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ON THE INCREMENTS OF MULTIDIMENSIONAL RANDOM FIELDS

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For a nondifferentiable random field $\{X_t : t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$, it is often easy to check that with probability 1 $\liminf_{s\to t} ||X_s - X_t||/\sigma(s, t) = 0$ and $\limsup_{s\to t} ||X_s - X_t||/\sigma(s, t) = \infty$ for a.e. t, where $\sigma^2(s, t) = E||X_s - X_t||^2$. In this note we discuss the "proportion" of s's near t for which $||X_s - X_t||/\sigma(s, t)$ is small or large.

Let $\{X_t, t \in \mathbb{R}^N\}$ be an M-valued stochastic process, $\mathbb{R}^N$ being N-dimensional Euclidean space and $(M, ||\cdot||)$ a separable Banach space; $(\Omega, \mathscr{F}, P)$ denotes the probability space. We assume that $(t, \omega) \to X_t(\omega)$ is measurable $\mathscr{B} \otimes \mathscr{F} \to \mathscr{M}$ ($\mathscr{B}$, $\mathscr{M}$ the Borel σ-fields in $\mathbb{R}^N$, M respectively), that $E||X_t||^2 < \infty$ for all t, and that $\sigma^2(s, t) = E||X_s - X_t||^2$ is jointly continuous.

In this paper we will consider the approximate local behavior of the normalized increments $||X_s - X_t||/\sigma(s, t)$. When $M = \mathbb{R}^d$, $d \leqq N$, and X_t is nondifferentiable, it is usually easy to check that with probability one,

$$(1) \qquad \liminf_{s\to t} \frac{||X_s - X_t||}{\sigma(s, t)} = 0 \qquad \text{and} \qquad \limsup_{s\to t} \frac{||X_s - X_t||}{\sigma(s, t)} = \infty$$

for every t or λ-a.e. t, where λ is Lebesgue measure on $\mathscr{B}$. However, behavior such as (1) provides no information on the "proportion" of s's near t for which $||X_s - X_t||/\sigma(s, t)$ is small or large. Indeed, (1) can be arranged by altering $X_\bullet(\omega)$ on a *countable* set.

Let μ be a positive, σ-finite Borel measure on $\mathscr{B}$, positive on open sets, and let $B(t, \varepsilon)$ be the open ball centered at $t \in \mathbb{R}^N$ of radius ε. Let $f(s)$ be a real, $\mathscr{B}$-measurable function. The *approximate lower limit* ap $\liminf_{s\to t} f(s)$ (relative to μ and the Euclidean topology) is the supremum of those v such that $\{s : f(s) < v\}$ has density 0 at t—i.e.,

$$\lim_{\varepsilon\downarrow 0} \frac{\mu\{B(t, \varepsilon) \cap \{s : f(s) < v\}\}}{\mu\{B(t, \varepsilon)\}} = 0 .$$

The *approximate upper limit* is defined analogously. (See [2] and [7] for the role of approximate limits, derivatives, etc., in classical and modern analysis.)

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For a large class of processes we will show that for each $t \in \mathbb{R}^N$, with probability one:

$$\text{(2)} \qquad \text{ap}\lim\inf_{s\to t}\frac{\|X_s - X_t\|}{\sigma(s,t)} = 0 \qquad \text{and} \qquad \text{ap}\lim\sup_{s\to t}\frac{\|X_s - X_t\|}{\sigma(s,t)} = \infty\,.$$

Obviously, (2) implies (1).

Before stating the main theorems, we give two examples; the first shows how these theorems complement certain results in [3], [4] and [6].

EXAMPLE 1. Let $\{X_t = (X_1(t), \cdots, X_d(t)), t \in \mathbb{R}^N\}$ be the d-dimensional Gaussian random field with i.i.d. components, zero means, and covariance $EX_j(t)X_j(s) = \|t\|^\alpha + \|s\|^\alpha - \|t-s\|^\alpha$, $0 < \alpha < 2$. It follows from Theorem 1 of [6] that, if $N - \alpha d/2 < 0$,

$$\text{(3)} \qquad \lim_{s\to t}\frac{\|X_s - X_t\|}{\|s-t\|^r} = \infty \qquad \text{for } \lambda\text{-a.e. } t\,, \quad \text{a.s.}$$

