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SUPPORTS OF INFINITELY DIVISIBLE MEASURES ON HILBERT SPACE[1]

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The supports of infinitely divisible measures on separable Hilbert spaces are characterized in terms of angular semigroups. Restricted to $\mathbb{R}^n$ this result extends results of Hudson and Mason. Restricted to $\mathbb{R}^1$ our result improves Tucker's result and Hudson and Tucker's results on such supports. Also investigated are the supports of stable measures on Hilbert space.

1. Introduction and summary. The purpose of this paper is to investigate the supports of infinitely divisible probability measures on separable Hilbert spaces. Since supports of Gaussian measures have been adequately discussed, we shall restrict our attention to measures without Gaussian component. Such measures have a characteristic function of the form

$$\hat{\mu}(y) = \exp\left\{i(x_0, y) + \int_H \left\{e^{i(x,y)} - 1 - \frac{i(x, y)}{1 + ||x||^2}\right\} dM(x)\right\}$$

or simply $\mu \sim [x_0, M]$. The measure M is called the Lévy measure of μ and satisfies (1) $M(\{x: ||x|| > \varepsilon\}) < \infty$ for every $\varepsilon > 0$; (2) $M(\{0\}) = 0$; and (3) $\int_{||x||<1} ||x||^2 \, dM(x) < \infty$. Any measure M satisfying (1), (2), and (3) will be referred to as a Lévy measure. If M is any Lévy measure and $x_0 \in H$, then there is an infinitely divisible measure μ with $\mu \sim [x_0, M]$. Background information on infinitely divisible measures on Hilbert spaces can be found in Parthasarathy (1967).

In (1975a), Hudson and Tucker investigated infinitely divisible probability measures on $\mathbb{R}^1$ whose Lévy measure M was absolutely continuous with respect to Lebesgue measure and satisfied $M(\mathbb{R}) = \infty$. Such measures were found to have supports which were necessarily of the form $(-\infty, a]$, $[a, \infty)$ or $\mathbb{R}$. In a subsequent paper (1975b), Hudson and Tucker proved that if μ is absolutely continuous (so $M(\mathbb{R}) = \infty$) then the support of μ is of the above form. This result was extended to $\mathbb{R}^n$ by Hudson and Mason (1975) where it was proven that if μ is infinitely divisible on $\mathbb{R}^n$ and if $M(\mathbb{R}^n) = \infty$ and M is absolutely continuous, then the support of μ is of the form $(A + G)^-$ where A is a closed set and G is a semigroup with 0 as a limit point. In this paper they posed the problem of determining the support of general infinitely divisible measure on $\mathbb{R}^n$.

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In a different direction, Tucker (1975) extended his and Hudson's results by proving that if μ is an infinitely divisible measure on $\mathbb{R}^1$ and if $M(-\varepsilon, \varepsilon) > 0$ for every $\varepsilon > 0$, then the support of μ is one of $(-\infty, a]$, $[a, \infty)$ or $\mathbb{R}$. In yet a different direction Yuan and Liang (1976) discussed supports of infinitely divisible measures on locally compact Abelian groups.

In this paper we prove a theorem which extends all of the above results on supports of infinitely divisible measures. We characterize such supports in terms of angular semigroups. Such sets have been extensively studied (see Hille and Phillips (1957)).

Considering stable measures on Hilbert space, we use our main theorem to extend a result of de Acosta (1975) on supports of stable measures. For stable measures with characteristic exponent $\alpha < 1$ we show the support is a cone, and answer a question which is implicitly raised by Kuelbs and Mandrekar (1974).

