The Annals of Probability
1978, Vol. 6, No. 1, 72–84

LOCAL NONDETERMINISM AND THE ZEROS OF GAUSSIAN PROCESSES[1]

BY JACK CUZICK

Columbia University

The concept of local nondeterminism introduced by Berman is generalized and applied to divided difference sequences generated by a Gaussian process. The resulting estimates are then used to find simple sufficient conditions for the finiteness of the moments of the number of crossings of level zero. In particular it is shown that under mild regularity conditions very little more is required to make all moments finite when the variance is finite. The results are extended to curves $\xi \in \mathscr{C}_2[0, T]$. Finally examples are given in which the variance is finite but the third moment is infinite.

1. Introduction. The concept of local nondeterminism was introduced by Berman (1973) and shown by him to be closely related to the existence of a continuous local time for Gaussian processes. In studying the moments of $N(0, T)$, the number of zero crossings in time interval $[0, T]$ for a Gaussian process, we require a form of local nondeterminism, not for the process, but for its associated divided difference sequences. We begin by defining ϕ-regularity for stationary Gaussian processes and show that this condition implies the generalized notion of local ϕ-nondeterminism for divided difference sequences of the appropriate order. This condition is then used to find simple sufficient conditions which guarantee the finiteness of all the moments of $N(0, T)$. In one important special case we establish the convergence of the (factorial) moment generating function of $N(0, T)$. This extends previous work in Cuzick (1975), to which we refer the reader for background on the problem.

To simplify exposition we shall take the separable Gaussian process $X(t)$ to be stationary with mean zero. Analogous results hold for processes with stationary increments if we use their spectral function (Berman (1973), Section 4). In Section 2 we develop material concerning local nondeterminism and divided differences. This material is applied in Section 3 to obtain simple sufficient conditions for finite zero crossing moments. Section 4 extends the results of Section 3 to crossings of general curves; the final section considers the more difficult question of necessary conditions for the existence of higher moments.

2. Local nondeterminism. Let $X(t)$ have spectral distribution function $F(\lambda)$ and nth spectral moment $\lambda_n = \int_{-\infty}^{\infty} |\lambda|^n \, dF(\lambda)$. We need the following notations and definitions:

Received March 22, 1976.

[1] This work supported in part by NSF grant MCS76-09179.

AMS 1970 *subject classifications.* Primary 60G17; Secondary 60G15, 60G25.

Key words and phrases. Zero crossings, curve crossings, local nondeterminism, Gaussian processes, point processes, prediction.

PROPERTY (R1). A collection of measures $\{\mu_\alpha\}_{\alpha \in R}$ satisfies Property (R1) if there exists a bounded interval $[A, B]$ such that for any $x_1, \cdots, x_n \in [A, B]$ there is a $\delta > 0$ with

$$\liminf_{\alpha \to \infty} \int_C d\mu_\alpha > 0 \tag{1}$$

where $C = [A, B] \backslash \bigcup_{i=1}^n S_\delta(x_i)$ and $S_\delta(x)$ is an open sphere of radius δ centered at x.

DEFINITION. Let ϕ be any function which is regularly varying at the origin with index $0 \leqq \beta \leqq 2$ and $\phi(0) = 0$. If $\beta = 2$, we require also that

$$\liminf_{0 < s \leqq 1; t \to 0} \phi(st)/s^2\phi(t) > 0 .$$

We say that a stationary Gaussian process $X(t)$ is *ϕ-regular* if all spectral moments are finite or $\lambda_{2n} < \infty$, $\lambda_{2n+2} = \infty$ and the collection of measures

$$dG_{t^{-1}}(\lambda) = \frac{dF(\lambda/t)}{t^{2n}\phi(t)}$$

satisfies Property (R1). If $\phi(t) = \sigma^2(t) = E(X^{(n)}(t) - X^{(n)}(0))^2$ satisfies the above conditions we say that X is *regular*.

Note that the extra condition when $\beta = 2$ is satisfied if $\phi(t) = t^2|\ln t|^\alpha$, $\alpha \geqq 0$ or $\phi(t) = \rho(0) - \rho(t)$ with $\rho(t)$ is nonnegative definite and increasing on $[0, \varepsilon]$, $\varepsilon > 0$.

We recall the notion of divided differences and of extended divided differences:

DEFINITION. The nth divided difference of a function $X(t)$ at the distinct points $t_1 < \cdots < t_{n+1}$, denoted $X[t_{n+1}, \cdots, t_1]$, is defined iteratively by

$$X[t_{n+1}, \cdots, t_1] = \frac{X[t_{n+1}, \cdots, t_2] - X[t_n, \cdots, t_1]}{t_{n+1} - t_1}$$

with

$$X[t_2, t_1] = \frac{X(t_2) - X(t_1)}{t_2 - t_1} .$$

When $X'(t)$ exists we define $X[t_1, t_1] = X'(t_1)$.

We further define the nth extended divided difference by

$$X_\phi[t_{n+1}, \cdots, t_1] = \frac{X[t_{n+1}, \cdots, t_2] - X[t_n, \cdots, t_1]}{\phi(t_{n+1} - t_1)}$$

for functions $\phi(t)$ which are positive on $(0, t_{n+1} - t_1)$ with $\phi(0) = 0$.

DEFINITION. The process $X(t)$ is *locally ϕ-nondeterministic* if for any k and any $\mathbf{t} = (t_1, \cdots, t_k)$ with $t_1 < t_2 < \cdots < t_k$

$$\liminf_{\mathbf{t} \downarrow 0} \det \operatorname{Cov}(X(t_1), X_{\phi^{1/2}}[t_1, t_2], \cdots, X_{\phi^{1/2}}[t_{k-1}, t_k]) > 0 . \tag{2}$$

We say that the nth divided differences of $X(t)$ are locally ϕ-nondeterministic if

$$\begin{aligned} &\liminf_{\mathbf{t} \downarrow 0} \det \operatorname{Cov}(X(t_1), X[t_1, t_2], \cdots, X[t_1, \cdots, t_{n+1}], \\ &\qquad X_{\phi^{1/2}}[t_1, \cdots, t_{n+2}], \cdots, X_{\phi^{1/2}}[t_{k-n-1}, \cdots, t_k]) > 0 \\ &\qquad\qquad n = 0, 1, \cdots; k \geqq n+2 . \end{aligned} \tag{3}$$

Again if $\phi(t) = E(X^{(n)}(t) - X^{(n)}(0))^2$ we suppress explicit mention of ϕ. When $n = 0$ and $\phi(t) = \sigma^2(t)$ our definition is equivalent to Berman's by his Lemma 2.2.