for any $r > \alpha/2$. On the other hand, if $N - \alpha d/2 = \varepsilon > 0$, the results of [3] yield

$$\text{(4)} \qquad \text{ap}\lim_{s\to t}\frac{\|X_s - X_t\|}{\|s-t\|^r} = \infty \qquad \text{for } \lambda\text{-a.e. } t\,, \quad \text{a.s.}$$

for $r = \alpha/2 + \varepsilon/d$. Now for standard one-dimensional Brownian motion, i.e., $\alpha, N, d = 1$, it is not hard to show, using the zero-one law, that

$$\text{(5)} \qquad \text{ap}\lim\inf_{s\to t}\frac{|X_s - X_t|}{|s-t|^{\frac{1}{2}}} = \infty \qquad \text{for } \lambda\text{-a.e. } t\,, \quad \text{a.s.}$$

Finally, by Theorem 3 of this paper we find that for any N and d (in particular when $N = \alpha d/2$),

$$\text{(6)} \qquad \text{ap}\lim\inf_{s\to t}\frac{\|X_s - X_t\|}{\|s-t\|^{\alpha/2}} = 0\,,$$

$$\text{ap}\lim\inf_{s\to t}\frac{\|X_s - X_t\|}{\|s-t\|^{\alpha/2}} = \infty \qquad \lambda\text{-a.e.}\,, \quad \text{a.s.}$$

It is clear that in some sense (6) represents the boundary case between (3) and (4).

EXAMPLE 2. Let $W(s, t)$, $s, t \in [0, 1]$, be the Yeh–Wiener process, i.e., the mean 0, real Gaussian field on $[0, 1]^2$ with $W(0, t) = 0$ a.s., $W(s, 0) = 0$ a.s., and

$$EW(s_1, t_1)W(s_2, t_2) = \min(s_1, s_2) \cdot \min(t_1, t_2)\,.$$

One can also view W as a one-parameter process with values in $M = C[0, 1]$: $W_s\colon \Omega \to C[0, 1]$, $W_s(\omega)(t) = W(s, t, \omega)$.

From Theorem 1, then,

$$\text{ap}\lim\sup_{(s,t)\to(a,b)}\frac{|W(s,t) - W(a,b)|}{\sigma_1((s,t),(a,b))} = \infty\,,$$

$$\text{ap}\lim\inf_{(s,t)\to(a,b)}\frac{|W(s,t) - W(a,b)|}{\sigma_1((s,t),(a,b))} = 0 \quad \text{a.s.}$$

for each $0 \leqq a, b \leqq 1$, where $\sigma_1^2((s, t), (a, b)) = st + ab - 2(s \wedge a)(t \wedge b)$. Also,

$$\text{ap lim sup}_{s \to a} \frac{\sup_{0 \leqq t \leqq 1} |W(s, t) - W(a, t)|}{|s - a|^{\frac{1}{2}}} = \infty ,$$

$$\text{ap lim inf}_{s \to a} \frac{\sup_{0 \leqq t \leqq 1} |W(s, t) - W(a, b)|}{|s - a|^{\frac{1}{2}}} = 0 \quad \text{a.s.}$$

Returning to the general case, our basic assumptions about X_t will be

(A1) There exist measurable functions ϕ_j: $\mathbb{R}^N \to M$ and independent random variables ξ_j, $j = 1, 2, \cdots$ such that for each t,

(a) $X_t(\omega) = \lim_{n \to \infty} \sum_{j=1}^n \xi_j(\omega)\phi_j(t)$ a.s. and
(b) $\lim_{s \to t} ||\phi_j(s) - \phi_j(t)||/\sigma(s, t) = 0$;

(A2) For each $t \in \mathbb{R}^N$ and $Q > 0$ there are numbers $\varepsilon, \delta > 0$ such that

$$\delta \leqq P(||X_s - X_t|| \leqq Q\sigma(s, t)) \leqq 1 - \delta$$

for all $s \in B(t, \varepsilon)$.

We note that the set of pairs (t, ω) for which $\sum_1^n \xi_j(\omega)\phi_j(t)$ converges to $X_t(\omega)$ is jointly measurable; hence, for almost every ω we have $X_t(\omega) = \sum_1^\infty \xi_j(\omega)\phi_j(t)$ for each $t \in A_\omega$, $\mu(A_\omega^c) = 0$.

