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# ITERATED LOGARITHM LAWS FOR ASYMMETRIC RANDOM VARIABLES BARELY WITH OR WITHOUT FINITE MEAN

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One-sided iterated logarithm laws of the form $\limsup (1/b_n) \sum_1^n X_i = 1$, a.s. and $\limsup (1/b_n) \sum_1^n X_i = -1$, a.s. are obtained for asymmetric independent and identically distributed random variables, the first when these have a vanishing but barely finite mean, the second when $E|X|$ is barely infinite. In both cases, $\liminf (1/b_n) \sum_1^n X_i = -\infty$, a.s. The constants $b_n/n$ are slowly varying, decreasing to zero in the first case and increasing to infinity in the second. Although defined via the distribution of $|X|$, $b_n$ represents the order of magnitude of $E|\sum_1^n X_i|$ when this is finite. Corresponding weak laws of large numbers are established and related to Feller's notion of "unfavorable fair games" and in the process a theorem playing the same role for the weak law as Feller's generalization of the strong law is proved.

**1. Introduction.** Following along the lines of Feller, independent, identically distributed (i.i.d.) random variables $\{X_n, n \geqq 1\}$ will be said to obey a generalized law of the iterated logarithm (LIL) if there exist constants $\{b_n, n \geqq 1\}$, $0 < b_n \uparrow \infty$ for which

$$(*) \qquad \text{(i)} \quad \limsup \frac{1}{b_n} \sum_{i=1}^n X_i = 1\,, \quad \text{a.s.} \qquad \text{or}$$

$$\text{(ii)} \quad \limsup \frac{1}{b_n} \sum_{i=1}^n X_i = -1\,, \quad \text{a.s.}$$

According to Theorem 4, the second alternative is nonvacuous.

When $EX = 0$, $EX^2 = \sigma^2 < \infty$, the renowned theorem of Hartman–Wintner [8] asserts that (i) obtains with $b_n = \sigma(2n \log_2 n)^{\frac{1}{2}}$. For *symmetric* i.i.d. random variables, Feller [6] has shown that (i) holds with[3] $b_n = 2^{\frac{1}{2}}(n^2/H(n) \log_2 n)^{-1}$ provided $EX^2 = \infty$, $E(X^+)^2/H(X^+) \log_2 X^+ < \infty$ where $H(x) = EX^2 I_{[|X| \leqq x]}$ and so $b_n/(n \log_2 n)^{\frac{1}{2}} \to \infty$. According to a result of Heyde [9] and Rogosin [13] for *symmetric* i.i.d. random variables to obey a generalized LIL (i) it is necessary that the distribution of $X$ belong to the domain of partial attraction of a normal distribution and according to Kesten [10] this is sufficient.

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[3] For the sake of comparisons, $n^2/H(n) \log \log n$ is taken here as monotone.

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The situation for *asymmetric* i.i.d. random variables with barely finite (and vanishing) mean or barely infinite mean turns out to be completely different. Here it will be proved (Theorem 3) that for suitable distributions with $EX = 0$, a one-sided generalized LIL (i) obtains, but not the corresponding two-sided LIL [15, 16]. Moreover, $b_n/n$ is slowly varying and decreases to zero. Likewise, for suitable distributions with $E|X| = \infty$, a one-sided generalized LIL (ii) but not a two-sided LIL obtains (Theorem 4). In this case, $b_n/n$ increases to infinity and is again slowly varying. In both situations the distribution may lie outside the domain of partial attraction of a normal distribution.

Furthermore, in contradistinction to the *symmetric* case where (i) entails [10, Lemma 4]

$$\frac{1}{b_n} \sum_1^n X_i \to_P 0 ,$$

in the asymmetric case (*) is compatible with

$$(**) \qquad \text{(i)} \quad \frac{1}{b_n} \sum_1^n X_i \to_P 1 \qquad \text{or} \qquad \text{(ii)} \quad \frac{1}{b_n} \sum_1^n X_i \to_P -1 ,$$

according to Theorem 2 and Corollary 3.

The normalizing constants $b_n$ which ensure (*) and (**) are simply defined via the distribution of $|X|$ (see (6)). These are also connected with the corresponding random walk $S_n = \sum_1^n X_i$, $n \geqq 1$ and Theorem 5 reveals in case (i) that the same conditions which guarantee $(1/b_n) \sum_1^n X_i \to_P (c-1)/(c+1)$, likewise ensure $(1/b_n)ES_n^+ \to \max[1/(1+c), c/(1+c)]$. The two limits agree only in the extreme case $c = \infty$ (see Theorem 2 for the meaning of $c$).

**2. Weak laws of large numbers.** The proofs of Theorems 3 and 4 are facilitated by first establishing corresponding weak laws of large numbers as in Theorem 2. In fulfillment of the latter, it is convenient to prove a result (Theorem 1) which bears the same relation to the classical weak law of large numbers as Feller's generalization [3] does to the strong law of large numbers.

THEOREM 1. *Let* $\{X, X_n, n \geqq 1\}$ *be* i.i.d. *random variables and let* $\{b_n, n \geqq 1\}$ *be constants such that* $0 < b_n \uparrow \infty$ *and either*

$$\text{(i)} \quad b_n/n \downarrow 0 , \qquad b_n/n^{\frac{1}{2}} \to \infty \qquad \textit{and} \qquad \sum_{j=1}^n \left(\frac{b_j}{j}\right)^2 = O\left(\frac{b_n^2}{n}\right) \qquad \textit{or}$$

$$\text{(ii)} \quad b_n/n \uparrow .$$

*Then*

$$(1) \qquad \frac{1}{b_n}\left(\sum_{j=1}^n X_j - nEXI_{[|X| \leqq b_n]}\right) \to_P 0$$

iff

$$(2) \qquad nP\{|X| > b_n\} = o(1) .$$

PROOF. The weak law of large numbers for independent random variables reduces in the i.i.d. case to the statement that (1) holds iff (2) and

