathbb{N}$$

and

$$P\left\{\omega : \tau(\omega) < \frac{2}{n\varepsilon_n}\right\} = 0\,, \qquad n \in \mathbb{N}$$

does not lead beyond the approximation order $O((\varepsilon_n)^{\frac{1}{2}})$ in (i) and (ii) of our theorem.

(iii) None of the three conditions (α), (β), (γ) can be dispensed with, either for assertion (i) or for assertion (ii) of our theorem. Assumption (α) cannot be omitted according to Example 4 of [2]. Assumption (γ) cannot be omitted according to our preceding example [even for $\tau_n = [n\tau]$ and a two-valued function τ]. Assumption (β) cannot be omitted according to the following consideration:

We shall show that under condition (γ) and with $\tau_n = [n\tau]$ (whence condition (α) is always fulfilled) each approximation order can be destroyed in (i) and (ii) of our theorem. Let $\delta_n \to 0$ and choose a subsequence δ_n, $n \in \mathbb{N}_0$, with $\sum_{n \in \mathbb{N}_0} \delta_n \leqq 1$. Let X_n, $n \in \mathbb{N}$, be a sequence of i.i.d. random variables with mean zero and $P\{X_1 = 0\} > 0$. Furthermore, let τ be a random variable independent of X_n, $n \in \mathbb{N}$, with

$$P\left\{\tau = \frac{1}{n}\right\} = \delta_n \qquad \text{for} \quad n \in \mathbb{N}_0\,.$$

Then

$$P\{\textstyle\sum_{\nu=1}^{[n\tau]} X_\nu = 0\} \geqq P\{[n\tau] = 1, X_1 = 0\} = \delta_n P\{X_1 = 0\}\,, \qquad n \in \mathbb{N}_0$$

and therefore the approximation order in (i) and (ii) (with $\tau_n = [n\tau]$) cannot be better than $O(\delta_n)$.

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FACHBEREICH MATHEMATIK
UNIVERSITÄT KÖLN
WEYERTAL 86-90
D-5000 KÖLN
GERMANY

FACHBEREICH STATISTIK
UNIVERSITÄT KONSTANZ
POSTFACH 7733
D-7750 KONSTANZ
GERMANY