THEOREM 1. *Suppose* $\{X_t\}$ *satisfies* (A1) *and* (A2). *Then for each* $t \in \mathbb{R}^N$, (2) *holds with probability one.*

NOTE. For t, ω, and $Q > 0$ fixed, the expression

$$g(\varepsilon) \equiv \frac{\mu\{s \in B(t, \varepsilon) \colon ||X_s(\omega) - X_t(\omega)|| \leqq Q\sigma(s, t)\}}{\mu\{B(t, \varepsilon)\}} \tag{7}$$

need not have a limit as $\varepsilon \to 0$. (See Remark (e).) To prove the theorem, we must show that for each t fixed, $\limsup_{\varepsilon \downarrow 0} g(\varepsilon) > 0$ and $\liminf_{\varepsilon \downarrow 0} g(\varepsilon) < 1$ for all $Q > 0$, with probability one.

PROOF. Fix $t \in \mathbb{R}^N$ and define (for a.e. ω):

$$X^m(t, \omega) = \sum_{j=m+1}^\infty \xi_j(\omega)\phi_j(t) , \qquad X_m(t, \omega) = \sum_{j=1}^m \xi_j(\omega)\phi_j(t) , \quad m \geqq 0 ,$$

and

$$K^m(Q, n, \omega) = [\mu\{B(t, 1/n)\}]^{-1}\mu\{s \in B(t, 1/n) \cap A_\omega \colon ||X^m(s, \omega) - X^m(t, \omega)|| \leqq Q\sigma(s, t)\} , \qquad n \geqq 1, Q > 0 .$$

(Since $\mu(A_\omega^c) = 0$ a.s., we will ignore A_ω^c in what follows.) Further, define

$$\bar{G}^m(Q, \omega) = \limsup_{n \to \infty} K^m(Q, n, \omega) , \qquad \underline{G}^m(Q, \omega) = \liminf_{n \to \infty} K^m(Q, n, \omega) .$$

We will show that, with probability one, $\bar{G}^0(Q, \omega) > 0$ for all Q and $\underline{G}^0(Q, \omega) < 1$ for all Q.

Now fix $m \geqq 0$, $Q > \eta > 0$, and $\omega \in \Omega$. By (A1)(b), there exists an integer n_0 such that $||s - t|| \leqq 1/n_0$ implies $||X_m(s, \omega) - X_m(t, \omega)|| \leqq \eta\sigma(s, t)$. Hence, for $n \geqq n_0$ and a.e. ω

$$\begin{aligned}
&\mu\left\{s \in B\left(t, \frac{1}{n}\right): \|X_s - X_t\| \leqq (Q - \eta)\sigma(s, t)\right\} \\
&\quad \leqq \mu\left\{s \in B\left(t, \frac{1}{n}\right): \|X^m(s) - X^m(t)\| - \|X_m(s) - X_m(t)\| \leqq (Q - \eta)\sigma(s, t)\right\} \\
&\quad \leqq \mu\left\{s \in B\left(t, \frac{1}{n}\right): \|X^m(s) - X^m(t)\| \leqq Q\sigma(s, t)\right\} \\
&\quad \leqq \mu\left\{s \in B\left(t, \frac{1}{n}\right): \|X^m(s) - X^m(t)\| + \|X_m(s) - X_m(t)\| \leqq (Q + \eta)\sigma(s, t)\right\} \\
&\quad \leqq \mu\left\{s \in B\left(t, \frac{1}{n}\right): \|X_s - X_t\| \leqq (Q + \eta)\sigma(s, t)\right\}.
\end{aligned}$$

