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ASYMPTOTIC BEHAVIOUR OF THE VARIANCE OF RENEWAL PROCESSES AND RANDOM WALKS

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For a sequence of independent identically distributed random variables $\{X_n\}$, $n = 1, 2, \cdots$, yielding the sums $S_n = X_1 + \cdots + X_n$ let $N(x) = \#\{n \geqq 1 : S_n \leqq x\}$. Results of Stone and the general renewal equation as treated by Feller are used to prove that under certain conditions on the common distribution function of the X_n's, the variance of $N(x)$ is asymptotically like $Ax + B + o(1)$ as $x \to \infty$ for specified constants A and B.

1. Introduction and statement of results. For independent identically distributed random variables (rv's) $X, X_1, X_2 \cdots$ with distribution function (df) F, let

$$S_n = X_1 + \cdots + X_n \tag{1.1}$$

denote the nth partial sum, and define

$$N(x) = \#\{n = 1, 2, \cdots : S_n \leqq x\}, \tag{1.2}$$

being the number of partial sums in the half-line $(-\infty, x]$. $N(x)$ is finite when $E|X| < \infty$ and the mean step-length $\lambda^{-1} \equiv EX$ of the random walk $\{S_n\}$ is positive. The so-called *renewal function*

$$H(x) = EN(x) = \textstyle\sum_1^\infty F^{n*}(x) \tag{1.3}$$

(F^{n*} is the n-fold convolution of F) is asymptotically linear, i.e., $H(x) = \lambda x + o(x)$ $(x \to \infty)$, provided that it is finite, which is the case provided $E(\min(0, X))^2 < \infty$ (Heyde, 1964). If it is further true that F is a *nonlattice* (or *nonarithmetic*) df and $EX^2 < \infty$, then (Smith, 1960) this property of asymptotic linearity can be refined further to assert that

$$H(x) = \lambda x + K_1 + o(1) \quad (x \to \infty) \tag{1.4}$$

$$K_1 \equiv \tfrac{1}{2}\lambda^2 EX^2 - 1. \tag{1.4a}$$

The main object of this note is to give sufficient conditions for an analogue of (1.4) to be true for the variance $\mathrm{Var}\, N(x)$, that is, for the validity of

$$V(x) \equiv \mathrm{Var}\, N(x) = A\lambda x + B + o(1) \quad (x \to \infty), \tag{1.5}$$

$$A \equiv 1 + 2K_1 = \lambda^2 EX^2 - 1 = \mathrm{Var}\,(X/EX), \tag{1.5a}$$

$$B \equiv K_1(1 + K_1) + 4K_2 - 2\lambda \textstyle\int_{-\infty}^0 H(u)\, du \tag{1.5b}$$

where

$$K_2 = (\lambda^2 EX^2/2)^2 - \lambda^3 EX^3/6.$$

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We also comment on the methods used, and indicate an analogue of (1.5) for stationary random walk point processes for which $N(x)$ is related to the corresponding Palm–Khintchine rv's. The methods appear to be capable of extension to higher moment analogues of (1.5).

For the simpler case where $N(\cdot)$ is a renewal process (i.e., $F(0-) = 0$, and hence $H(x) = 0$ for $x < 0$), Smith (1959) used Laplace–Stieltjes transform techniques in a more general discussion of higher order moments. Equation (1.5) simplifies, and can be found in a form involving central moments in Cox (1962, Equation (17) on page 58).

2. Statement of result and preliminaries. The main results are given under one of the following two sets of conditions.

CONDITIONS A_ε. F is a nonlattice df with positive mean, finite third absolute moment, and for some $\varepsilon \geqq 0$,

$$H(x) = \lambda x + K_1 + o(x^{-1-\varepsilon}) \quad (x \to \infty)\,, \tag{2.1a}$$

$$= o(|x|^{-1-\varepsilon}) \quad (x \to -\infty)\,. \tag{2.1b}$$

CONDITIONS B_ρ. F is a strongly nonlattice df (i.e., $\phi(\theta) \equiv Ee^{i\theta x} \neq 1$ for $\theta \neq 0$ and $\liminf_{|\theta|\to\infty} |1 - \phi(\theta)| > 0$) with positive mean and finite ρth order absolute moment for some $\rho \geqq 2$.

