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LIMIT THEOREMS FOR NONERGODIC SET-VALUED MARKOV PROCESSES

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Certain Markov processes on the state space of subsets of the integers have $\varnothing$ as a trap, but have an equilibrium $\nu \neq \delta_\varnothing$. In this paper we prove weak convergence to a mixture of $\delta_\varnothing$ and ν from any initial state for some of these processes. In particular, we prove that the basic symmetric one-dimensional contact process of Harris has only $\delta_\varnothing$ and ν as extreme equilibria when the infection rate is large enough in comparison to the recovery rate.

1. Introduction. Let Ξ be the set of all subsets of the d-dimensional integer lattice Z_d. In this paper we study certain Ξ-valued continuous time Feller processes (ξ_t); namely, the *associate* and *additive* lattice interactions formulated by Harris in [1], [2] and [3]. The reader is assumed to be familiar with [1]—[3], from which much of the notation and terminology as well as numerous results will be drawn. Denote by Ξ_0 the set of all *finite* subsets of Z_d. Recall from [2] that two Ξ-valued Feller processes (ξ_t) and $(\hat{\xi}_t)$ are *associate* if

$$(1) \qquad P_\xi(\xi_t \cap \hat{\xi} = \varnothing) = \hat{P}_{\hat{\xi}}(\hat{\xi}_t \cap \xi = \varnothing) \qquad \text{for all} \quad \xi, \hat{\xi} \in \Xi_0 .$$

(The Feller property extends (1) to the situation where one of ξ and $\hat{\xi}$ is infinite.) Let $\mathscr{M}$ comprise all probability measures on Ξ, $\mathscr{M}_0$ the subclass of measures which are invariant with respect to translation in Z_d. Write $\mu P^t = P_\mu(\xi_t \in \cdot)$, for any initial distribution $\mu \in \mathscr{M}$. Set $\delta_\xi =$ the delta measure at ξ, and abbreviate $\xi P^t = \delta_\xi P^t$. Finally, let $\Rightarrow$ denote weak convergence as $t \to \infty$. Then whenever associate processes (ξ_t) and $(\hat{\xi}_t)$ exist, one can draw the following conclusions (cf. [2]):

(a) $\varnothing$ is a trap for (ξ_t) and $(\hat{\xi}_t)$;
(b) $Z_d P^t \Rightarrow \nu$ for some equilibrium ν;
(c) with $\hat{\tau} =$ the hitting time of $\varnothing$ for $(\hat{\xi}_t)$,

$$\hat{P}_{\hat{\xi}}(\hat{\tau} < \infty) = 1 \qquad \text{for all} \quad \hat{\xi} \in \Xi_0$$

iff

$$\nu = \delta_\varnothing$$

iff

$$\mu P^t \Rightarrow \delta_\varnothing \qquad \text{for all} \quad \mu \in \mathscr{M} .$$

(ξ_t) is called *ergodic* if any of the conditions in (c) holds, and *nonergodic* otherwise. In the latter case, $\nu(\{\varnothing\}) = 0$. Harris proved (Theorem (9.2) of [2]) that

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for many processes with associates,

$$\mu P^t \Rightarrow \mu(\{\varnothing\})\delta_\varnothing + (1 - \mu(\{\varnothing\}))\nu \qquad \text{for all} \quad \mu \in \mathscr{M}_0 . \tag{2}$$

A natural question arises: for which (ξ_t) and individual initial configurations ξ is there a limit theorem

$$\xi P^t \Rightarrow \alpha\delta_\varnothing + (1 - \alpha)\nu \tag{3}$$

for some constant α? This is the central problem of the present paper. As in [3], say that ξ is *R-dense*, $R > 0$, if ξ intersects every ball of radius R, and that ξ is *dense* if it is R-dense for some R. In Section 2 we show how Harris' theorem (2) can be modified to yield

$$\xi P^t \Rightarrow \nu \qquad \text{for all dense} \quad \xi \in \Xi , \tag{4}$$

when applied to many interactions. In particular, (4) holds for a wide variety of *contact processes* [1]. Section 3 contains the main results of the paper. For some one-dimensional additive processes (ξ_t), it is proved that if τ is the hitting time of $\varnothing$, then

$$\mu P^t \Rightarrow P_\mu(\tau < \infty)\delta_\varnothing + P_\mu(\tau = \infty)\nu \qquad \text{for all} \quad \mu \in \mathscr{M}. \tag{5}$$

Evidently (5) implies that all equilibria for (ξ_t) are mixtures of $\delta_\varnothing$ and ν.