Consequently, $\bar{G}^0(Q - \eta, \omega) \leqq \bar{G}^m(Q, \omega) \leqq \bar{G}^0(Q + \eta, \omega)$ and $\underline{G}^0(Q - \eta, \omega) \leqq \underline{G}^m(Q, \omega) \leqq \underline{G}^0(Q + \eta, \omega)$. Since $\bar{G}^0(\cdot, \omega)$ and $\underline{G}^0(\cdot, \omega)$ are nondecreasing in Q, each has at most countably many discontinuities. It follows (upon letting $\eta \downarrow 0$) that for a.e. ω, $\bar{G}^0(Q, \omega) = \bar{G}^m(Q, \omega)$ and $\underline{G}^0(Q, \omega) = \underline{G}^m(Q, \omega)$ for all m, for all but at most countably many Q's. Now $\bar{G}^m$ and $\underline{G}^m$ being jointly measurable, we then find, for λ-a.e. $Q > 0$, that $\bar{G}^0(Q, \omega) = \lim_{m\to\infty} \bar{G}^m(Q, \omega)$, $\underline{G}^0(Q, \omega) = \lim_{m\to\infty} \underline{G}^m(Q, \omega)$ a.s.

Therefore, for such Q, $\bar{G}^0$ and $\underline{G}^0$ are measurable with respect to the tail σ-field for the sequence $\{\xi_j(\omega)\}$.

Finally,

$$\begin{aligned}
E\underline{G}^0(Q, \omega) &= E \liminf_{n\to\infty} K^0(Q, n, \omega) \leqq \liminf_{n\to\infty} EK^0(Q, n, \omega) \\
&= \liminf_{n\to\infty} [\mu\{B(t, 1/n)\}]^{-1} \int_{B(t,1/n)} P(\|X_s - X_t\| \\
&\leqq Q\sigma(s, t))\mu(ds) \leqq 1 - \delta_Q
\end{aligned}$$

for some $0 < \delta_Q < 1$, where the last inequality uses (A2). As a result, for λ-a.e. $Q > 0$, $G^0(Q, \omega) < 1$ a.s., from which it follows—by Fubini's theorem and the monotonicity of $\underline{G}^0(\cdot, \omega)$—that $\underline{G}^0(Q, \omega) < 1$ for all Q a.s. As for $\bar{G}^0$, $E\bar{G}^0 = E \limsup_{n\to\infty} K^0(Q, n, \omega) \geqq \limsup_{n\to\infty} EK^0(Q, n, \omega) \geqq \delta_Q > 0$, which leads to $\bar{G}^0(Q, \omega) > 0$ for all Q a.s. This completes the proof.

We intend to apply Theorem 1 to multidimensional Gaussian random fields $X(t) = \{X_i(t)\}_{i=1}^d$, $t \in \mathbb{R}^N$ (i.e., $M = \mathbb{R}^d$). Roughly speaking, we will show that (A1) and (A2) hold whenever $\sigma(s, t)/\|s - t\| \to \infty$ and the components $X_i(t, \omega)$ are independent and each covariance has a suitable "spectral representation." The following lemma extends Theorem 1 of Klein [5]. We omit the proof, it being an easy modification of Klein's.

LEMMA 2. *Let $\{X_t, t \in \mathbb{R}^N\}$ be a real-valued, second-order process with mean* 0 *and let $R(t, s) = EX_tX_s$. Suppose that there exists a locally compact Hausdorff space L and a σ-finite, regular Borel measure π on L, finite on compacts, such that*: (a) *there exists a family of real functions on $L\{g(t, \cdot), t \in \mathbb{R}^N\}$ such that the finite linear* *combinations of the* $g(t, \cdot)$ *are dense in* $L^2(\pi)$ *and* (b) $(\partial/\partial s_k)g(s, \lambda)$ *exists and is jointly continuous for all* $(s, \lambda) \in \mathbb{R}^N \times L$ *and* $1 \leqq k \leqq n$, *and* $R(t, s) = \int_L g(t, \lambda)g(s, \lambda)\pi(d\lambda)$.

Then there exist real functions $\phi_j(t)$, $t \in \mathbb{R}^N$, *and orthogonal random variables* $\xi_j(\omega)$, $j = 1, 2, \cdots$, *such that*:

(i) $\phi_j(t) = E\xi_j X_t$ *for all* t;

(ii) $\limsup_{s\to t} |\phi_j(s) - \phi_j(t)|/||s - t|| < \infty$ *for all* t;

(iii) $\{\phi_j\}$ *forms a complete orthonormal system in the reproducing kernel Hilbert space* $H(R)$; *and*

(iv) $E|X_t - \sum_{j=1}^m \xi_j \phi_j(t)|^2 \to 0$ *as* $m \to \infty$ *for all* t.