$$\frac{n}{b_n^2}\sigma^2(XI_{[|X|\leqq b_n]}) = o(1) \tag{3}$$

obtain. Thus, it suffices to show under (i) or (ii) that (3) is a consequence of (2). Now setting $b_0 = 0$ and noting that $b_n/n^{\frac{1}{2}} \to \infty$ in either case, the left side of (3) is dominated by

$$\begin{aligned}
\frac{n}{b_n^2} EX^2 I_{[|X|\leqq b_n]} &= \frac{n}{b_n^2} \sum_{j=1}^n EX^2 I_{[b_{j-1}<|X|\leqq b_j]} \\
&\leqq \frac{n}{b_n^2} \sum_{j=1}^n b_j^2[P\{|X| > b_{j-1}\} - P\{|X| > b_j\}] \\
&= \frac{n}{b_n^2} [b_1^2 P\{|X| > 0\} - b_n^2 P\{|X| > b_n\} \\
&\quad + \textstyle\sum_{j=1}^{n-1} (b_{j+1}^2 - b_j^2) P\{|X| > b_j\}] \\
&= \frac{1}{nc_n^2} \sum_{j=1}^{n-1} [(j+1)^2 c_{j+1}^2 - j^2 c_j^2] P\{|X| > b_j\} + o(1) \\
&\leqq \frac{1}{nc_n^2} \sum_{j=1}^{n-1} [j(c_{j+1}^2 - c_j^2) + 3c_{j+1}^2] jP\{|X| > b_j\} + o(1)
\end{aligned}$$

where $c_n = b_n/n$. However, $a_{nj} = (1/nc_n^2)[j(c_{j+1}^2 - c_j^2) + 3c_{j+1}^2]$, $1 \leqq j < n$, $a_{nn} = 0$ is a nonnegative Toeplitz matrix since $\lim_{n\to\infty} a_{nj} = 0$, all $j$ and $\sum_{j=1}^n a_{nj} \leqq 4$ in case (ii) while $\sum_{j=1}^n a_{nj} \leqq (3/nc_n^2) \sum_{i=2}^n c_i^2 = O(1)$ in case (i). Thus, the last displayed expression is $o(1)$ in view of (2). □

COROLLARY 1. *If $b_n/n$ is slowly varying at $\infty$ and decreases to zero, then* (2) *is necessary and sufficient for* (1).

PROOF. Slow variation yields $\sum_{j=1}^n (b_j/j)^2 \sim n(b_n/n)^2$. Moreover, the well-known representation [5, page 274] of a slowly varying function $L$ ensures that for $L$ positive, increasing $\log L(x) = o(\log x)$ as $x \to \infty$ and hence $L(x) = o(x^\alpha)$, all $\alpha > 0$. Thus, $n/b_n = o(n^{\frac{1}{2}})$ or $b_n/n^{\frac{1}{2}} \to \infty$. □

For any unbounded random variable $X$, define

$$\begin{aligned}
&\text{(i)} \quad \bar{\mu}(x) = \textstyle\int_x^\infty P\{|X| > y\}\, dy \qquad \text{when} \quad E|X| < \infty \\
&\text{(ii)} \quad \mu(x) = \textstyle\int_0^x P\{|X| > y\}\, dy \qquad \text{if} \quad E|X| = \infty\,.
\end{aligned} \tag{4}$$

If $F_{|X|}$ denotes the distribution of $|X|$, integration by parts yields[4]

$$\begin{aligned}
&\text{(i)} \quad \bar{\mu}(x) = -xP\{|X| > x\} + \textstyle\int_x^\infty y\, dF_{|X|}(y) \\
&\text{(ii)} \quad \mu(x) = xP\{|X| > x\} + \textstyle\int_0^x y\, dF_{|X|}(y)
\end{aligned} \tag{5}$$

[4] Relation (5) holds for *all* $x > 0$ provided the Lebesgue-Stieltjes integral from $a$ to $b$ is interpreted as being over the half open interval $(a, b]$.

and moreover both $x/\bar{\mu}(x)$ and $x/\mu(x)$ are increasing. Thus, if

$$(6)\qquad \text{(i)}\quad b_x = \left(\frac{x}{\bar{\mu}(x)}\right)^{-1} \quad \text{when} \quad E|X| < \infty$$

$$\text{(ii)}\quad b_x = \left(\frac{x}{\mu(x)}\right)^{-1} \quad \text{if} \quad E|X| = \infty\,,$$

the function $b_x$ is well defined for $x >$ some constant $x_0$ (depending on $F$), necessarily $0 < b_x \uparrow \infty$, and the fundamental relationships

$$(7)\qquad \text{(i)}\quad b_x = x\bar{\mu}(b_x) \qquad \text{or} \qquad \text{(ii)}\quad b_x = x\mu(b_x)$$

follow immediately. Note, however, via (7), that $(b_x/x) \downarrow 0$ in case (i) whereas $(b_x/x) \uparrow \infty$ in case (ii).

COROLLARY 2. *Let* $\{X, X_n, n \geqq 1\}$ *be unbounded* i.i.d. *random variables with* $\bar{\mu}$ *or* $\mu$ *slowly varying at* $\infty$ *according as* $E|X|$ *is finite or infinite. If* $\{b_n, n \geqq n_0\}$ *are constants defined by* (6), *then*

$$\frac{1}{b_n}\left(\sum_{i=1}^n X_i - nEXI_{[|X|\leqq b_n]}\right) \to_P 0\,.$$

PROOF. In case (i), as already noted, $b_n/n \downarrow 0$. Moreover,

$$(8)\qquad \bar{\mu}(x) \quad \text{slowly varying} \Rightarrow \bar{\mu}(b_x) \quad \text{slowly varying},$$

since if $\varepsilon > 1$, $b_{\varepsilon x} \leqq \varepsilon b_x$, implying

$$1 \geqq \frac{\bar{\mu}(b_{\varepsilon x})}{\bar{\mu}(b_x)} \geqq \frac{\bar{\mu}(\varepsilon b_x)}{\bar{\mu}(b_x)} \to 1$$

and clearly also $\bar{\mu}(b_{\varepsilon x}) \sim \bar{\mu}(b_x)$ when $\varepsilon < 1$. Thus (8) guarantees that $b_n/n$ is slowly varying.