Let us now establish some notation to be used in the sequel. By a semigroup, we mean a set S satisfying $S + S \subseteq S$. If additionally $0 \in S$, we call S a monoid. An angular semigroup is an open semigroup which has 0 as a limit point. A point x will be called a point of increase of a measure μ if $\mu(U) > 0$ for every open set U containing x. All measures under consideration will be σ-finite. The support of a measure μ will be denoted $S(\mu)$. It is the collection of all points of increase of μ. The convolution of two measures α and β will be denoted $\alpha * \beta$ and α^{*k} denotes the kth convolution power of α. The monoid generated by $S(\alpha)$ is denoted $G(\alpha)$ and is known to be $G(\alpha) = (\bigcup_{k\geq 0} S(\alpha^{*k}))^-$. The interior of a set A is denoted by A^0, and the closure of A by A^-. Finally, I_A denotes the indicator function of the set A, i.e., $I_A(x) = 0$ if $x \notin A$, and $I_A(x) = 1$ if $x \in A$. When the measures μ_n converge weakly to the measure μ, we write $\mu_n \Rightarrow \mu$.

2. Infinitely divisible measures on Hilbert space. In this section we characterize the support of infinitely divisible measures without Gaussian component on a Hilbert space.

THEOREM 2.1. *Suppose μ is an infinitely divisible measure on H with Lévy representation $\mu \sim [x_0, M]$.*

(1) *If $\int_{\|x\|<1} \|x\| \, dM(x) < \infty$, then*

$$S(\mu) = a + G(M) \qquad \text{where} \quad a = x_0 - \int \frac{x}{1 + \|x\|^2} \, dM(x) \, .$$

(2) *If $\int_{\|x\|<1} \|x\| \, dM(x) = \infty$, then $S(\mu) = (A + G(M))^-$ where A is a closed set.*

(3) *The monoid $G(M)$ has the following properties:*

(a) *If $M(B) > 0$ for every open set B containing 0, then $G(M)$ has 0 as a limit point;*

(b) *if $S(M)^0$ has 0 as a limit point, then $G(M)^0$ is an angular semigroup.*

To prove this theorem we utilize the following lemma due to Yuan and Liang (1976). They prove it for locally compact Abelian groups; however, it is not difficult to emulate their proof for the Hilbert space case.

LEMMA 2.2. *Suppose* $\mu = \alpha_n * \mu_n$ *for all* $n \geqq 1$ *and* $\mu_n \Rightarrow \mu$. *Then* $S(\mu) = \bigcap_{j\geqq 1} (\bigcup_{n\geqq j} S(\mu_n))^-$.

PROOF OF THEOREM 2.1. Let us suppose first that $\int_{\|x\|<1} \|x\| \, dM(x) < \infty$ so that with a as given in the theorem we have $\hat{\mu}(y) = \hat{\delta}_a(y)\hat{\mu}_0(y)$ where $\hat{\mu}_0(y) = \exp\{\int (e^{i(y,x)} - 1) \, dM(x)\}$. Clearly $S(\mu) = a + S(\mu_0)$ so it remains to show $S(\mu_0) = G(M)$ to complete the proof of (1).

If $M(H) = k < \infty$, then we have $\mu_0 = \sum_{n=0}^{\infty} (e^{-k} M^{*n}/n!)$ and $S(\mu_0) = (\bigcup_{n\geqq 0} S(M^{*n}))^- = G(M)$.

If $M(H) = \infty$, then let M_n be the Lévy measure (equivalent to M) defined via the Radon–Nikodym derivative

$$
\frac{dM_n}{dM}(x) = \|x\| \quad \text{if} \quad \|x\| < 1/n \qquad (2.1)
$$
$$
= 1 \quad \text{if} \quad \|x\| \geqq 1/n
$$