REMARK. If $\liminf_{t \downarrow 0} \sigma^2(t)/\phi(t) = 0$, then $X(t)$ cannot be ϕ-regular or locally ϕ-nondeterministic.

In studying local nondeterminism, we are rescaling a collection of random variables so that the limiting variables do not become linearly dependent along any subsequence of the $\mathbf{t}$ as $|t_k - t_1| \to 0$. This rescaling is reflected by a rescaling of the spectral measure. This is made clear in the following basic result.

THEOREM 1. *If X is ϕ-regular and $\lambda_{2n} < \infty$, $\lambda_{2n+2} = \infty$, then the* nth *divided differences of X are locally ϕ-nondeterministic.*

Before proving Theorem 1 we need the following two lemmas.

LEMMA 1. *Let f_n be entire functions such that $f_n \to f$ uniformly on compact sets. If there exists a family of measures μ_n on the real line with Property* (R1), *then $\int |f_n|^2 \, d\mu_n \to 0$ implies $f \equiv 0$.*

PROOF. The uniform convergence implies that f is entire so that if f is not identically zero, it can have only a finite number of zeros, $x_1, \cdots, x_m$, on any bounded interval $[A, B]$. Choose $[A, B]$ and $\delta > 0$ so that $C = [A, B] \backslash \bigcup_{i=1}^m S_\delta(x_i)$ satisfies (1). Then since $\inf_{x \in C} |f(x)|^2 = \varepsilon > 0$ and $f_n \to f$ uniformly on C, we have that

$$\liminf_{n\to\infty} \int |f_n|^2 \, d\mu_n \geqq \frac{\varepsilon}{2} \liminf_{n\to\infty} \int_C d\mu_n > 0 .$$

This contradicts our assumptions and thus f must vanish everywhere.

LEMMA 2. *If $\phi(t)$ is regularly varying with index β, then for all δ, $\varepsilon > 0$, there exists a $t^* > 0$ such that*

$$(1 - \varepsilon)s^{\beta+\delta} \leqq \frac{\phi(st)}{\phi(t)} \leqq (1 + \varepsilon)s^{\beta-\delta}$$

for all $0 < s \leqq 1$ and $0 < t \leqq t^*$.

PROOF. The regularly varying function $\phi(t)$ has a representation

$$\phi(t) = c(t) \exp \int_t^1 -\frac{b(u)}{u} \, du$$

where $c(t) \to c > 0$ and $b(t) \to \beta$ as $t \to 0$. Choose $t^* \leqq 1$ so small that $\beta - \delta \leqq b(u) \leqq \beta + \delta$ for $0 \leqq u \leqq t^*$ and $1 - \varepsilon \leqq c(st)/c(t) \leqq 1 + \varepsilon$ for $0 < s \leqq 1$ and $0 < t \leqq t^*$. Then

$$\frac{\phi(st)}{\phi(t)} = \frac{c(st)}{c(t)} \exp \int_{st}^t -\frac{b(u)}{u} \, du \leqq (1 + \varepsilon)s^{\beta-\delta} .$$

The lower bound follows analogously.

PROOF OF THEOREM 1. The method of proof is to show that there does not exist a subsequence $\mathbf{t}^{(l)}$ such that the limit in (2) or (3) is zero. We shall assume that $\lambda_2 < \infty$, $\lambda_4 = \infty$ and give a proof for first divided differences. The general case is similar.

Choose $\varepsilon > 0$ so that $\sigma^2(t)$ is positive in $(0, \varepsilon)$. Let $t_1 < t_2 < \cdots < t_{k+2}$ and define

$$Y_0 = X[t_1, t_2]$$
$$Y_j = X_{\phi^{\frac{1}{2}}}[t_j, t_{j+1}, t_{j+2}] \qquad j = 1, \cdots, k\,.$$

We must show that

$$\liminf_{t \downarrow 0} \det \operatorname{Cov}(X(t_1), Y_0, \cdots, Y_k) \geqq C_k > 0\,.$$

Using the integral representation for divided differences (Isaacson and Keller (1966), page 250) we have

$$(4) \qquad Y_0 = \int_0^1 X'(t_1 + s\Delta_1)\,ds$$
$$Y_j = \int_0^1 \frac{X'(t_j + s\Delta_j) - X'(t_{j+1} + s\Delta_{j+1})}{\phi^{\frac{1}{2}}(\Delta_j + \Delta_{j+1})}\,ds \qquad j = 1, \cdots, k$$

where $\Delta_j = t_{j+1} - t_j$. According to Berman ((1973), page 82) it is enough to show that if

$$(5) \qquad \lim_{t^{(l)} \downarrow 0} \operatorname{Var}\,[aX(t_1) + \textstyle\sum_{j=0}^k b_j Y_j] = 0$$

for some subsequence $\mathbf{t}^{(l)}$ and fixed a and b_j, then $a = b_j = 0$ for $j = 0, \cdots, k$. Since the process is stationary we may take $t_1 = 0$. Using (4) the variance in (5) can be rewritten in spectral form as

$$(6) \qquad \int_{-\infty}^{\infty} |g_h(\lambda)|^2 \frac{dF(\lambda/h)}{h^2\phi(h)}$$

where

$$(7) \qquad \begin{aligned} g_h(\lambda) = {} & ah\phi^{\frac{1}{2}}(h) + b_0\lambda \int_0^1 e^{i(s\Delta_1/h)\lambda}\phi^{\frac{1}{2}}(h)\,ds \\ & + \int_0^1 \textstyle\sum_{j=1}^k b_j\lambda\,[\exp[i((t_j + s\Delta_j)/h)\lambda] - \exp[i((t_{j+1} + s\Delta_{j+1})/h)\lambda]] \\ & \times \left[\frac{\phi^{\frac{1}{2}}(h)}{\phi^{\frac{1}{2}}(\Delta_j + \Delta_{j+1})}\right] ds \end{aligned}$$

and $h = \max_{1 \leqq j \leqq k+1} \Delta_j$ so that $\Delta_j/h \leqq 1$. We now distinguish two cases depending on the index β of the regularly varying function ϕ.

Case I. $\beta < 2$.