Our methods apply to the following four representative interactions on Z:

EXAMPLE 1. Basic contact process [1]. The "flip rates" at each site x when the system is in state ξ are

at x	*with rate*
$1 \to 0$	1
$0 \to 1$	$\lambda\|\{x-1, x+1\} \cap \xi\| \quad (\lambda \geqq 0)$.

($|\xi|$) denotes the cardinality of ξ.)

EXAMPLE 2. One-sided contact process [2]. The rates are

at x	*with rate*
$1 \to 0$	1
$0 \to 1$	$\lambda\|\{x-1\} \cap \xi\| \quad (\lambda \geqq 0)$.

EXAMPLE 3. Biased voter model [8]. The rates are

at x	*with rate*
$1 \to 0$	$\|\{x-1, x+1\} \cap \xi^c\|$
$0 \to 1$	$\lambda\|\{x-1, x+1\} \cap \xi\| \quad (\lambda > 1)$.

EXAMPLE 4. Simple exclusion with births [8]. (ξ_t) is a simple exclusion process with $p(x, x+1) = p$, $p(x, x-1) = q$, $p(x, y) = 0$ otherwise, but modified so that a particle is created at $x + 1$ with rate r and at $x - 1$ with rate s whenever $x \in \xi$. Here p, q, r and s are strictly positive parameters. See [8] for details, and more general versions of this process.

Harris proved (2) for Examples 1 and 2 in [2]. For Example 3 and certain cases of Example 4 (e.g., $p = q$ or $r = s$), Schwartz showed that $\delta_\varnothing$ and δ_Z are the only extreme equilibria. For a generalization of Example 3 she also proved (2), and results of type (3) starting from certain ξ. Unbiased voter models which generalize Example 3 with $\lambda = 1$ have been treated in some detail by Holley and Liggett [4].

In Examples 1 and 2, monotonicity arguments establish the existence of *critical constants*, call them λ^* and λ_0^* respectively, below which ergodicity takes place, and above which it does not. Simple comparisons show $\lambda^* \geqq 1$ ([1]), $\lambda_0^* \geqq 2$. A powerful new method of Holley and Liggett [5] yields $\lambda^* \leqq 2$, $\lambda_0^* \leqq 4$.

The limit laws which will be derived below have the following implications for the four examples.

EXAMPLE 1. For all λ, (4) holds. If $\lambda > \lambda_0^*$, then (5) holds. In particular, the basic contact process has only $\delta_\varnothing$ and ν as extreme invariant measures when $\lambda > \lambda_0^*$.

EXAMPLE 2. (4) holds for any λ. Also, $\xi P^t \Rightarrow \delta_\varnothing$ whenever $\xi \in \Xi_0$. However ξP^t need not converge if $|\xi| = \infty$, ξ is not dense, and $\lambda > \lambda^*$.

EXAMPLE 3. For any $\lambda > 1$,

$$\mu P^t \Rightarrow P_\mu(\tau < \infty)\delta_\varnothing + P_\mu(\tau = \infty)\delta_Z \qquad \text{for all} \quad \mu \in \mathscr{M}. \tag{6}$$

EXAMPLE 4. For any p, q, r and s such that $p + r > q$ and $q + s > p$, (6) holds, with $P_\mu(\tau < \infty) = \mu(\{\varnothing\})$.

Results such as these are to be expected, based on the work of Vasershtein and Leontovich [9], Vasil'ev [10] and Vasil'ev, et al. [11] with closely related discrete time processes. Some of the key ideas in this paper are derived from [9]—[11]. But certain of their techniques, especially those based on "contour" estimates, cannot be translated to the continuous setting. Consequently the arguments have to be altered. We rely heavily on the "graphical representations" for additive processes [3], which are especially simple in Examples 1 through 4.

2. Convergence to ν from dense initial states. By modifying slightly the argument leading to Theorem 9.2 in [2], it is possible to obtain convergence to ν from dense ξ for various processes (ξ_t) which have associates. We illustrate this with a result for *contact processes*, as formulated on page 185 of [2]. A Ξ-valued process is *local homogeneous* if its semigroup is in the class $\mathscr{K}$ defined in (9.1) of [2]. Also, with $S_i(\{0\})$ as in (9.8) of [2], let $S = \bigcup_{i=1}^\infty S_i(\{0\})$.