and let $\mu_n = e^{-k} \sum_{r=0}^{\infty} (M_n^{*r}/r!)$ where $k = M_n(H)$. Clearly $M_n(H) < \infty$ and $S(M_n) = S(M)$ since M_n and M have the same points of increase. We also may easily verify that $(M - M_n)$ is a Lévy measure so that $\mu = \alpha_n * \mu_n \; n \geqq 1$ for some measure α_n defined via the characteristic function $\hat{\alpha}_n(y) = \exp\{\int (e^{i(y,x)} - 1)(M - M_n)(dx)\}$. Since $\hat{\alpha}_n(y) \to 1$ and $\hat{\mu}_n(y) \to \hat{\mu}_0(y)$, it follows that $\mu_n \Rightarrow \mu_0$. Thus by Lemma 2.2, $S(\mu_0) = \bigcap_{j\geqq 1} (\bigcup_{n\geqq j} S(\mu_n))^- = \bigcap_{j\geqq 1} (\bigcup_{n\geqq j} G(M_n))^- = \bigcap_{j\geqq 1} (\bigcup_{n\geqq j} G(M)) = G(M)$. This proves (1).

Suppose now that $\int_{\|x\|<1} \|x\| \, dM(x) = \infty$. Define M_n by (2.1) and $a_n = x_0 - \int x/(1 + \|x\|^2) \, dM_n(x) = x_0 - \int_{\|x\|<1/n} x\|x\|/(1 + \|x\|^2) \, dM(x) - \int_{\|x\|>1/n} x/(1 + \|x\|^2) \, dM(x)$. Let $\mu_n \sim [x_0, M_n]$. Then $\int_{\|x\|<1} \|x\| \, dM_n(x) < \infty$ so $S(\mu_n) = a_n + G(M_n) = a_n + G(M)$ by (1). Since $\mu = \alpha_n * \mu_n$ for each n, we have, upon taking $n = 1$, $S(\mu) = (S(\alpha_1) + a_1 + G(M))^- = (A + G(M))^-$ which proves (2). Note that in fact by Lemma 2.2 we have $S(\mu) = \bigcap_{j\geqq 1}(\bigcup_{n\geqq j} \{a_n + G(M)\})^- = \bigcap_{j\geqq 1}(A_n + G(M))^-$ where A_n is the (countable) set $A_n = \bigcup_{n\geqq j} \{a_j\}$.

To prove (3a) we note that $M(B) > 0$ for every neighborhood B of 0 together with $M(\{0\}) = 0$ implies that 0 is a limit point of $S(M)$. To prove (3b), we merely note that $G(M)^0$ is an open semigroup containing $S(M)^0$.

REMARK. Hudson and Mason (1975) showed that in $\mathbb{R}^n$, by assuming M is absolutely continuous with respect to Lebesgue measure, and also assuming $M(\mathbb{R}^n) = \infty$, then $S(\mu) = (A + G(M))^-$. This result is extended by Theorem 2.1 since M absolutely continuous and $M(\mathbb{R}^n) = \infty$ implies 0 is a limit point of $S(M)^0$ and $M(B) > 0$ for every neighborhood B containing 0.

(It should be mentioned that Hudson and Mason actually prove much more than just the support statement above. In fact they show in addition that under the above assumptions on the Lévy measure M, the corresponding infinitely divisible measure μ is actually equivalent to Lebesgue measure on $S(\mu)$. Also they prove that the Lebesgue measure of the boundary of $(A + G)$ is zero if G is an angular semigroup and A is a closed set, a characterization which is interesting in its own right.)

The following corollary proves the result of Tucker (1975) on support of infinitely divisible measures on $\mathbb{R}^1$. Tucker's proof was difficult and his method could not be extended to $\mathbb{R}^n$. The argument used here is vastly simpler and shows how the results of topological algebra often simplify calculations.

COROLLARY 2.3. *Suppose* $\mu \sim [x_0, M]$ *is an infinitely divisible measure on* $\mathbb{R}$.

(1) *If M has both positive and negative points of increase, then $S(\mu)$ is either* $a + \alpha\mathbb{Z}$ *or* $\mathbb{R}$ (here $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$).

(2) (Tucker) *If* $M(-\varepsilon, \varepsilon) > 0$ *for every* $\varepsilon > 0$, *then $S(\mu)$ is one of* $(-\infty, a]$, $[a, \infty)$, *or* $\mathbb{R}$.