Furthermore, slow variation of $\bar{\mu}$ also ensures

$$(9)\qquad \bar{\mu}(x) \sim E|X|I_{[|X|>x]}\,,$$

or equivalently, via (5),

$$(9)'\qquad xP\{|X| > x\} = o(\bar{\mu}(x))\,,$$

since for any $\varepsilon$ in (0, 1)

$$\frac{x(1-\varepsilon)}{\bar{\mu}(x)} P\{|X| > x\} \leqq \frac{1}{\bar{\mu}(x)} \int_{\varepsilon x}^{x} P\{|X| > y\}\, dy = \frac{\bar{\mu}(\varepsilon x)}{\bar{\mu}(x)} - 1 = o(1)\,.$$

But (9)′ conjoined with (7) yields

$$nP\{|X| > b_n\} = \frac{b_n}{\bar{\mu}(b_n)} P\{|X| > b_n\} = o(1)$$

and the conclusion follows from Corollary 1.

In case (ii), $b_n/n \uparrow$ and in similar fashion slow variation of $\mu$ yields (9)′ with $\bar{\mu}$ replaced by $\mu$, so that (2) follows analogously and Theorem 1 yields the conclusion. □

REMARK 1. Clearly, slow variation of $\mu$ implies that of $E|X|I_{[|X|\leqq x]}$, and it follows from the proof of Theorem 2 [5, page 275] that conversely slow variation of $E|X|I_{[|X|\leqq x]}$ ensures the analogue of (9)$'$ and hence slow variation of $\mu$. Likewise, $\bar{\mu}$ is slowly varying iff $E|X|I_{[|X|>x]}$ is slowly varying. Of course, slow variation of $x[1 - F_{|X|}(x)]$ implies that of $\mu$ or $\bar{\mu}$, but not conversely [5].

REMARK 2. In case (ii), slow variation of $\mu$ implies that of $b_n/n$.

THEOREM 2. *Let* $\{X, X_n, n \geqq 1\}$ *be unbounded* i.i.d. *random variables and* $\{b_n, n \geqq n_0\}$ *constants defined by* (6). *Then according as*

$$(10)\quad \text{(i)}\quad \lim_{x\to\infty} \frac{EX^- I_{[X^->x]}}{EX^+ I_{[X^+>x]}} = c \in [0, \infty]\,, \quad \bar{\mu} \ \textit{slowly varying}, \quad EX = 0 \quad \textit{or}$$

$$\text{(ii)}\quad \lim_{x\to\infty} \frac{EX^- I_{[X^-\leqq x]}}{EX^+ I_{[X^+\leqq x]}} = c \in [0, \infty]\,, \quad \mu \ \textit{slowly varying}, \quad E|X| = \infty\,,$$

$$(11)\qquad \frac{1}{b_n} \sum_{i=1}^n X_i \to_P \frac{c-1}{c+1} \quad \textit{or} \quad \frac{1-c}{1+c}\,.$$

PROOF. When $c = \infty$, $(c-1)/(c+1)$ and $(1-c)/(1+c)$ are to be interpreted as 1 and $-1$ respectively. However, in proving (11) it may be supposed that $c < \infty$ since $c = \infty$ is the analogue of $c = 0$ when $X$ is replaced by $-X$. Now, in case (i), via (10) (i),

$$(12)\qquad \begin{aligned} EXI_{[|X|\leqq x]} &= -EX^+ I_{[X^+>x]} + EX^- I_{[X^->x]} \\ &= (c-1) EX^+ I_{[X^+>x]} + o(EX^+ I_{[X^+>x]}) \end{aligned}$$

and

$$(13)\qquad \begin{aligned} E|X|I_{[|X|>x]} &= EX^+ I_{[X^+>x]} + EX^- I_{[X^->x]} \\ &= (1+c) EX^+ I_{[X^+>x]} + o(EX^+ I_{[X^+>x]})\,. \end{aligned}$$

However, recalling that slow variation of $\bar{\mu}$ ensures (9), it follows that as $x \to \infty$,

$$\frac{1}{\bar{\mu}(x)} \cdot EX^+ I_{[X^+>x]} \leqq \frac{1}{\bar{\mu}(x)} E|X|I_{[|X|>x]} = 1 + o(1)$$

and so

$$\frac{1}{\bar{\mu}(x)} \cdot o(EX^+ I_{[X^+>x]}) = o(1)\,.$$

Consequently, from (12), (13), (9), and (7)

$$\begin{aligned} \frac{n}{b_n} EXI_{[|X|\leqq b_n]} &= \frac{n}{b_n}\left[\frac{c-1}{c+1} E|X|I_{[|X|>b_n]} + o(\bar{\mu}(b_n))\right] \\ &= \frac{1}{\bar{\mu}(b_n)}\left[\frac{c-1}{c+1}\bar{\mu}(b_n) + o(\bar{\mu}(b_n))\right] \to \frac{c-1}{c+1} \end{aligned}$$

as $n \to \infty$ and (11) follows from Corollary 2.