Extract a subsequence so that $\Delta_j/h \to \delta_j$ and $t_j/h \to \tau_j$ with $0 \leqq \delta_j \leqq 1$ for all j. If $\delta_j + \delta_{j+1} = 0$, then the term corresponding to Y_j, $j \geqq 1$ drops out since by Lemma 2

$$\lim_{(\Delta_j + \Delta_{j+1})/h \downarrow 0} \frac{|\exp[i((s\Delta_j + (1-s)\Delta_{j+1})/h)\lambda] - 1|}{\phi^{\frac{1}{2}}(\Delta_j + \Delta_{j+1})}\,\phi^{\frac{1}{2}}(h) = 0$$

uniformly on compact sets. The first two terms in (7) also go to zero uniformly on compact sets.

We now apply Lemma 1 to (6) and find that

$$\begin{aligned} g(\lambda) &= \textstyle\sum_{j \in S} b_j\lambda \int_0^1 \{\exp[i(\tau_j + s\delta_j)\lambda] - \exp[i(\tau_{j+1} + s\delta_{j+1})\lambda]\}\{\delta_j + \delta_{j+1}\}^{-\beta/2}\,ds \\ &\equiv 0 \end{aligned}$$

where $S = \{1 \leqq j \leqq k : \delta_j + \delta_{j+1} > 0\}$. Note that S is never empty. Thus the Fourier transform $\hat{g}$ of $g(\lambda)/i\lambda^2$ is identically zero and

$$\hat{g}(x) = \sum_{j \in S} b_j f_j(x) \tag{8}$$

where

$$f_j(x) = (\delta_j + \delta_{j+1})^{-\beta/2} \int_0^1 I_{[\tau_j + s\delta_j, \tau_{j+1} + s\delta_{j+1}]}(x)\, ds .$$

Note that $\operatorname{supp} f_j = [\tau_j, \tau_{j+2}]$. Let $m = \sup_{i \in S} \{i\}$. If $\delta_{m+1} > 0$, then f_{m+1} is the only term in (8) which is positive on $[\tau_{m+1}, \tau_{m+2}]$, and it is strictly positive on that open interval. Thus $b_{m+1} = 0$. If $\delta_{m+1} = 0$, then $\delta_m > 0$ and $f_m(\tau_{m+1}) = 0$ but $f_{m+1}(\tau_{m+1}) > 0$, so that again $b_{m+1} = 0$. Proceeding inductively we have $b_j = 0$ for all $j \in S$. Now let

$$h' = \max_{j \notin S} \Delta_j$$

and in the same manner find at least one more $b_j = 0$. Continue this procedure until we have all $b_j = 0$, $j \geqq 1$. Finally we obtain $\operatorname{Var}(aX(t_1) + b_0 X'(t_1)) = 0$ so that $a = b_0 = 0$.

CASE II. When $\beta = 2$ we must be more careful about the degenerate intervals. We have that

$$\int_0^1 \{\exp[i((t_j + s\Delta_j)/h)\lambda] - \exp[i((t_{j+1} + s\Delta_{j+1})/h)\lambda]\} \frac{\phi^{\frac{1}{2}}(h)}{\phi^{\frac{1}{2}}(\Delta_j + \Delta_{j+1})}\, ds \tag{9}$$

$$= \int_0^1 \left[\frac{\exp[i((t_j + s\Delta_j)/h)\lambda] - \exp[i((t_{j+1} + s\Delta_{j+1})/h)\lambda]}{((s\Delta_j + (1-s)\Delta_{j+1})/h)} \right] \times \left[\frac{(s\Delta_j + (1-s)\Delta_{j+1})\phi^{\frac{1}{2}}(h)}{h\phi^{\frac{1}{2}}(\Delta_j + \Delta_{j+1})} \right] ds . \tag{10}$$

By our extra assumption in the definition of ϕ-regularity when $\beta = 2$, the second term in (10) is bounded so that we can extract a subsequence such that

$$\int_0^1 \frac{(s\Delta_j + (1-s)\Delta_{j+1})\phi^{\frac{1}{2}}(h)}{h\phi^{\frac{1}{2}}(\Delta_j + \Delta_{j+1})}\, ds \to K$$

for some $0 \leqq K < \infty$. Thus if $(\Delta_j + \Delta_{j+1})/h \to 0$, (9) will converge to $i\lambda K e^{i\tau_j \lambda}$ along this subsequence. Thus as before, the integrand (7) converges to the null function. In this case the Fourier transform of $g(\lambda)/i\lambda^2$ is a generalized function which consists of $\hat{g}(x)$ as before plus possibly weighted delta functions localized at some of the τ_j. This does not affect the fact that $b_j \equiv 0$ for $j \in S$ and the proof continues as before.

REMARK. It is known that X is locally nondeterministic when $\sigma^2(t)$ is locally concave (Berman 1973). It is not known if the concavity of $\sigma^2(t)$ implies local nondeterminism for divided differences. This point is overlooked by Mirošin (1973, 1974a, 1974b) and invalidates his proofs of results on crossing moments. This point has been discussed more fully in Cuzick (1975).

We now give some results which help to characterize ϕ-regularity.

THEOREM 2 (Berman 1973, Lemma 7.1). *If* $\sigma^2(t) = E(X^{(n)}(t) - X^{(n)}(0))^2$ *is regularly varying with index* $0 < \beta < 2$ *for some n then* $X(t)$ *is regular.*

THEOREM 3. *If*

$$\lim_{t\downarrow 0} \int_\varepsilon^\infty \frac{dF(\lambda/t)}{\sigma^2(t)} = 0 \qquad \text{for all} \quad \varepsilon > 0\,, \tag{11}$$

then X is not *locally nondeterministic. If* $\lambda_2 < \infty$, *then* (11) *holds.*

PROOF. For the first part it is enough to show that

$$\det \operatorname{Cov}(X_\sigma[0, t], X_\sigma[t, 2t]) \tag{12}$$

goes to zero as $t \to 0$. Equation (12) can be written in spectral form as

$$2\,\text{Real}\left\{\int_0^\infty (e^{i\lambda} - 1)^2(e^{-i\lambda} - 1)\frac{dF(\lambda/t)}{\sigma^2(t)}\right\}. \tag{13}$$