THEOREM 1A. *Under Conditions A_ε, (1.5) holds with the error term $o(x^{-\varepsilon})$ $(x \to \infty)$.*

THEOREM 1B. *Under Conditions B_ρ, with $\rho \geqq 3$, (1.5) holds with the error term equal to*

$$(\lambda^3/3) \int_x^\infty (u - x)^2(2u - x)\, dF(u) + o(x^{2-\rho}) \quad (x \to \infty)\,. \tag{2.2}$$

The preliminary results we need concern an expression for $EN^2(x)$, an asymptotic expansion for H based on Fourier techniques, and the general form of the renewal theorem. First, elementary computation (for example, by expressing $N(x)$ as the sum of indicator rv's) shows that

$$EN^2(x) = H(x) + 2 \int_{-\infty}^x H(x - u)\, dH(u)\,, \tag{2.3}$$

finite or infinite; an elementary argument in which the integrand is replaced by linear bounds shows $EN^2(x)$ to be finite if and only if $\int_{-\infty}^0 H(u)\, du < \infty$, which is the case when $E|\min(0, X)|^3 < \infty$. This condition is satisfied under the conditions of Theorem 1, so we wish to study the asymptotic behaviour of

$$V(x) = H(x) + 2 \int_{-\infty}^x H(x - u)\, dH(u) - [H(x)]^2\,. \tag{2.4}$$

In using (2.1) (and, as it happens, Conditions B_ρ with $\rho \geqq 3$), the main problem in discussing (2.4) centres on the integral it contains.

Next, Smith (1967) (in his Corollary 5.1 and below his equation (A.10)) has strengthened results of Stone (1965) (his equation (16)), both of them using

Fourier techniques, in showing that when F satisfies Conditions B_ρ,

$$H(x) - \lambda x - K_1 + \lambda^2 S_+(x) = o(x^{1-\rho}) \quad (x \to \infty), \tag{2.5a}$$

$$H(-x) - \lambda^2 S_-(x) = o(|x|^{1-\rho}) \quad (x \to \infty), \tag{2.5b}$$

where

$$S_+(x) = \int_x^\infty du \int_u^\infty (1 - F(v))\, dv = \tfrac{1}{2} \int_x^\infty (u - x)^2\, dF(u), \tag{2.6a}$$

$$S_-(-x) = \int_{-\infty}^{-x} du \int_{-\infty}^{u} F(v)\, dv = \tfrac{1}{2} \int_{-\infty}^{-x} (u + x)^2\, dF(u). \tag{2.6b}$$

Consequently, under Conditions B_ρ with $\rho \geqq 3$, the conditions A_ε are satisfied with $\varepsilon = \rho - 3$; however, the form $o(|x|^{2-\rho})$ of the error term at equations (2.1) is weaker than the explicit expressions given at equations (2.5).

Finally, we recall that for nonlattice df's F with finite absolute first moment and positive mean λ^{-1}, the *solution* Z of the *general renewal equation*

$$Z(x) = z(x) + \int_{-\infty}^{\infty} Z(x - y)\, dF(y) \tag{2.7}$$

in which we speak of z as the *generator*, satisfies

$$\lim_{x\to\infty} Z(x) = \lim_{x\to\infty} \int_{-\infty}^{\infty} z(x - y)\, dU(y) = \lambda \int_{-\infty}^{\infty} z(y)\, dy, \tag{2.8}$$

where $U(y) = \sum_0^\infty F^{n*}(y) = H(y) + I(y)$ and $I(y) = 0$ or 1 as $y <$ or $\geqq 0$, provided that the generator z is directly Riemann integrable. This result, at (2.8), is the general renewal theorem, a proof for the case where $F(0-) = 0$ being given in Chapter XI of Feller (1966); the more general result, established by ladder variable techniques, is alluded to in the "Problems" at the end of that Chapter XI. (It is worth noting that a probabilistic proof of the Blackwell renewal theorem, which underlies (2.8), has recently been given by Lindvall (1977). The referee draws attention to the first statement and proof of (2.8) under the assumption of direct Riemann integrability being in Smith (1961).)