PROOF. The proof follows from the characterization of topological semigroups on $\mathbb{R}^1$. According to Hille and Phillips (1957), the only closed additive semigroup in $\mathbb{R}$ with both positive and negative members having 0 as a limit point is $\mathbb{R}$, and the only closed semigroup in $\mathbb{R}$ with both positive and negative members which does not have 0 as a limit point is $\alpha\mathbb{Z}$, the group of all integral multiples of some fixed number α (cf. Theorems 8.6.1 and 8.6.2, page 264 of [2]). Thus to prove (1), if $G(M)$ contains 0 as a limit point, then $G(M) = \mathbb{R}$ and $(A + G(M))^- = S(\mu) = \mathbb{R}$ by whichever is applicable of (1) or (2) of Theorem 2.1. On the other hand, if $G(M)$ does not have 0 as a limit point, then in particular $S(M)$ cannot have 0 as a limit point and consequently $M(\mathbb{R}) < \infty$. Applying (1) of Theorem 2.1 to this case yields $S(\mu) = a + G(M) = a + \alpha\mathbb{Z}$.

To see that (2) holds, we use the fact that the only closed semigroups in $\mathbb{R}$ with 0 as a limit point are $(-\infty, 0]$, $[0, \infty)$, or $\mathbb{R}$ (cf. [2]). Clearly $M((-\varepsilon, \varepsilon)) > 0$ for all $\varepsilon > 0$ implies 0 is a limit point of $G(M)$. Since $(A + (-\infty, 0])^- = (-\infty, a]$ and $(A + [0, \infty))^- = [a, \infty)$ and $(A + \mathbb{R})^- = \mathbb{R}$, the applicable part of Theorem 2.1 implies the result.

Let us now turn to a characterization of supports of stable distributions on Hilbert spaces. A measure μ on H is called stable if for every integer k there exists a positive constant a_k and a vector b_k such that $\mu^{*k} = T_{a_k}\mu * \delta_{b_k}$. Here $T_{a_k}\mu(A) = \mu(A/a_k)$. Such measures are infinitely divisible and are either purely Gaussian, or have characteristic function $\mu \sim [\beta, M]$ where the measure M satisfies

$$M(A) = \int_H I_A(r, s)\frac{dr}{r^{1+\alpha}}\,\sigma(ds) \tag{2.2}$$

where $x = rs$, $r = ||x||$, $s = x/||x||$ and σ is a measure on the unit ball B of H, $\sigma(W) = \alpha M(\{x \in H : ||x|| \geqq 1, x/||x|| \in W\})$, and $0 < \alpha < 2$. The index α is called the characteristic exponent of μ since it can be shown that the characteristic function of μ is given by

$$\hat{\mu}(y) = \exp\{i(y, \gamma) - \int_B |(x, s)|^\alpha \sigma(ds) + iC(\alpha, y)\} \tag{2.3}$$

where

$$C(\alpha, y) = \tan \pi\alpha/2 \int_B (y, s)|(y, s)|^{\alpha-1}\sigma(ds) \quad (\alpha \neq 1)$$
$$= 2/\pi \int_B (y, s) \ln |(y, s)|\sigma(ds) \quad \alpha = 1.$$

See Kuelbs (1973) for the details of these formulas.

The problem to which the next corollary addresses itself concerns the structure of $S(\mu)$ when μ is stable with characteristic exponent $\alpha < 1$. The case $\alpha > 1$ was considered by de Acosta (1975) in a more general setting; however the case $\alpha \leqq 1$ was left unanswered. For symmetric stable measures the corollary answers a question raised implicitly by the work of Kuelbs and Mandrekar (1974) as to whether the support of μ with $\alpha < 1$ is the linear subspace generated by $S(\sigma)$ when μ is symmetric.