The integral on $[\varepsilon, \infty)$ goes to zero by assumption. On $[0, \varepsilon]$ we have that (13) is less than or equal

$$2\varepsilon \int_0^\varepsilon |e^{i\lambda} - 1|^2 \frac{dF(\lambda/t)}{\sigma^2(t)} \leqq 2\varepsilon \int_0^\infty |e^{i\lambda} - 1|^2 \frac{dF(\lambda/t)}{\sigma^2(t)} = \varepsilon\,.$$

For the second part we know that $\sigma^2(t) = \lambda_2 t^2(1 + o(1))$. Since

$$\int_\alpha^\infty dF(\lambda) \leqq \alpha^{-2} \int_\alpha^\infty \lambda^2\, dF(\lambda)$$

it follows that

$$\int_\varepsilon^\infty \frac{dF(\lambda/t)}{\sigma^2(t)} = \int_{\varepsilon t^{-1}}^\infty \frac{dF(\lambda)}{\sigma^2(t)} \leqq \frac{\int_{\varepsilon t^{-1}}^\infty \lambda^2\, dF(\lambda)}{\sigma^2(t)(\varepsilon t^{-1})^2}$$
$$\sim \lambda_2^{-1}\varepsilon^{-2} \int_{\varepsilon t^{-1}}^\infty \lambda^2\, dF(\lambda) \to 0 \qquad \text{as} \quad t \to 0\,.$$

REMARK. The Ornstein–Uhlenbeck process, which is Markov and has $\beta = 1$, is locally nondeterministic in the strongest possible way. As β moves farther away from unity, local nondeterminism becomes less pronounced. For example, if F has a density of the form $f(\lambda) = (1 + \lambda^3)^{-1}$, then $\lambda_2 = \infty$ but the process is not locally nondeterministic since $\sigma^2(t) \sim t^2|\ln t|$ and (11) holds. However, the process is ϕ-regular (and hence locally ϕ-nondeterministic) when $\phi(t) = t^2$. At the other extreme, if $f(\lambda) = (1 + \lambda|\ln \lambda|^\alpha)^{-1}$ with $\alpha > 1$, then $\sigma^2(t) \sim |\ln t|^{-\alpha+1}$ and X is not regular but is $\sigma^2(t)|\ln t|^{-1}$ regular. Also if $\sigma^2(t) = |\ln t|^{-\beta}$ in a neighborhood of zero then X is not regular but is $|\ln t|^{-(\beta+1)}$ regular. However, in these last two cases $\sigma^2(t)$ is concave near zero and thus X is locally nondeterministic.

3. Moments of zero crossings. We can now apply the results of Section 2 to study the moments of the number of zero crossings. Let the random variable $N(\xi, T)$ denote the number of crossings of the smooth curve ξ by the differentiable process X in time interval $[0, T]$, and let $M_k(\xi, T)$ denote the kth factorial moment of $N(\xi, T)$. We refer to zero crossings by $N(0, T)$ and $M_k(0, T)$. In Cuzick (1975) it was shown that a necessary and sufficient condition for $M_k(0, t) < \infty$

is that for some $\varepsilon > 0$

$$(14) \qquad \int_0^\varepsilon \cdots \int_0^\varepsilon \left[\frac{\prod_{i=1}^k \sigma_i^2}{\det R_k}\right]^{\frac{1}{2}} d\Delta_1 \cdots d\Delta_{k-1} < \infty$$

where $\sigma_i^2 = \mathrm{Var}\,(X'(t_i)\,|\,X(t_j), j = 1, \cdots, k)$ and $\det R_k = \det \mathrm{Cov}\,(X(t_1), \cdots, X(t_k))$. The following two lemmas provide our basic estimates.

LEMMA 3. *If X is ϕ-regular with $\lambda_2 < \infty$, $\lambda_4 = \infty$ and $\sigma^2(t) = E(X'(t) - X'(0))^2$ is increasing in a neighborhood of zero, then for $|t_k - t_1|$ near zero, there exists a positive constant C_k such that*

(i) $C_k[\prod_{i=1}^{k-1} \Delta_i^2][\prod_{i=2}^{k-1} \phi(\Delta_{i-1} + \Delta_i)] \leqq \det \mathrm{Cov}\,(X(t_1), \cdots, X(t_k))$

(ii)
$$\sigma_1^2 \leqq \sigma^2(\Delta_1)$$
$$\sigma_i^2 \leqq \min\,(\sigma^2(\Delta_{i-1}), \sigma^2(\Delta_i)) \leqq 4\,\frac{\sigma^2(\Delta_{i-1})\sigma^2(\Delta_i)}{\sigma^2(\Delta_{i-1} + \Delta_i)} \qquad i = 2, \cdots, k-1$$
$$\sigma_k^2 \leqq \sigma^2(\Delta_{k-1})\,.$$

PROOF. (i) Using the divided difference manipulations from Cuzick (1975) we have

$$\det \mathrm{Cov}\,(X(t_1), \cdots, X(t_k)) = [\prod_{i=1}^{k-1} \Delta_i^2][\prod_{i=2}^{k-1} \phi(\Delta_{i-1} + \Delta_i)] \det \mathrm{Cov}\,(X(t_1), X[t_1, t_2], X_{\phi^{\frac{1}{2}}}[t_1, t_2, t_3], \cdots, X_{\phi^{\frac{1}{2}}}[t_{k-2}, t_{k-1}, t_k])\,.$$

The lower bound follows by the assumption of ϕ-regularity on X.

(ii) The upper bounds follow by noting that for $i < k$

$$\begin{aligned}\sigma_i^2 &= \mathrm{Var}\,(X'(t_i)\,|\,X(t_j), j = 1, \cdots, k)\\ &= \sigma^2(\Delta_i)\,\mathrm{Var}\,(X_\sigma[t_i, t_i, t_{i+1}]\,|\,X(t_j), j = 1, \cdots, k)\\ &\leqq \sigma^2(\Delta_i)\,\mathrm{Var}\,(X_\sigma[t_i, t_i, t_{i+1}]) \leqq \sigma^2(\Delta_i)\,.\end{aligned}$$

The last step follows from the fact that

$$\begin{aligned}\mathrm{Var}\,(X_\sigma[t_i, t_i, t_{i+1}]) &= \mathrm{Var}\left(\int_0^1 \frac{X'(t_i) - X'(t_i + s\Delta_i)}{\sigma(\Delta_i)}\,ds\right)\\ &\leqq \int_0^1 \mathrm{Var}\left(\frac{X'(t_i) - X'(t_i + s\Delta_i)}{\sigma(\Delta_i)}\right) ds \leqq 1\end{aligned}$$

since σ is increasing near zero. By considering the conditional variance of $X_\sigma[t_{i-1}, t_i, t_i]$ when $i > 1$, the other half of the inequality is obtained. The remaining inequality follows since $\sigma^2(t)$ is increasing near zero and $\sigma^2(t)/\sigma^2(2t) \geqq \frac{1}{4}$ for all t (Lukacs (1970), page 69).