In case (ii), $\mu(x) \geqq E|X|I_{[|X|\leqq x]}$ via (5), whence as $x \to \infty$

$$\frac{1}{\mu(x)} \cdot o(EX^+ I_{[X^+\leqq x]}) = o(1)\,.$$

Thus, proceeding analogously

$$\frac{n}{b_n} EXI_{[|X| \le b_n]} = \frac{n}{b_n}\left[\frac{1-c}{1+c} E|X|I_{[|X|\le b_n]} + o(\mu(b_n))\right]$$
$$= \frac{1}{\mu(b_n)}\left[\frac{1-c}{1+c}\mu(b_n) + o(\mu(b_n))\right] \to \frac{1-c}{1+c}$$

and the conclusion again stems from Corollary 2. □

REMARK 3. If, in case (i), $E|X| < \infty$ but $EX \neq 0$, (11) is replaced by

$$\frac{1}{b_n}\sum_{i=1}^n X_i - \frac{n}{b_n} EX \to_P \frac{c-1}{c+1}. \tag{14}$$

The contrast between (i) and (ii) is highlighted by the cases $c = 0, \infty$:

COROLLARY 3. (i) *If* $EX = 0$ *and* $\bar{\mu}$ *is slowly varying then* $(1/b_n)\sum_1^n X_i \to_P -1$ *or* 1 *according as* $\lim_{x\to\infty} EX^- I_{[X^- > x]}/EX^+ I_{[X^+ > x]} = 0$ *or* $\infty$. (ii) *If* $E|X| = \infty$ *and* $\mu$ *is slowly varying, then* $(1/b_n)\sum_{i=1}^n X_i \to_P 1$ *or* $-1$ *according as* $\lim_{x\to\infty} EX^- I_{[X^- \le x]}/EX^+ I_{[X^+ \le x]} = 0$ *or* $\infty$.

In [2] and [4, page 262], Feller presented examples of "unfavorable, fair games," that is, random walks $S_n = \sum_{i=1}^n X_i$, $n \geqq 1$ with $X_n \geqq 0$ and $(S_n/n) \to_P EX < \infty$, but such that for some $b_n = o(n)$

$$\lim_{n\to\infty} P\{S_n - nEX < -(1-\varepsilon)b_n\} = 1, \qquad \varepsilon > 0.$$

Actually, in both examples

$$\frac{S_n - nEX}{b_n} \to_P -1$$

and (i) of Corollary 3 (see Remark 3) reveals that it is the slow variation of $\bar{\mu}$ that underlies the phenomenon noted by Feller. Part (ii) of Corollary 3 likewise generalizes the St. Petersburg game for which Feller showed [4] that $S_n/n \operatorname{Log} n \to_P 1$ where Log denotes logarithm to the base 2.

## 3. Generalized laws of the iterated logarithm.

Two preliminary lemmas will be needed in establishing the main results.

The first lemma is reminiscent of comparable results in [7], [14], [12].

LEMMA 1. *Let* $S_n = \sum_{i=1}^n X_i$ *where* $\{X_n, n \geqq 1\}$ *are independent random variables with* $EX_n = 0$, $EX_n^2 = \sigma_n^2$, $s_n^2 = \sum_1^n \sigma_i^2$. *If* $P\{X_n \leqq C_n\} = 1$, $n \geqq 1$ *where* $0 < C_n \uparrow$, *then for any* $x > 0$

$$P\{\max_{1\le j\le n} S_j \geqq x\} \leqq \exp\left\{-tx + s_n^2\left(\frac{e^{tC_n} - 1 - tC_n}{C_n^2}\right)\right\}. \tag{15}$$

PROOF. Since any twice differentiable function $q(t) = q(0) + tq'(0) + \int_0^t \int_0^y q''(u)\,du\,dy$, it follows for any random variable $X$ with finite variance and $X \leqq C$ that for $t > 0$

$$Ee^{tX} = 1 + tEX + \int_0^t \int_0^y EX^2 e^{uX}\,du\,dy \leqq 1 + tEX + EX^2 \int_0^t \int_0^y e^{uC}\,du\,dy$$

and so $EX_j = 0$ and $C_j \uparrow$ ensure that for $1 \leqq j \leqq n$ and $t > 0$

$$Ee^{tX_j} \leqq \exp\left\{\sigma_j^2\left(\frac{e^{tC_j} - 1 - tC_j}{C_j^2}\right)\right\} \leqq \exp\left\{\sigma_j^2\left(\frac{e^{tC_n} - tC_n - 1}{C_n^2}\right)\right\}.$$

Consequently, observing that $\exp\{tS_j\}$, $1 \leqq j \leqq n$ is a submartingale, the submartingale inequality guarantees that for $t > 0$

$$P\{\max_{1\leqq j\leqq n} S_j \geqq x\} = P\{\max_{1\leqq j\leqq n} e^{tS_j} \geqq e^{tx}\} \leqq e^{-tx}Ee^{tS_n}$$
$$\leqq \exp\left\{-tx + s_n^2\left(\frac{e^{tC_n} - tC_n - 1}{C_n^2}\right)\right\}. \qquad \square$$

Case (ii) of the next lemma is essentially the first lemma and proposition of [1].

LEMMA 2. *Let $b_x$ be defined by* (6) *and suppose in case* (i) *that* $\bar{\mu}(x) = \int_x^\infty P\{|X| > y\}\,dy$ *is slowly varying at* $\infty$. *Then*

(16) $$\sum_{n=1}^\infty P\{|X| > \varepsilon b_n\} = \infty, \qquad \varepsilon > 0.$$

PROOF. Set $a(x) = x/\bar{\mu}(x)$ and $\varepsilon > 1$. For all large $x$, slow variation of $\bar{\mu}$ yields

$$a(\varepsilon x) \geqq a(x) \geqq \frac{a(\varepsilon x)}{2\varepsilon}$$

and it follows that $Ea(|X|) < \infty$ iff $Ea(\varepsilon|X|) < \infty$, all $\varepsilon > 0$.

Now if $E|X|/\bar{\mu}(|X|) < \infty$ and $F$ denotes the distribution of $|X|$ then (9)′ holds, implying

(17) $$E|X|/\int_{|X|}^\infty y\,dF(y) < \infty.$$

However, (17) contradicts the Abel–Dini theorem in view of $E|X| < \infty$. Consequently, $Ea(|X|) = \infty$ which, as just seen, is equivalent to $Ea(|X|/\varepsilon) = \infty$, $\varepsilon > 0$, which, in turn, is tantamount to (16).