If, in addition, $\{X_t\}$ *is Gaussian, then* $\xi_1, \xi_2, \cdots$ *may be taken independent and standard normal.*

As we shall see in the proof of Theorm 3, the conclusions of Lemma 2 hold for any real, mean 0, Gaussian random field $\{X_t, t \in \mathbb{R}^N\}$ with stationary increments.

THEOREM 3. *Let* $\{X_t, t \in \mathbb{R}^N\} = \{(X_1(t), X_2(t), \cdots, X_d(t)), t \in \mathbb{R}^N\}$ *be a mean* 0, *d-dimensional, Gaussian random field with independent components. Assume further that* $\sigma(s, t)/||s - t|| \to \infty$ *as* $s \to t$ *for all* t, *and that each component* $\{X_j(t), t \in \mathbb{R}^N\}$ *has stationary increments. Then for each* $t \in \mathbb{R}^N$, (2) *holds with probability* 1.

PROOF. First, we show that (i)—(iv) of Lemma 2 hold for $\{X_j(t) - X_j(0), t \in \mathbb{R}^N\}$ for each $j = 1, 2, \cdots, d$. For simplicity, let $j = 1$.

It is well known (see e.g., [1], Theorem 3.1) that there exists a unique measure π on $\mathbb{R}^N \backslash \{0\}$

$$E(X_1(t) - X_1(0))(X_1(s) - X_1(0)) = \int_{\mathbb{R}^N \backslash \{0\}} (e^{it\cdot\lambda} - 1)(e^{is\cdot\lambda} - 1)\pi(d\lambda) \qquad \text{for all } s, t$$

where π is a Lévy measure, i.e., with $D = \{\lambda \in \mathbb{R}^N : ||\lambda|| \geqq 1\}$ and $C = \{\lambda \in \mathbb{R}^N : 0 < ||\lambda|| \leqq 1\}$,

$$\pi(D) < \infty \qquad \text{and} \qquad \int_C ||\lambda||^2 \pi(d\lambda) < \infty .$$

To apply Lemma 2, we choose $L = \mathbb{R}^N \backslash \{0\}$; now π is finite on compacts since these are bounded away from ∞ *and* 0. We need to show that the functions of the form $\sum_{j=1}^m c_j(e^{it_j\cdot\lambda} - 1)$ are dense in $L^2(\pi)$. To this end, let $f(\lambda) \in L^2(\pi)$ be orthogonal to each of the functions $\lambda \to (e^{it\cdot\lambda} - 1)$, $t \in \mathbb{R}^N$, or what is the same,

$$\int_L f^+(\lambda)(e^{it\cdot\lambda} - 1)\pi(d\lambda) = \int_L f^-(\lambda)(e^{it\cdot\lambda} - 1)\pi(d\lambda) \qquad \text{for all } t \tag{8}$$

where f^+, f^- are the positive and negative parts of f. Since f^+ and f^- are in $L^2(\pi)$:

$$\int_C ||\lambda||^2 f^\pm(\lambda)\pi(d\lambda) \leqq \int_C ||\lambda|| f^\pm(\lambda)\pi(d\lambda) \leqq \left(\int_C ||\lambda||^2 \pi(d\lambda)\right)^{\frac{1}{2}} \left(\int_C (f^\pm(\lambda))^2 \pi(d\lambda)\right)^{\frac{1}{2}} < \infty$$

and

$$\int_D f^\pm(\lambda)\pi(d\lambda) \leqq \left(\int_D (f^\pm(\lambda))^2 \pi(d\lambda)\right)^{\frac{1}{2}} (\pi(D))^{\frac{1}{2}} < \infty .$$

Hence $f^+(\lambda)\pi(d\lambda)$ and $f^-(\lambda)\pi(d\lambda)$ are Lévy measures. By (8) and the uniqueness of Lévy measures, $f^+ = f^-$ π-a.e.