The next lemma sharpens Lemma 4.2 of Cuzick (1975).

LEMMA 4. *If $\phi(st)/s^\beta\phi(t) \leqq K$ for $0 < s \leqq 1$ and $t \in (0, \varepsilon]$ for some $\beta > 0$ and $0 < \varepsilon < 1$, then*

$$(15) \qquad I_k(\phi) = \int_0^\varepsilon \cdots \int_0^\varepsilon \prod_{i=1}^k \left(\frac{dx_i}{x_i}\right) \frac{\prod_{i=1}^k \phi(x_i)}{\prod_{i=2}^k \phi(x_{i-1} + x_i)} \leqq \frac{\phi(\varepsilon)}{\varepsilon^\beta}\left(\frac{16K}{\beta}\right)^k\,.$$

PROOF. As in Cuzick (1975), we can assume $\phi(x) = x^\beta$ and $\varepsilon = 1$. In this case, from the proof of Theorem 3 in Mirošin (1973) we see that

$$I_k(\phi) \leqq \left(\frac{2}{\beta}\right)^k [\prod_{i=1}^{k-1} (\sum_{j=1}^{i} \gamma_j)(1 - \sum_{j=1}^{i} \gamma_j)]^{-1} \int_0^1 x^{-\gamma}\, dx$$

for any $\gamma_j > 0$, $j = 1, \cdots, k-1$, such that $\gamma = \sum_{j=1}^{k-1} \gamma_j < 1$. The result follows by taking $\gamma_1 > \frac{1}{4}$ and $\gamma < \frac{1}{2}$.

We can now state the major results for crossing moments.

THEOREM 4. *Assume that X is ϕ-regular with $\lambda_2 < \infty$, $\lambda_4 = \infty$ and $\sigma^2(t)$ is increasing at zero. Then a sufficient condition for $M_k(0, T) < \infty$ is that*

$$\int_0^\varepsilon \cdots \int_0^\varepsilon \prod_{i=1}^{k-1} \left(\frac{d\Delta_i}{\Delta_i}\right) \frac{\prod_{i=1}^{k-1} \sigma^2(\Delta_i)}{\prod_{i=2}^{k-1} [\phi(\Delta_{i-1} + \Delta_i)\sigma^2(\Delta_{i-1} + \Delta_i)]^{\frac{1}{2}}} \tag{16}$$

be finite.

PROOF. Immediate from (14) and Lemma 3.

THEOREM 5. *Under the assumptions of Theorem 4 and the additional assumptions that ϕ is regularly varying with index $0 < \beta \leqq 2$ and*

$$\limsup_{t \downarrow 0} \frac{\sigma^2(t)}{\phi(t)L(t)} < \infty \tag{17}$$

for some positive slowly varying function $L(t)$, then all the $M_k(0, T)$ are finite. If $X(t)$ is regular and σ^2 has positive index, then all crossing moments are finite.

PROOF. Using the remark before Theorem 1 and (17), there exist positive constants K_1 and K_2 such that

$$K_1\phi(t) \leqq \sigma^2(t) \leqq K_2\phi(t)L(t)\,.$$

These inequalities allow us to eliminate $\sigma^2(t)$ from (16) and then it is enough to establish the finiteness of

$$\int_0^\varepsilon \cdots \int_0^\varepsilon \prod_{i=1}^{k-1} \left(\frac{d\Delta_i}{\Delta_i}\right) \left[\frac{\prod_{i=1}^{k-1} \phi(\Delta_i)}{\prod_{i=1}^{k-2} \phi(\Delta_{i-1} + \Delta_i)}\right]^\delta \left[\frac{\prod_{i=1}^{k-1} \phi(\Delta_i)}{\prod_{i=1}^{k-2} \phi(\Delta_{i-1} + \Delta_i)}\right]^{1-\delta} \max_i L(\Delta_i)^{k-1}$$

for some $0 < \delta \leqq 1$. Using Lemma 2, for ε small enough, the product of the last two terms is less than

$$\max_i \phi(\Delta_i)^{1-\delta} L(\Delta_i)^{k-1}\,,$$

which is bounded for any $0 < \delta < 1$. The finiteness of the remaining integral follows from Lemmas 3 and 4.

When $X(t)$ is regular and σ^2 has positive index, we can drop the requirement that σ^2 be increasing. In this case Lemma 2 implies that $\sigma^2(st)/\sigma^2(t) < 1 + \varepsilon$ for $0 \leqq s \leqq 1$ and t small enough, so that the inequalities in Lemma 3(ii) still hold if we multiply the right-hand sides by $(1 + \varepsilon)^2$.

The case in which $\beta = 0$ is more difficult and our results are less complete.

THEOREM 6. *If X has a spectral density given by $f(\lambda) = (1 + \lambda^3|\ln \lambda|^\alpha)^{-1}$, then for $k \geqq 2$ and $\alpha > \frac{1}{2}(3k - 2)$, we have that $M_k(0, T) < \infty$.*

PROOF. The result is known when $k = 2$. We assume $k \geqq 3$. Calculation shows that $\sigma^2(t) \sim |\ln t|^{-\alpha+1}$ and is monotone near zero. If we let $\phi(t) = |\ln t|^{-\alpha}$ then X is ϕ-regular and $\sigma^2(t)/\phi(t) \sim |\ln t|$. Using these estimates directly on (16) and splitting the integrand as in Theorem 5 it is enough to show that for some $0 < \delta \leqq 1$, the following expression is finite:

$$\int_0^\varepsilon \cdots \int_0^\varepsilon \prod_{i=1}^{k-1} \left(\frac{d\Delta_i}{\Delta_i}\right) \left[\frac{\prod_{i=1}^{k-1} \phi(\Delta_i)}{\prod_{i=2}^{k-1} \phi(\Delta_{i-1} + \Delta_i)}\right]^\delta \max_i \left[\phi(\Delta_i)^{1-\delta}|\ln \Delta_i|^{k/2}\right].$$