In case (ii), it suffices to choose $a(x) = x/\mu(x)$ and note that $a(x) \uparrow$ whereas $x^{-1}a(x) \downarrow$. Then (16) follows by an analogous argument employing the other part of the Abel–Dini theorem corresponding to $E|X| = \infty$. $\square$

THEOREM 3. *Let $\{X, X_n, n \geqq 1\}$ be unbounded* i.i.d. *random variables with $EX = 0$ and let $\bar{\mu}$ be as in* (4). *If*

(18) $$E\frac{X^+}{\bar{\mu}(X^+)} < \infty,$$

(19) $$\bar{\mu}(x) \sim \bar{\mu}(x \log_2 x) \qquad \text{as} \quad x \to \infty,$$

*then*

(20) $$\limsup_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^n X_i = 1, \quad \text{a.s.},$$
$$\liminf_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^n X_i = -\infty, \quad \text{a.s.}$$

*where $\{b_n, n \geqq n_0\}$ are constants defined by* (6).

PROOF. In view of (18),

$$EX^+I_{[X^+>x]} = E\bar\mu(X^+)\frac{X}{\bar\mu(X^+)}I_{[X>x]} \leqq \bar\mu(x)E\frac{X^+}{\bar\mu(X^+)}I_{[X^+>x]} = o(\bar\mu(x)) .$$

Thus, since (19) entails slow variation of $\bar\mu$ and hence (9),

$$EX^-I_{[X^->x]} = E|X|I_{[|X|>x]} - EX^+I_{[X^+>x]} = \bar\mu(x) + o(\bar\mu(x))$$

implying as $x \to \infty$ that

$$\frac{EX^-I_{[X^->x]}}{EX^+I_{[X^+>x]}} = \frac{(1+o(1))\bar\mu(x)}{EX^+I_{[X^+>x]}} \to \infty .$$

Thus, Corollary 3 guarantees

$$\frac{1}{b_n}\sum_{i=1}^n X_i \to_P 1 \tag{21}$$

which, in turn, ensures

$$\limsup_{n\to\infty}\frac{1}{b_n}\sum_{i=1}^n X_i \geqq 1 , \quad \text{a.s.} \tag{22}$$

To prove the reverse inequality of (22), note at the outset via (19) (recall (8)) that

$$\bar\mu(x) , \quad \bar\mu(b_x) \quad \text{are slowly varying} \tag{23}$$

$$\bar\mu\left(\frac{x}{(\log_2 x)^2}\right) \sim \bar\mu(x) \sim \bar\mu(x(\log_2 x)^2) .$$

Set

$$d_n = \frac{b_n}{(\log_2 b_n)^2} \sim \frac{b_n}{(\log_2 n)^2}$$

and note that in view of (23)

$$-\sum_{i=1}^n EXI_{[X<-d_i]} = \sum_{i=1}^n EX^-I_{[X^->d_i]} \leqq \sum_{i=1}^n E|X|I_{[|X|>d_i]} \tag{24}$$
$$\sim nE|X|I_{[|X|>d_n]} \sim n\bar\mu(d_n) \sim n\bar\mu(b_n) = b_n .$$

Define

$$Y_n = X_nI_{[|X_n|\leqq d_n]} - EXI_{[|X|\leqq d_n]}$$
$$W_n = X_nI_{[X_n>b_n]}$$
$$Z_n = X_nI_{[d_n<X_n\leqq b_n]} - EXI_{[d_n<X\leqq b_n]} .$$

By virtue of $EX = 0$ and (24)

$$\sum_{i=1}^n X_i = \sum_{i=1}^n (Y_i + Z_i + W_i + X_iI_{[X_i<-d_i]} - EXI_{[X<-d_i]} - EXI_{[X>b_i]}) \tag{25}$$
$$\leqq \sum_{i=1}^n Y_i + \sum_{i=1}^n Z_i + \sum_{i=1}^n W_i + b_n(1+o(1)) .$$

Consequently, if each of the sums on the right side of (25), when normalized by $b_n$, has a nonpositive upper limit, a.s. it follows that

$$\limsup\frac{1}{b_n}\sum_{i=1}^n X_i \leqq 1 , \quad \text{a.s.} \tag{26}$$

Now, (18) is tantamount to

$$(27) \qquad \sum_{n=1}^{\infty} P\{X^+ > b_n\} < \infty$$

whence $\sum_1^n W_i = O(1)$ with probability one and so

$$\frac{1}{b_n} \sum_1^n W_i \to_{\text{a.s.}} 0 .$$

Next, for any $\alpha$ in $(1, 2^{\frac{1}{2}})$ and all large $x$, slow variation of $b_x/x$ guarantees that $b_{2x} > \alpha 2^{\frac{1}{2}} b_x$ and hence $b_{2^k x} \geqq b_x(\alpha 2^{\frac{1}{2}})^k$, $k \geqq 0$. Thus, for all large $n$,

$$\sum_{j=n}^{\infty} b_j^{-2} = \sum_{k=0}^{\infty} \sum_{i=2^k n}^{2^{k+1}n-1} b_i^{-2} \leqq \sum_{k=0}^{\infty} 2^k n (b_{2^k n})^{-2} \leqq \sum_{k=0}^{\infty} \frac{n}{b_n^2 \alpha^{2k}} \leqq \frac{C(n-1)}{b_n^2}$$

where $C = 2\alpha^2/(\alpha^2 - 1)$. Consequently, for some positive constant $C_1$,

$$\begin{aligned} \sum_{i=2}^{\infty} \frac{1}{b_i^2} EX^2 I_{[d_i < X \leqq b_i]} &\leqq \sum_{n=1}^{\infty} EX^2 I_{[b_{n-1} < X \leqq b_n]} \sum_{j=n}^{\infty} b_j^{-2} \\ &\leqq C_1 + C \sum_{n=2}^{\infty} (n-1) P\{b_{n-1} < X \leqq b_n\} \\ &= C_1 + C \sum_{n=1}^{\infty} P\{X > b_n\} < \infty \end{aligned}$$

in view of (27). Thus, $\sum_{i=1}^{\infty} Z_i/b_i$ converges a.s. by the Khintchine-Kolmogorov theorem, and invoking Kronecker's lemma,