We use (2.8) in attempting to evaluate the integral in (2.4) which, in view of (1.4), is presumably like

$$2 \int_{-\infty}^{x} [\lambda(x - u) + K_1]\, dH(u) \tag{2.9}$$

$$= 2K_1 H(x) + 2\lambda \int_{-\infty}^{x} H(u)\, du$$

$$= 2K_1 H(x) + \lambda^2 x^2 + 2K_1 \lambda x + 2\lambda \int_{-\infty}^{x} [H(u) - \lambda L(u) - K_1 I(u)]\, du \tag{2.10}$$

where $I(\cdot)$ is as below (2.8) and

$$L(x) = \max(0, x) = \int_0^x I(u)\, du. \tag{2.11}$$

Now it can be checked that the functions H, λL, and $K_1 I$ are solutions of the renewal equation (2.7) with generators F,

$$G(x) \equiv \lambda \int_{-\infty}^{x} [I(u) - F(u)]\, du, \tag{2.12}$$

and $K_1(I - F)$ respectively, so $H - \lambda L - K_1 I$ is the solution corresponding to the generator $I - G - (K_1 + 1)(I - F)$. Also, for the solution Z with generator z, the solution $\int_{-\infty}^{x} Z(u)\, du = \int_0^\infty Z(x - u)\, du = (Z * L)(x)$ corresponds to the generator

$$(2.13)\qquad \int_0^\infty Z(x-u)\,du - \int_{-\infty}^\infty dF(y)\int_0^\infty Z(x-y-u)\,du = \int_0^\infty z(x-u)\,du = \int_{-\infty}^x z(u)\,du = (z * L)(x)\,,$$

provided of course that this last expression is integrable on the half line $(-\infty, x]$. When $z = I - G - (K_1+1)(I-F) = I - G - \frac{1}{2}\lambda^2 EX^2(I-F)$, $\int_{-\infty}^\infty z(u)\,du = 0$, and when F has a finite third absolute moment, $\int_{-\infty}^\infty |\int_{-\infty}^x z(u)\,du|\,dx$ is finite, so the integrand is directly Riemann integrable at required at (2.13) and as needed for application of (2.8). Thus,

$$(2.14)\qquad \begin{aligned} \lim_{x\to\infty} \lambda \int_{-\infty}^x [H(u) - \lambda L(u) - K_1 I(u)]\,du &= \lambda^2 \int_{-\infty}^\infty dx \int_{-\infty}^x z(u)\,du \\ &= \lambda^2 \int_{-\infty}^0 dx \int_{-\infty}^x z(u)\,du - \lambda^2 \int_0^\infty dx \int_x^\infty z(u)\,du \\ &= -\lambda^2 \int_{-\infty}^\infty u z(u)\,du \\ &= (\lambda^2 EX^2/2)^2 - \lambda^3 EX^3/6 \equiv K_2\,. \end{aligned}$$

It is worth noting that the argument above is a special case of the following result for which the proof is straightforward and is omitted.

THEOREM 2. *Suppose the general renewal equation (2.7) for which* $EX^2 = \int_{-\infty}^\infty x^2\,dF(x) < \infty$ *has a generator z for which both it and its tail integral* $\int_{-\infty}^x z(u)\,du - CI(x)$ *are directly Riemann integrable* ($C \equiv \int_{-\infty}^\infty z(u)\,du$). *Then in addition to* $Z(x) \to \lambda C$ $(x\to\infty)$, *we also have*

$$(2.15)\qquad \int_{-\infty}^y \{Z(x) - \lambda C I(x)\}\,dx \to \tfrac{1}{2}\lambda C EX^2 - \lambda \int_{-\infty}^\infty u z(u)\,du \quad (y\to\infty)\,.$$

Lemma 5.3.5 of Jagers (1975) states a result similar to the above for the case that $x \geqq 0$ a.s., with the direct Riemann integrability condition on z replaced by z being of locally bounded variation and $z(t)\to 0$ for $t\to\infty$.

3. Proof of Theorems 1A and 1B. We suppose now that Conditions A_ε hold, and write $A(x) = H(x) - \lambda L(x) - K_1 I(x)$, so $A(x) = o(|x|^{-1-\varepsilon})$ for $x\to\infty$. We then write the integral in (2.4) as

$$(3.1)\qquad \begin{aligned} \int_{-\infty}^x H(x-u)\,dH(u) &= \int_{-\infty}^x \{\lambda(x-u) + K_1\}\,dH(u) + \int_{-\infty}^x A(x-u)\,dH(u) \\ &= K_1 H(x) + \lambda \int_{-\infty}^x H(u)\,du + \int_{-\infty}^{0+} A(x-u)\,dH(u) \\ &\quad + \lambda \int_0^x A(x-u)\,du + \int_{0+}^x A(x-u)\,dA(u)\,. \end{aligned}$$