The last term is bounded if $\alpha > \frac{1}{2}k(1 - \delta)^{-1}$. For any such δ it is enough to show

$$\int_0^\varepsilon \cdots \int_0^\varepsilon \prod_{i=1}^{k-1} \left(\frac{d\Delta_i}{\Delta_i}\right) \left[\frac{\prod_{i=1}^{k-1} \phi(\Delta_i)}{\prod_{i=2}^{k-1} \phi(\Delta_{i-1} + \Delta_i)}\right]^\delta < \infty. \tag{18}$$

Now make the substitution $\Delta_i = \xi(x_i) = \exp(-x_i^{-(\alpha\delta)^{-1}})$. Since ξ is monotone, $\xi^{-1}(\xi(x) + \xi(y)) \geqq \frac{1}{2}(x + y)$ so that $(\phi(\Delta_{i-1} + \Delta_i))^\delta \geqq \frac{1}{2}(x_{i-1} + x_i)$. Neglecting constants we are led to the following majorant of (18):

$$\int_0^{\varepsilon^*} \cdots \int_0^{\varepsilon^*} \prod_{i=1}^{k-1} \left(\frac{dx_i}{x_i}\right) \left[\frac{\prod_{i=1}^{k-1} x_i^{1-1/\alpha\delta}}{\prod_{i=2}^{k-1} (x_{i-1} + x_i)}\right],$$

where $\varepsilon^* = \xi(\varepsilon)$. For any $0 < \delta^* \leqq 1$, this integral is less than

$$\int_0^{\varepsilon^*} \cdots \int_0^{\varepsilon^*} \prod_{i=1}^{k-1} \left(\frac{dx_i}{x_i}\right) \left[\frac{\prod_{i=1}^{k-1} x_i}{\prod_{i=2}^{k-1} (x_{i-1} + x_i)}\right]^{\delta^*} \max_i \left(x_i^{(1-\delta^*)-(k-1)/\alpha\delta}\right).$$

If $\alpha > (k - 1)/\delta$, then some $\delta^* > 0$ can be chosen so the last term remains bounded. When the last term is bounded the remaining integral is finite for any $\delta^* > 0$ by Lemma 4. Thus we require that $\alpha > \min_{0<\delta\leqq 1} \max((k - 1)/\delta, k/2(1 - \delta))$. This minimum occurs when $\delta = 2(k - 1)/(3k - 2)$ and thus we require that $\alpha > \frac{1}{2}(3k - 2)$.

When $\lambda_4 < \infty$ we can relax the condition that ϕ have positive index of variation. We obtain

THEOREM 7. *Suppose $\lambda_4 < \infty$, $\lambda_6 = \infty$ and X is ϕ-regular. Let $\sigma^2(t) = E(X''(t) - X''(0))^2$ be increasing near zero and assume (17) holds. Then $M_k(0, T) < \infty$ for all k.*

REMARK. The assumption $\lambda_6 = \infty$ is no restriction since it is known that when all spectral moments are finite, then all crossing moments are finite. In other cases the above conditions can be achieved by considering the appropriate derivative of X.

PROOF. As in Lemma 3, we use divided difference manipulations and iterate one further time to find that under the assumptions of the theorem

$$\det \mathrm{Cov}\,(X(t_1), \cdots, X(t_k)) \geqq C_k\left\{\prod_{i=1}^{k-1} \Delta_i^2\right\}\left\{\prod_{i=2}^{k-1} (\Delta_{i-1} + \Delta_i)^2\right\}\left\{\prod_{i=3}^{k-1} (\Delta_{i-2} + \Delta_{i-1} + \Delta_i)\right\}$$

and

$$\sigma_1^2 \leqq \Delta_1^2\sigma^2(\Delta_1 + \Delta_2) \leqq \Delta_1^2$$
$$\sigma_i^2 \leqq \min(\Delta_{i-1}^2, \Delta_i^2)\sigma^2(\Delta_{i-1} + \Delta_i) \qquad i = 2, \cdots, k-1$$
$$\sigma_k^2 \leqq \Delta_{k-1}^2\sigma^2(\Delta_{k-2} + \Delta_{k-1}) \leqq \Delta_{k-1}^2 .$$

Substituting these estimates in (14), it is enough to show that

$$\int_0^\varepsilon \cdots \int_0^\varepsilon \prod_{i=1}^{k-1}\left(\frac{d\Delta_i}{\Delta_i}\right)\left[\frac{\prod_{i=1}^{k-1}\Delta_i}{\prod_{i=2}^{k-1}(\Delta_{i-1}+\Delta_i)}\right]\left[\frac{\prod_{i=2}^{k-1}\phi(\Delta_{i-1}+\Delta_i)}{\prod_{i=3}^{k-1}\phi(\Delta_{i-2}+\Delta_{i-1}+\Delta_i)}\right]^{\frac{1}{2}}$$
$$\times\ [\max_i \Delta_i L(\Delta_i)^{\frac{1}{2}(k-2)}] < \infty .$$

The last two terms are bounded and the remaining integral is finite as before.

We conclude this section by considering the rate of growth of the factorial moments. Let $(X_1, \cdots, X_k)$ be multivariate normal with mean **0** and component variances σ_i^2. An application of the Hölder inequality shows that

$$E|\prod_{i=1}^k X_i| \leqq [\prod_{i=1}^k \sigma_i^2]^{\frac{1}{2}} E|Z|^k$$

where Z is a standard normal variable. From this and Cuzick (1975, (2.2)) it can be seen that $M_k(0, \varepsilon)$ is bounded by $E|Z|^k$ times the expression given at (14). For the special case in which $\sigma^2(t) = C|t| + o(t)$ it was shown in the above paper that the constant C_k in Lemma 3 satisfies $C_k \geqq 3^{-k}$. Finally using the estimate (15), we see that there exists a constant γ such that $M_k(0, \varepsilon) \leqq \gamma^k E|Z|^k$. Thus in this special case the factorial moments of $N(0, T)$ grow no faster than the absolute moments of a normal variable and thus the probability generating function of $N(0, T)$ converges in a neighborhood of unity.