$$\frac{1}{b_n} \sum_{i=1}^{n} Z_i \to_{\text{a.s.}} 0.$$

Thirdly, in view of

$$(28) \qquad \begin{aligned} EX^2 I_{[|X| \leqq x]} &= -\int_0^x y\, d\{E|X| I_{[|X|>y]}\} \\ &= -xE|X| I_{[|X|>x]} + \int_0^x E|X| I_{[|X|>y]}\, dy = o(xE|X| I_{[|X|>x]}) , \end{aligned}$$

it follows, setting $S_n = \sum_{i=1}^n Y_i$ and $s_n^2 = \sum_{i=1}^n \sigma_{Y_i}^2$, that

$$\begin{aligned} s_n^2 &\leqq \sum_{j=1}^n EX^2 I_{[|X| \leqq d_j]} \leqq nEX^2 I_{[|X| \leqq d_n]} = o(nd_n E|X| I_{[|X|>d_n]}) \\ &= o(nd_n \bar{\mu}(d_n)) = o(d_n b_n) . \end{aligned}$$

Hence, if $g(x) = x^{-2}(e^x - 1 - x)$ and $\delta$ is a positive constant,

$$\frac{s_n^2(\log_2 n)^2}{b_n^2} g\left(\frac{2d_n \log_2 n}{\delta b_n}\right) = o\left(\frac{d_n(\log_2 n)^2}{b_n} g\left(\frac{2}{\delta \log_2 n}\right)\right) = o(1)$$

and so choosing $t = (2(\log_2 n)/\delta b_n)$, $x = \delta b_n$ in (15) of Lemma 1,

$$(29) \qquad P\left\{\max_{1 \leqq j \leqq n} S_j > \frac{\varepsilon}{4} b_n\right\} \leqq \frac{2}{(\log n)^2}, \qquad \varepsilon > 0 .$$

Therefore, setting $n_k = 3^k$, $k \geq 0$ and choosing $K$ large and such that $b_{n_{k-1}}/b_{n_k} = \bar{\mu}(b_{n_{k-1}})/3\bar{\mu}(b_{n_k}) > \frac{1}{4}$ for $k \geqq K$, it follows from (29) that

$$\begin{aligned} \sum_{k=K}^{\infty} P\{\max_{n_{k-1} < n \leqq n_k} S_n > \varepsilon b_{n_{k-1}}\} &\leqq \sum_{k=K}^{\infty} P\left\{\max_{1 \leqq n \leqq n_k} S_n > \frac{\varepsilon}{4} b_{n_k}\right\} \\ &\leqq \sum_{k=K}^{\infty} \frac{2}{(k \log 3)^2} < \infty . \end{aligned}$$

Thus, by the Borel–Cantelli lemma, for all $\varepsilon > 0$

$$P\{S_n > \varepsilon b_n, \text{ i.o.}\} \leqq P\{\max_{n_{k-1}<n\leqq n_k} S_n > \varepsilon b_{n_{k-1}}, \text{ i.o. } (k)\} = 0$$

implying

$$\limsup_{n\to\infty} \frac{S_n}{b_n} \leqq 0\,, \quad \text{a.s.}$$

Consequently, (26) obtains and in conjunction with (22) yields the first half of (20).

To prove the second half of (20), note that when $E|X| < \infty$

$$\lim_{n\to\infty} \frac{b_n}{b_{n-1}} = 1\,, \tag{30}$$

since $1 \geqq b_{n-1}/b_n = (1-(1/n))\bar{\mu}(b_{n-1})/\bar{\mu}(b_n) \geqq 1-(1/n)$. Hence, if

$$\limsup_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^n (-X_i) =_{\text{a.s.}} C\,, \quad \text{finite,} \tag{31}$$

this would entail

$$\limsup_{n\to\infty} \left(\frac{-X_n}{b_n}\right) = \limsup_{n\to\infty} \left(\frac{-\sum_{i=1}^n X_i}{b_n} + \frac{\sum_{i=1}^{n-1} X_i}{b_n}\right) \leqq C+1 \tag{32}$$

via (30) and the portion of the theorem already proved. But (32) and the Borel–Cantelli lemma clearly imply

$$\sum_{n=1}^\infty P\{X_n^- > \varepsilon b_n\} < \infty\,, \qquad \varepsilon > C+1\,,$$

while (27) ensures

$$\sum_{n=1}^\infty P\{X^+ > \varepsilon b_n\} < \infty\,, \qquad \varepsilon \geqq 1\,.$$

Consequently, both series would converge for $\varepsilon > \max[1, C+1]$ which is incompatible with Lemma 2. Therefore, (31) is untenable and the zero-one law guarantees

$$\limsup_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^n (-X_i) = \infty\,, \quad \text{a.s.}$$

which is tantamount to the latter part of (20). □

THEOREM 4. *Let* $\{X, X_n, n \geqq 1\}$ *be* i.i.d. *random variables with* $E|X| = \infty$ *and let* $\mu$ *be as in* (4). *If*

$$E\frac{X^+}{\mu(X^+)} < \infty \tag{33}$$

$$\mu(x) \sim \mu(x \log_2 x) \qquad \text{as} \quad x \to \infty \tag{34}$$

*then*

$$\limsup_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^n X_i = -1\,, \quad \text{a.s.,} \tag{35}$$

$$\liminf_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^n X_i = -\infty\,, \quad \text{a.s.}$$

*where* $\{b_n, n \geqq n_0\}$ *are constants defined by* (6).