By Lemma 2, for each $j = 1, 2, \cdots, d$ there are independent standard normal rv's $\{\xi_{jk}\}_{k=1}^\infty$ and real functions $\{\tilde{\phi}_{jk}\}_{k=1}^\infty$ on $\mathbb{R}^N$ such that, for each t, $\sum_{k=1}^n \xi_{jk}(\omega)\tilde{\phi}_{jk}(t)$ converges to $X_j(t, \omega) - X_j(0, \omega)$ in mean square, and hence pointwise. Since the components $\{X_j\}_{j=1}^d$ are independent, we can and do assume *all* the ξ_{jk}'s independent. To see that (A1) holds, define $\phi_{jk}: \mathbb{R}^N \to \mathbb{R}^d$ by $\phi_{jk}(t) = (0, \cdots, 0, \tilde{\phi}_{jk}(t), 0, \cdots, 0)$ where $\tilde{\phi}_{jk}(t)$ occupies the jth coordinate. Then for each t,

$$X_t(\omega) - X_0(\omega) = \sum_{j=1}^d \sum_{k=1}^\infty \xi_{jk}(\omega)\phi_{jk}(t) \quad \text{a.s.}$$

so that (A1) holds for $\{X_t - X_0, t \in \mathbb{R}^N\}$, which is clearly enough for Theorem 1. (A1)(b) follows from conclusion (ii) of Lemma 2 together with the assumption that $\sigma(s, t)/||s - t|| \to \infty$ as $s \to t$.

By Theorem 1, to finish the proof, we need only check that (A2) holds. Let $\sigma_j^2(s, t) = E(X_j(t) - X_j(s))^2$, $1 \leqq j \leqq d$, and let $\psi_j(s) = \sigma_j^2(s, t)/\sigma^2(s, t)$ for $||s - t||$ small; here t is fixed. Then

$$\frac{||X_s - X_t||^2}{\sigma^2(s, t)} = \sum_{j=1}^d Y_j^2(\omega)\psi_j(s)$$

where $Y_1, Y_2, \cdots, Y_d$ are independent and standard normal, and of course $\sum \psi_j(s) = 1$. Hence, for any $Q > 0$,

$$P\{||X_s - X_t|| \leqq Q\sigma(s, t)\} = P\{\sum_{j=1}^d Y_j^2\psi_j(s) \leqq Q^2\}\,.$$

But,

$$0 < P\{\max_{1\leqq j\leqq d} |Y_j| \leqq Q\} \leqq P\{\sum_{j=1}^d Y_j^2\psi_j(s) \leqq Q^2\}$$
$$\leqq P\{\min_{1\leqq j\leqq d} |Y_j| \leqq Q\} < 1\,,$$

which gives (A2).

REMARKS. (a) The zero-one law for subgroups implies that, for Gaussian processes, the event $\text{ap}\limsup_{s\to t} ||X_s - X_t||/\sigma(s, t) < \infty$ has probability 0 or 1.

(b) Theorem 2 remains valid if the components $\{X_j(t), t \in \mathbb{R}^N\}$ have "mth order stationary increments" (see [7]).

(c) If, in Theorem 3, the components are identically distributed, but not necessarily independent, one still obtains $\text{ap}\limsup_{s\to t} ||X_s - X_t||/\sigma(s, t) = \infty$ a.s. for all t, as follows:

$$\mu\{s \in B(t, \varepsilon): \sum_{j=1}^d |X_j(s) - X_j(t)|^2 \leqq Q^2 d\sigma_1^2(s, t)\}$$
$$\leqq \mu\{s \in B(t, \varepsilon): |X_1(s) - X_1(t)| \leqq Q(d)^{\frac{1}{2}}\sigma_1(s, t)\}\,;$$

hence, since $\sigma_1^2(s, t)/||s - t|| \to \infty$,

$$\text{ap}\limsup_{s\to t} \frac{||X_s - X_t||}{\sigma(s, t)} \geqq \text{ap}\limsup_{s\to t} \frac{|X_1(s) - X_1(t)|}{\sigma_1(s, t)} = \infty \quad \text{a.s.}$$

(d) Let $X_t = (X_1(t), \cdots, X_d(t))$, $t \in \mathbb{R}^N$, where $X_1, \cdots, X_d$ are independent, symmetric stable processes of index $\alpha_1, \cdots, \alpha_d$, each with independent increments.