The obstruction to a more general result in this direction is the lower bound for C_k. The same type of lower bound would be achieved if it could be shown that

$$\limsup_{t\downarrow 0} \operatorname{Var}(X_{\phi^{\frac{1}{2}}}[t_1, t_2, t_3] \mid X_{\phi^{\frac{1}{2}}}[t_2, t_3, t_4], \cdots, X_{\phi^{\frac{1}{2}}}[t_{k-2}, t_{k-1}, t_k]) \geqq C$$

with C independent of k. Some work in this direction can be found in Cuzick (1977).

4. Crossings of general curves. The preceding results are not restricted to zero crossings but also apply to a large family of curves. By a modification of the argument in Cramér and Leadbetter (1967) it can be shown that for smooth curves ξ

$$M_k(\xi, T) = \int_0^T \cdots \int_0^T E(|\prod_{i=1}^k X'(t_i) - \xi'(t_i)| \mid X(t_j) = \xi(t_j), j = 1, \cdots, k)$$
$$\times\ P_{\mathbf{t}}(\xi(t_1), \cdots, \xi(t_k))\, dt_1 \cdots dt_k$$

where $P_{\mathbf{t}}(\mathbf{x})$ is the joint density of $(X(t_1), \cdots, X(t_k))$. The introduction of general curves presents little additional difficulty when they are at least as smooth as the sample functions of X. Thus there are many generalizations of the following

THEOREM 8. *If the conditions of Theorem* 5 *are satisfied and*

$$\xi \in \mathscr{C}_2[0, T], \quad \textit{then} \quad M_k(\xi, T) < \infty \quad \textit{for all} \quad k .$$

PROOF. Belyaev (1966) has shown that a sufficient condition for $M_k(\xi, T)$ to be finite is that all the integrals

$$\int_0^T \cdots \int_0^T \frac{|\mu_{\pi_1} \cdots \mu_{\pi_j}|\sigma_{\pi_{j+1}} \cdots \sigma_{\pi_k}}{(\det R_k)^{\frac{1}{2}}} \, d\Delta_1 \cdots d\Delta_{k-1}$$

are finite for $1 \leqq i \leqq k$ and any permutation π of $\{1, \cdots, k\}$. Here $\mu_i = E(X'(t_i) - \xi'(t_i) \mid X(t_j) = \xi(t_j), j = 1, \cdots, k)$. By our previous methods, the theorem is true if the μ_i satisfy the bounds given in the following

LEMMA 5. *Under the conditions of Theorem* 8, *there exists a constant K such that*

$$|\mu_1| \leqq K\sigma^2(\Delta_1)\chi(\Delta_1, \cdots, \Delta_{k-1})$$

$$|\mu_i| \leqq K\frac{\sigma^2(\Delta_{i-1})\sigma^2(\Delta_i)}{\sigma^2(\Delta_{i-1} + \Delta_i)}\chi(\Delta_1, \cdots, \Delta_{k-1}) \qquad i = 2, \cdots, k-1$$

$$|\mu_k| \leqq K\sigma^2(\Delta_{k-1})\chi(\Delta_1, \cdots, \Delta_{k-1})$$

where $\chi(\Delta_1, \cdots, \Delta_{k-1}) = \prod_{i=2}^{k-1} (\sigma^2(\Delta_{i-1} + \Delta_i)/\phi(\Delta_{i-1} + \Delta_i))$.

PROOF. From Belyaev (1966) we know that

$$\mu_i = -\frac{\det\left(\begin{array}{c|c} \xi'(t_i) & \xi(t_1), \cdots, \xi(t_k) \\ \hline \mathbf{Z}' & R_k \end{array}\right)}{\det R_k} \tag{19}$$

where $\mathbf{Z} = (E(X'(t_i)X(t_1)), \cdots, E(X'(t_i)X(t_k)))$.

Using the divided difference manipulations in a manner similar to Lemma 3 we can rewrite the numerator of (19) for $i < k$ as

$$\sigma^2(\Delta_i)(\prod_{i=1}^{k-1} \Delta_i^2)(\prod_{i=2}^{k-1} \sigma^2(\Delta_{i-1} + \Delta_i)) \det\left(\begin{array}{c|c} \xi_\sigma[t_i, t_i, t_{i+1}] & \boldsymbol{\xi} \\ \hline \mathbf{Z}^* & R_k^* \end{array}\right) \tag{20}$$

where

$$\boldsymbol{\xi} = (\xi(t_1), \xi[t_1, t_2], \xi_\sigma[t_1, t_2, t_3], \cdots, \xi_\sigma[t_{k-2}, t_{k-1}, t_k]),$$

$$(\mathbf{Z}^*)' = E(X_\sigma[t_i, t_i, t_{i+1}]\mathbf{Y}'),$$

$$R_k^* = E(\mathbf{Y}\mathbf{Y}'), \quad \text{and}$$

$$\mathbf{Y}' = (X(t_1), X[t_1, t_2], X_\sigma[t_1, t_2, t_3], \cdots, X_\sigma[t_{k-2}, t_{k-1}, t_k]).$$

The determinant in (20) is bounded since all its entries are. Finally, using the lower bounds of Lemma 3 for the denominator in (19), it follows that

$$|\mu_i| \leqq K\sigma^2(\Delta_i)\chi(\Delta_1, \cdots, \Delta_{k-1}) \qquad i = 1, \cdots, k-1.$$

The bounds

$$|\mu_i| \leqq K\sigma^2(\Delta_{i-1})\chi(\Delta_1, \cdots, \Delta_{k-1}) \qquad i = 2, \cdots, k$$

follow analogously. The proof is completed as with Lemma 3.

5. Necessary conditions for $M_k(0, T) < \infty$. Geman (1972) has shown that a necessary and sufficient condition for $M_2(0, T)$ is that

$$\int_0^\varepsilon \frac{\sigma^2(t)\, dt}{t} < \infty \qquad \text{for some} \quad \varepsilon > 0.$$

This has been extended in Cuzick (1974) to curves ξ whose derivatives satisfy a Lipschitz condition of order $|\ln t|^{-\beta}$ with $\beta > 1$. By Theorem 6, little more is required of $\sigma^2(t)$ to make higher moments finite. However, necessary conditions are more difficult for higher moments. The main difficulty lies in obtaining sharp lower bounds for the σ_i^2. This is a question of obtaining lower bounds for interpolation variances and is not handled satisfactorily by local nondeterminism, which is concerned with extrapolation. This point is not recognized by Mirošin (1974b) and the proof of the estimates in his Lemma 3 is invalid. We can, however, by ad hoc methods, establish some cases in which $N_2(0, T) < \infty$, $N_3(0, T) = \infty$.