PROOF. Choose $0 < c_x \uparrow \infty$ such that $EX^+I_{[X^+ \leqq c_x]} = o(\mu(x))$. Then (33) entails

$$EX^+I_{[X^+\leqq x]} \leqq EX^+I_{[X^+\leqq c_x]} + E\mu(X^+)\frac{X^+}{\mu(X^+)}I_{[c_x<X^+\leqq x]} \tag{36}$$
$$\leqq o(\mu(x)) + \mu(x)E\frac{X^+}{\mu(X^+)}I_{[c_x<X\leqq x]} = o(\mu(x)) .$$

Thus, since (34) implies slow variation of $\mu$ and hence the analogue of (9),

$$EX^-I_{[X^-\leqq x]} = E|X|_{[|X|\leqq x]} - EX^+I_{[X^+\leqq x]}$$
$$= \mu(x) + o(\mu(x)) ,$$

and therefore as $x \to \infty$,

$$\frac{EX^-I_{[X^-\leqq x]}}{EX^+I_{[X^+\leqq x]}} = \frac{(1+o(1))\mu(x)}{EX^+I_{[X^+\leqq x]}} \to \infty .$$

Thus, according to Corollary 3,

$$\frac{1}{b_n}\sum_{i=1}^n X_i \to_P -1 \tag{37}$$

and so

$$\limsup_{n\to\infty}\frac{1}{b_n}\sum_{i=1}^n X_i \geqq -1 , \quad \text{a.s.} \tag{38}$$

Define $d_n$, $Y_n$, $Z_n$ and $W_n$ exactly as in Theorem 3. In view of (36),

$$EXI_{[-d_n\leqq X\leqq b_n]} = EXI_{[-d_n\leqq X\leqq 0]} + EXI_{[0<X\leqq b_n]}$$
$$\leqq -E|X|I_{[|X|\leqq d_n]} + 2EX^+I_{[X^+\leqq b_n]}$$
$$= -\mu(d_n)(1+o(1)) + o(\mu(b_n)) ,$$

implying via (34) that

$$\sum_{i=1}^n EXI_{[-d_i\leqq X\leqq b_i]} \leqq -n\mu(b_n)(1+o(1)) = -b_n(1+o(1)) .$$

Thus, the analogue of (25) of Theorem 3 is

$$\sum_{i=1}^n X_i = \sum_{i=1}^n (Y_i + Z_i + W_i + X_iI_{[X_i<-d_i]} + EXI_{[-d_i\leqq X\leqq b_i]}) \tag{39}$$
$$\leqq \sum_{i=1}^n Y_i + \sum_{i=1}^n Z_i + \sum_{i=1}^n W_i - b_n(1+o(1)) .$$

Exactly as in Theorem 3, $(1/b_n)\sum_1^n W_i \to_{\text{a.s.}} 0$. Moreover, slow variation of $\mu$ entails that of $E|X|I_{[|X|\leqq x]}$, whence

$$s_n^2 = \sum_{j=1}^n \sigma_{Y_j}^2 \leqq nEX^2I_{[|X|\leqq d_n]} = n\int_0^{d_n} x\,d\{E|X|I_{[|X|\leqq x]}\}$$
$$= n[d_nE|X|I_{[|X|\leqq d_n]} - \int_0^{d_n} E|X|I_{[|X|\leqq x]}\,dx]$$
$$= o(nd_nE|X|I_{[|X|\leqq d_n]}) = o(nd_n\mu(d_n)) = o(d_nb_n)$$

and so the argument of Theorem 3 carries over, mutatis mutandis, yielding $\limsup_{n\to\infty}(1/b_n)\sum_{j=1}^n Y_j \leqq 0$, a.s.

Furthermore, since $(b_n/n) \uparrow$

$$\sum_{n=2}^{\infty} \frac{1}{b_n^2} EX^2 I_{[d_n < X \leqq b_n]} \leqq C \sum_{n=2}^{\infty} EX^2 I_{[b_{n-1} < X \leqq b_n]} \sum_{j=n}^{\infty} \frac{1}{b_j^2}$$
$$\leqq C_1 \sum_{n=2}^{\infty} (n-1) P\{b_{n-1} < X \leqq b_n\}$$
$$= C_1 \sum_{n=1}^{\infty} P\{X > b_n\} < \infty ,$$

whence $(1/b_n) \sum_{j=1}^{n} Z_j \to_{\text{a.s.}} 0$.

Thus, (39) elicits the reverse inequality of (38) and the first half of (35) follows.

Finally, (37) and $X_n/b_n \to_P 0$ ensure $(1/b_n) \sum_{i=1}^{n-1} X_i \to_P -1$, and thus the argument of Theorem 3 carries over to the last half of (35), since $b_{n-1}/b_n = b_n^{-1} \sum_1^{n-1} X_i / b_{n-1}^{-1} \sum_1^{n-1} X_i \to_P 1$. □

**4. Relation of the normalizing constants to the distribution of the sum.** In Section 2, it was shown that if $EX = 0$, $\bar{\mu}$ is slowly varying and (10) holds, then $(1/b_n) \sum_1^n X_i \to_P (c-1)/(c+1)$. Theorem 5 asserts under identical conditions that $(1/b_n)E(\sum_1^n X_i)^+ \to \max[1/(1+c), c/(1+c)]$.

THEOREM 5. *Let* $\{X, X_n, n \geqq 1\}$ *be unbounded* i.i.d. *random variables with* $EX = 0$ *and* $\{b_n, n \geqq n_0\}$ *constants defined by* (6). *If* $\bar{\mu}$ *is slowly varying and*

$$\lim_{x\to\infty} \frac{EX^- I_{[X^- > x]}}{EX^+ I_{[X^+ > x]}} = c \in [0, \infty] , \tag{40}$$

*then, setting* $S_n = \sum_1^n X_i$,

$$\lim_{n\to\infty} \frac{1}{b_n} ES_n^+ = \max\left[\frac{1}{1+c}, \frac{c}{1+c}\right] . \tag{41}$$

PROOF. Define

$$S_n' = \sum_{j=1}^{n} X_j I_{[|X_j| > b_n]} , \qquad S_n'' = \sum_{j=1}^{n} X_j I_{[|X_j| \leqq b_n]} .$$

Recalling (28),

$$E(S_n'' - ES_n'')^2 \leqq nEX^2 I_{[|X| \leqq b_n]} = no(b_n \bar{\mu}(b_n)) = o(b_n^2) .$$