COROLLARY 2.4. *Let μ be a stable measure on H with characteristic exponent $\alpha < 1$.*

(a) *$S(\mu) = a + C$ where $C = G(M)$ is a closed cone in H* (i.e., $C + C \subseteqq C$ *and* $tC \subseteqq C$ *for all* $t > 0$).

(b) *If μ is symmetric, then $S(\mu)$ is a closed linear subspace of H. In fact $S(\mu)$ is the linear subspace generated by $S(\sigma)$ in the representation* (2.2) *and* (2.3).

PROOF. Using $dM = r^{-\alpha-1}\, dr\sigma(ds)$ where σ is a measure on B, one easily sees that if $\alpha < 1$, then

$$\int_{||x||<1} ||x||\, dM(x) = \sigma(B) \int_0^1 r^\alpha\, dr < \infty$$

so that applying (1) of Theorem 2.1 it follows that $S(\mu) = a + G(M)$. It remains to show $tG(M) \subseteqq G(M)$ for all $t > 0$. This, however, follows from the radial nature of M since if x is a point of increase of M and $t > 0$, and $B_{\varepsilon,x} = \{y: ||y - x|| < \varepsilon\}$, then

$$M(B_{\varepsilon,tx}) = \int_B \int_0^\infty I_{B_{\varepsilon,tx}}(r, s) \frac{dr}{r^{\alpha+1}} \sigma(ds) = \int_B \int_0^\infty I_{B_{\varepsilon/t,x}}(r/t, s) \frac{dr}{r^{\alpha+1}} \sigma(ds)$$
$$= \frac{1}{t^\alpha} \int_B \int_0^\infty I_{B_{\varepsilon/t,x}}(y, s) \frac{dy}{y^{\alpha+1}} \sigma(ds) = \frac{1}{t^\alpha} M(B_{\varepsilon/t,x}) > 0\,.$$

Thus tx is also a point of increase of M. Now, if x is a point of increase of M^{*2}, then

$$M^{*2}(B_{\varepsilon,tx}) = \int M(B_{\varepsilon,tx} - y)\, dM(y) = \int M(B_{\varepsilon,tx-y})\, dM(y)$$
$$= \frac{1}{t^\alpha} \int M(B_{\varepsilon/t,x-y/t})\, dM(y)$$

where the last equality follows from the equality established for $M(B_{\varepsilon,tx})$. One easily establishes via (2.2) that $M(tA) = (1/t^\alpha)M(A)$. Using this in the above we have $M^{*2}(B_{\varepsilon,tx}) = (1/t^\alpha) \int M(B_{\varepsilon/t,x-y/t})\, dM(y) = (1/t^\alpha) \int M(B_{\varepsilon/t,x-u})\, dM(tu) = (1/t^\alpha)^2 \int M(B_{\varepsilon/t,x-u})\, dM(u) = (1/t^\alpha)^2 M^{*2}(B_{\varepsilon/t,x}) > 0$ for all $t > 0$. Thus tx is also a point of increase of M^{*2}. By induction, we may utilize a similar argument to show that if $x \in G(M) = (\bigcup_{k\geqq 0} S(M^{*k}))^-$, then $tx \in G(M)$ and $G(M)$ is a cone.

To prove (b), we recall that if μ is symmetric, then $S(\mu) = G(M) = -G(M)$ is a semigroup. By (a), if $t > 0$ then $tG(M) \subseteqq G(M)$ and hence $G(M)$ is a linear subspace (note that this holds even if $\alpha \geqq 1$). Using the radial property of M again it is not difficult to see that $G(M)$ is the linear subspace generated by $S(\sigma)$ where σ is the measure in (2.2).

REMARK. The above analysis carries over to studies of supports of infinitely divisible measures on locally compact groups which are either symmetric or satisfy $\int |\chi(x) - 1| \, dM(x) < \infty$ for all characters χ. The proof of this would follow Theorem 2.1 exactly. The resulting theorem would extend the results of Yuan and Liang (1976) on supports of infinitely divisible measures on locally compact Abelian groups.

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