THEOREM 9. *If* $\sigma^2(t) = E(X'(t) - X'(0))^2 = |\ln t|^{-\alpha}$ *for* $\alpha \leqq \frac{3}{2}$ *and* $|t| < \delta$ *with* $\delta > 0$, *then* $M_3(0, T) = \infty$.

PROOF. Assume $EX(t)^2 = EX'(t)^2 = 1$ so that X has covariance function

$$\rho(t) = 1 - \tfrac{1}{2}t^2 + \Psi(t) \qquad \text{and}$$

$$\Psi(t) = \tfrac{1}{2}\int_0^t \int_0^s \sigma^2(u)\,du = \tfrac{1}{4}t^2|\ln t|^{-\alpha}(1 + \tfrac{3}{2}\alpha|\ln t|^{-1} + 0|\ln t|^{-2})\,.$$

For simplicity we rename $\Delta_1 = x$, $\Delta_2 = y$. For a small positive ε we consider the region $y \leqq x \leqq y(1 + \varepsilon)$ and by direct calculation obtain the following seven estimates in that region for small x:

(21A) $\quad E(X'(t_1) - X[t_1, t_2])^2 \sim |\ln x|^{-\alpha}$

(21B) $\quad E(X'(t_1) - X[t_1, t_2])(X[t_1, t_2] - X[t_2, t_3]) = 0|\ln x|^{-(\alpha+1)}$

(21C) $\quad E(X[t_1, t_2] - X[t_2, t_3])^2 \sim |\ln x|^{-(\alpha+1)}$

(21D) $\quad E(X'(t_2) - X[t_2, t_3])(X[t_1, t_2] - X[t_2, t_3]) = 0|\ln x|^{-(\alpha+1)}$

(21E) $\quad E(X'(t_1) - X[t_1, t_2])(X[t_1, t_2]) = 0|\ln x|^{-(\alpha+1)}$

(21F) $\quad E(X'(t_2) - X[t_1, t_2])(X[t_1, t_2]) = 0|\ln x|^{-(\alpha+1)}$

(21G) $$\frac{E(X[t_1, t_2] - X[t_2, t_3])^2}{E(X[t_1, t_2] - X[t_2, t_3])(X[t_1, t_2])} \geqq 2\,.$$

We shall only prove (21G). It can be rewritten as

(22) $$1 + \frac{E(X[t_2, t_3])^2 - E(X[t_1, t_2]X[t_2, t_3])}{E(X[t_1, t_2])^2 - E(X[t_1, t_2]X[t_2, t_3])} = 1 + \frac{-2\Psi(y)/y^2 + (\Psi(x + y) - \Psi(x) - \Psi(y))/xy}{-2\Psi(x)/x^2 + (\Psi(x + y) - \Psi(x) - \Psi(y))/xy}$$

which is greater than or equal 2 when $y \leqq x$ since Ψ is convex and $\Psi(y))/y^2$ is strictly increasing at zero.

All of the terms in (14) can be written in terms of determinants or ratios of determinants. For example,

$$\begin{aligned}\sigma_1^2 &= \operatorname{Var}(X'(t_1) \mid X(t_1), X(t_2), X(t_3)) \\ &\sim \operatorname{Var}(X'(t_1) - X[t_1, t_2] \mid X[t_1, t_2] - X[t_2, t_3], X[t_1, t_2]) \\ &= \frac{\det \operatorname{Cov}(X'(t_1) - X[t_1, t_2], X[t_1, t_2] - X[t_2, t_3], X[t_1, t_2])}{\det \operatorname{Cov}(X[t_1, t_2] - X[t_2, t_3], X[t_1, t_2])}\,.\end{aligned}$$

In this manner, using the estimates (21), we obtain for $y \leqq x \leqq (1+\varepsilon)y$ that

$$\det R_3 \sim x^2 y^2 |\ln x|^{-(\alpha+1)}$$
$$\sigma_i^2 \sim |\ln x|^{-\alpha} \qquad i = 1, 2, 3 .$$

Using these estimates the integral (14) will be infinite if

$$\int_0^\varepsilon \frac{|\ln x|^{-(2\alpha-1)/2}}{x}\, dx = \infty .$$

This will occur if $\alpha \leqq \frac{3}{2}$.

Acknowledgment. The author is indebted to the referee for comments on an earlier draft.

REFERENCES

BELYAEV, YU. K. (1966). On the number of intersections of a level by a Gaussian stochastic process I. *Teor. Verojatnost. i Primenen* **11** 120–128.

BERMAN, S. M. (1973). Local nondeterminism and local times of Gaussian processes. *Indiana Math. J.* **23** 69–94.

CRAMÉR, H. and LEADBETTER, M. R. (1967). *Stationary and Related Stochastic Processes.* Wiley, New York.

CUZICK, J. M. (1974). On the moments of the number of curve crossings by a stationary Gaussian process. Ph. D. Dissertation, Claremont Graduate School.

CUZICK, J. M. (1975). Conditions for finite moments of the number of zero crossings for Gaussian processes. *Ann. Probability* **3** 849–858.

CUZICK, J. M. (1977). A lower bound for the prediction error of stationary Gaussian processes. *Indiana University Math. J.* **26** 577–584.

GEMAN, D. (1972). On the variance of the number of zeros of a stationary Gaussian process. *Ann. Math. Statist.* **43** 977–982.

ISAACSON, E. and KELLER, H. B. (1966). *Analysis of Numerical Methods.* Wiley, New York.

LUKACS, E. (1970). *Characteristic Functions.* Hafner, New York.

MIROŠIN, R. N. (1973). A sufficient condition for finiteness of the number of zeros of a differentiable Gaussian process. *Teor. Verojatnost. i Primenen* **18** 454–463.

MIROŠIN, R. N. (1974a). Sufficient conditions for finite moments of zeros of a stationary Gaussian process. *Vestnik Leningrad. Univ.* **1** 48–65.

MIROŠIN, R. N. (1974b). A necessary condition for the moments of the number of zeros of a differentiable Gaussian process to be finite. *Teor. Verojatnost. i Primenen* **19** 596–603.

DEPARTMENT OF MATHEMATICAL STATISTICS
COLUMBIA UNIVERSITY
NEW YORK, NEW YORK 10027