Thus, noting that $ES_n = 0$ (whence $E|S_n| = 2ES_n^+$), $|E|S_n| - E|S_n' - ES_n'|| \leqq E|S_n'' - ES_n''| = o(b_n)$ and so to establish (41), it suffices to verify that $(1/b_n)E|S_n' - ES_n'| \to 2\max[1/(1+c), c/(1+c)]$. To this end, observe that as in the proof of Corollary 2,

$$nP\{|X| > b_n\} = o(1) ; \tag{42}$$

while via the proof of Theorem 2,

$$EX^+ I_{[X^+ > b_n]} \sim \frac{1}{1+c} \bar{\mu}(b_n) .$$

Thus,

$$
\begin{aligned}
E|S_n' - ES_n'| &\leqq E|S_n'| + |ES_n'| \\
&\leqq n[(1+o(1))\tilde{\mu}(b_n) + |EX^+I_{[X^+>b_n]} - EX^-I_{[X^->b_n]}|] \qquad (43)\\
&= n\tilde{\mu}(b_n)\left[1 + o(1) + \left|\frac{1}{1+c} - \frac{c}{1+c}\right|\right] \\
&= 2b_n\left(\max\left[\frac{1}{1+c}, \frac{c}{1+c}\right] + o(1)\right).
\end{aligned}
$$

To obtain the reverse inequality of (43), set

$$
B_n = \bigcap_{j=1}^n [|X_j| \leqq b_n], \qquad A_{n,j} = \{[|X_j| > b_n] \bigcap_{i\neq j}^n [|X_i| \leqq b_n]\},
$$

and note that via (42)

$$
\begin{aligned}
\textstyle\sum_{j=1}^n P\{A_{n,j}\} = nP\{A_{n,1}\} &\leqq nP\{|X| > b_n\} = o(1) \\
P\{B_n\} &\to 1
\end{aligned}
$$

as $n \to \infty$. Thus,

$$
\begin{aligned}
E|S_n' - ES_n'| &\geqq E|S_n' - ES_n'|I_{B_n} + \textstyle\sum_{j=1}^n E(|S_n'| - |ES_n'|)I_{A_{n,j}} \\
&= |ES_n'|[P\{B_n\} - \textstyle\sum_{j=1}^n P\{A_{n,j}\}] + \textstyle\sum_{j=1}^n E|X_j|I_{A_{n,j}} \\
&= |ES_n'|(1+o(1)) + nE|X_1|I_{A_{n,1}} \\
&= |ES_n'|(1+o(1)) + nE|X|I_{[|X|>b_n]}P\{\bigcap_{i=2}^n [|X_i| \leqq b_n]\} \\
&= (1+o(1))[n\tilde{\mu}(b_n) + |ES_n'|] \\
&= 2b_n\left(\max\left[\frac{1}{1+c}, \frac{c}{1+c}\right] + o(1)\right)
\end{aligned}
$$

exactly as in (43) and the theorem follows.

Since the common hypothesis (with $c = \infty$) of Theorems 2 and 5 is implied by that of Theorem 3, there follows

COROLLARY 4. *Under the hypotheses of Theorem* 3,

$$
\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{b_n} \textstyle\sum_{i=1}^n X_i =_{\text{a.s.}} 1 = \lim_{n\to\infty} E\frac{S_n^+}{b_n} \\
&S_n/b_n \to_p 1 .
\end{aligned}
$$

## REFERENCES


[1] ERICKSON, K. B. (1973). The strong law of large numbers when the mean is undefined. *Trans. Amer. Math. Soc.* **185** 371-381.

[2] FELLER, W. (1945). Note on the law of large numbers and "fair games." *Ann. Math. Statist.* **16** 301-304.

[3] FELLER, W. (1946). A limit theorem for random variables with infinite moments. *Amer. J. Math.* **68** 257-262.

[4] FELLER, W. (1966). *An Introduction to Probability Theory and its Applications* **1**, 3rd ed. Wiley, New York.

[5] FELLER, W. (1968). *An Introduction to Probability Theory and its Applications* **2**, 1st ed. Wiley, New York.

[6] FELLER, W. (1968). An extension of the law of the iterated logarithm to variables without variance. *J. Math. Mech.* **18** 343-356.

[7] FREEDMAN, D. (1975). On tail probabilities for martingales. *Ann. Probability* **3** 100-118.

[8] HARTMAN, P. and WINTNER, A. (1941). On the law of the iterated logarithm. *Amer. J. Math.* **63** 169-176.

[9] HEYDE, C. C. (1969). A note concerning behavior of iterated logarithm type. *Proc. Amer. Math. Soc.* **23** 85-90.

[10] KESTEN, H. (1972). Sums of independent random variables without moment conditions. *Ann. Math. Statist.* **43** 701-732.

[11] KLASS, M. J. (1974). A survey of the $S_n/n$ problem. *Proc. Summer Res. Inst. Statist. Inference for Stochastic Processes* **2** 179-201 (Indiana Univ.). Academic Press, New York.

[12] KLASS, M. J. (1976). Toward a universal law of the iterated logarithm, Part I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **36** 165-178.

[13] ROGOZIN, B. A. (1968). On the existence of exact upper sequences. *Teor. Verojatnost. i Primenen* **13** 701-707. (*Theor. Probability Appl.* **13** 667-672.)

[14] STEIGER, W. L. (1969). A best possible Kolmogorov-type inequality for martingales and a characteristic property. *Ann. Math. Statist.* **40** 764-769.

[15] TEICHER, H. (1974). On the law of the iterated logarithm. *Ann. Probability* **2** 714-728.

[16] TEICHER, H. (1975). A necessary condition for the iterated logarithm law. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **31** 343-349.

[17] WICHURA, M. (1974). Functional laws of the iterated logarithm for the partial sums of i.i.d. random variables in the domain of attraction of a completely asymmetric stable law. *Ann. Probability* **2** 1108-1138.

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