Define $\phi_j(s, t) \geqq 0$ by $E(e^{iu[X(s)-X(t)]}) = \exp\{-\phi_j^{\alpha_j}(s, t)|u|^{\alpha_j}\}$ and let $\phi(s, t) = \sum_{j=1}^d \phi_j(s, t)$. Then ap lim $\inf_{s\to t} \|X_s - X_t\|/\phi(s, t) = 0$ and ap lim $\sup_{s\to t} \|X_s - X_t\|/\phi(s, t) = \infty$. To see this note that the zero-one law still applies by the independence of the increments and (A2) holds by the choice of ϕ.

(e) We will show that for one-dimensional Brownian motion $\{W_t : 0 \leqq t \leqq 1\}$ the limit in (7) does not exist for $t = 0$ (and hence for any $t \geqq 0$). $A =_D B$ will mean that A and B have the same distribution, and m will denote Lebesgue measure.

Fix $Q > 0$ and let

$$\begin{aligned} Z_\varepsilon^Q &= \frac{1}{\varepsilon} m\{0 \leqq s \leqq \varepsilon : |W(s)| \leqq Qs^{\frac{1}{2}}\} \\ &= \frac{1}{\varepsilon} \int_0^\varepsilon 1_{[0,Q]}(|W(s)|/s^{\frac{1}{2}})\, ds = \int_0^1 1_{[0,Q]}(|W(\varepsilon s)|/(\varepsilon s)^{\frac{1}{2}})\, ds \\ &=_D \int_0^1 1_{[0,Q]}(|W(s)|/s^{\frac{1}{2}})\, ds\,, \qquad \text{since} \quad \{W(\varepsilon s)/\varepsilon^{\frac{1}{2}} : 0 \leqq s \leqq 1\} \\ &=_D \{W(s) : 0 \leqq s \leqq 1\}\,. \end{aligned}$$

Hence the rv's Z_ε^Q, $0 < \varepsilon \leqq 1$, are identically distributed. If Z_ε^Q converges a.s. (as $\varepsilon \downarrow 0$), say to Z^Q, we have $Z^Q =_D Z_\varepsilon^Q = m\{0 \leqq s \leqq 1 : |W(s)| \leqq Qs^{\frac{1}{2}}\}$. Also, by the independence of the increments of $\{W_s\}$, the zero-one law implies that Z^Q is constant a.s., and hence a.s.:

$$\begin{aligned} Z^Q = EZ^Q &= \int_0^1 P(|W(s)| \leqq Qs^{\frac{1}{2}})\, ds \\ &= 2\Phi(Q) - 1\,, \end{aligned}$$

where $\Phi(Q)$ is the standard normal distribution function. As a result, then, $Z_\varepsilon^Q = 2\Phi(Q) - 1$ a.s. for all ε, and consequently

$$\int_0^\varepsilon 1_{[0,Q]}(W(s)/s^{\frac{1}{2}})\, ds = \varepsilon(2\Phi(Q) - 1) \qquad \text{for all} \quad \varepsilon\,, \quad \text{a.s.}$$

Differentiating both sides above with respect to ε, we arrive at a contradiction. Hence, for each $Q > 0$,

$$P\{\lim_{\varepsilon\downarrow 0} Z_\varepsilon^Q \ \text{ exists}\} = 0\,.$$

REFERENCES

[1] DUDLEY, R. M. (1973). Sample functions of the Gaussian process. *Ann. Probability* **1** 66–103.

[2] FEDERER, H. (1969). *Geometric Measure Theory*. Springer-Verlag, Berlin.

[3] GEMAN, D. (1977). On the approximate local growth of multidimensional random fields. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **38** 237–251.

[4] GEMAN, D. and HOROWITZ, J. (1976). Local times for real and random functions. *Duke Math. J.* **43** 809–828.

[5] KLEIN, R. (1976). A representation theorem on stationary Gaussian processes and some local properties. *Ann. Probability* **4** 844–849.

[6] KÔNO, N. (1975). Sur la minoration asymptotique et la caractère transitoire des trajectoires des functions aléatoires gaussiennes à valeurs dans $\mathbb{R}^d$. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **33** 95–112.

[7] SAKS, S. (1964). *Theory of the Integral.* Dover, New York.

[8] YAGLOM, A. M. (1962). *An Introduction to the Theory of Stationary Random Functions.* Prentice-Hall, Englewood Cliffs, N.J.

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