The last integral here equals

$$(3.2)\qquad \int_0^{x/2} A(x-u)\,dA(u) + A(x)A(0) - (A(x/2))^2 + \int_0^{x/2} A(x-u)\,dA(u)\,,$$

which is $o(x^{-\varepsilon})$ because $A(\cdot)$, being the difference of two monotonic functions whose variation on $(0, x)$ is $O(x)$ $(x\to\infty)$, is itself of bounded variation on a finite interval with total variation at most $O(x)$ for $x\to\infty$. Also, $\int_{-\infty}^{0+} A(x-u)\,dH(u) = o(x^{-\varepsilon})$. The remaining two integrals equal

$$(3.3)\qquad \begin{aligned} &\lambda \int_0^x (\lambda u + K_1)\,du + 2\lambda \int_{-\infty}^\infty A(u)\,du - 2\lambda \int_x^\infty A(u)\,du - \lambda \int_{-\infty}^0 A(u)\,du \\ &\qquad = \tfrac{1}{2}\lambda^2 x^2 + K_1 \lambda x + 2K_2 - \lambda \int_{-\infty}^0 H(u)\,du + o(x^{-\varepsilon})\,. \end{aligned}$$

Substituting for H and H^2 in (2.4) from (2.1) and for the integral from (3.3) now yields the result asserted in Theorem 1A concerning (1.5).

The proof of Theorem 1B is similar to the above, except that the error term is more explicit. We omit the algebraic detail.

4. Analogues for the stationary random walk. Suppose that a point process $N_1(\cdot)$ is generated by a random walk: if the so-called counting function $N_1(t, t+x]$, the number of points in the half-open interval $(t, t+x]$, has joint distributions (for sets of values of x) that are independent of t, it may be called a stationary *random walk point process* (Daley and Oakes, 1974); the counting function $N_0(\cdot)$ for the corresponding Palm–Khintchine distribution in which a point of the process occurs at 0 is not stationary, realizations of $N_0(\cdot)$ being in 1-1 correspondence with realizations $\{S_n : n = 0, \pm 1, \cdots\}$ of a two-sided random walk that visits the origin. If this random walk has a nonlattice df, then we may expect to be able to obtain an asymptotically linear expression for $\operatorname{Var} N_0(t, t+x)$ $(x \to \infty)$ which for $t \to \infty$ should agree with that of

$$
\begin{aligned}
\operatorname{Var} N_1(t, t+x] &= \operatorname{Var} N_1(0, x] \\
&= \lambda \int_0^x \{1 + 2(H(u) - H(-u) - \lambda u)\}\, du \qquad (4.1)\\
&= (1 + 2K_1)\lambda x + 2K_2 - 4\lambda \int_{-\infty}^0 H(u)\, du + o(x^{-\varepsilon})
\end{aligned}
$$

when Conditions A_ε hold.

An expression with a more complicated error term than (4.1) can be found when Conditions B_ρ hold.

5. Concluding remarks. A drawback to equation (1.5b) is the presence there of the integral which we have not been able to evaluate. It arises from the integral in (2.4) which, it should be noted, is not the convolution of H with itself: if it were, it would equal $\int_{-\infty}^{\infty} H(x-u)\, dH(u)$, and an explicit asymptotic evaluation of this expression (with error $o(1)$) can be found as at the end of Section 2 above. Also, since $\int_{-\infty}^{\infty} H(x-u)\, dH(u) = \sum_1^\infty (n-1)F^{n*}(x)$, results of Smith (1967) could then be used.

The treatment of the integral of (2.4) in Section 3 should be noted with care: it is not enough to approximate $dH(x)$ by $\lambda\, dx$, as the renewal theorem suggests, even when the order of the error term is known.

In his first edition Feller (1966) has the comments: "this method [of using the general renewal equation to establish that $H(x) - \lambda x \to K_1$] can be used for better estimates when higher moments exist" (page 372); and "the asymptotic expansion of U [equivalently, of H] may be further refined if F has moments of higher order" (page 357). We have not been able to find a method of using the renewal equation to study $x^\rho[H(x) - \lambda L(x) - K_1] \equiv x^\rho A(x)$ as $x \to \infty$ for any nonzero ρ, and it may be pertinent to note that these comments are omitted from the second edition (1971) of Feller's Volume II. What we have been able to do at and above Theorem 2 is to show that the general renewal equation can be used to study the asymptotic behaviour of the convolution product of $A(x)$ and

x, but in the absence of any knowledge such as ultimate monotonic behaviour of $A(x)$ we cannot then infer anything about H itself.

It appears to remain an open problem as to how to use real variable methods to derive any refinements of the asymptotic behaviour of $H(\cdot)$ analogous to the results of Stone (1965) and Smith (1967) who used Fourier methods.

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