PROPOSITION. *Let (ξ_t) be a local homogeneous contact process with an associate, such that $\mu > 0$ and $\lambda(\xi) > 0$ whenever $\xi \neq \varnothing$. If Z_d has no proper subgroup containing S, then $\xi_0 P^t \Rightarrow \nu$ for any dense initial state ξ_0.*

PROOF. Let ξ_0 be R-dense. Inspection of the proof of Harris' theorem shows that one need only derive (9.15) of [2] for $\mu = \delta_{\xi_0}$. In other words, it suffices to check that for any fixed $t > 0$,

$$\sup_{x \in Z_d} P_{\xi_0}(x \notin \xi_t) = \rho < 1 . \tag{7}$$

By the aperiodicity, there is an $i \geqq 1$ and a $y \in Z_d$ such that $B_R + y \subset S_i(\{0\})$. ($B_R$ is the ball of radius R centered at the origin.) Then $\xi_0 \cap S_i(\{x\}) \neq \varnothing$ for each x. Hence

$$\begin{aligned} P_{\xi_0}(x \notin \xi_t) &\leqq \sup_{\xi : \xi \cap S_i(\{x\}) \neq \varnothing} P_\xi(x \notin \xi_t) \\ &= \sup_{\xi : \xi \cap S_i(\{0\}) \neq \varnothing} P_\xi(0 \notin \xi_t) \\ &= \max_{\varnothing \neq \xi \subset S_i(\{0\})} P_\xi(0 \notin \xi_t) = \rho < 1 . \end{aligned}$$

The last inequality holds because (ξ_t) is monotone, in the sense of [3], and $\rho < 1$ because $\lambda(\xi) > 0$ for $\xi \neq \varnothing$. So (7) is established, completing the proof.

REMARK. Proposition 1 applies to Examples 1 and 2, and also handles contact processes, "one-sided" or not, in higher dimensions. In much the same manner one can prove an extension of Harris' theorem (9.2) in [2]. Namely, retain his condition (a), but replace (b) by the assumption:

(b′) There are finite sets $\Lambda_R \subset Z_d$, $R = 1, 2, \cdots$, such that $P_\xi(0 \in \xi_t) > 0$ whenever $t > 0$ and $\varnothing \neq \xi \subset \Lambda_R$ for some R.

Then conclusion (iii) of the theorem may be changed to

(iii′) If $\nu \neq \delta_\varnothing$, then $\nu(\{\varnothing\}) = 0$, and if μ is any measure such that

$$\lim_{R \to \infty} \sup_{x \in Z_d} [\mu(\{\xi : \xi \cap (\Lambda_R + x) = \varnothing\}) - \mu(\{\varnothing\})] = 0 , \tag{8}$$

then

$$\mu P^t \Rightarrow \mu(\{\varnothing\})\delta_\varnothing + (1 - \mu(\{\varnothing\}))\nu .$$

To obtain the original Theorem (9.2), set $\Lambda_R = B_R$ and note that (8) is automatic when μ is translation invariant. The extension gives an alternate proof of Theorem (9.17), and also applies to one-sided examples in higher dimensions. A similar result for a certain class of discrete-time processes was mentioned by Vasershtein and Leontovich in [9].

Consider now Example 2 starting from more general initial states ξ. If $\xi \in \Xi_0$, then $\xi P^t \Rightarrow \delta_\varnothing$ for any λ because the "infected" set ξ_t wanders off to the right unless it dies out. Also, if ξ is dense on any half line $(-\infty, x]$ then $\xi P^t \Rightarrow \nu$. This follows from the additive nature of (ξ_t) and the fact that each individual process $(\xi_t{}^x)$ (cf. [3]) either dies out or tends to $+\infty$. Finally, by taking ξ to be a union of disjoint "intervals," $\xi = \bigcup_{i=1}^\infty [x_i, y_i]$, where $0 = y_0 > x_0 > y_1 > \cdots$, so that $y_i - x_i \to \infty$ and $x_i - y_{i+1} \to \infty$ quite rapidly, one can show that ξP^t does not converge as $t \to \infty$, but rather converges to ν along one subsequence and to $\delta_\varnothing$ along another. This phenomenon was noted by Vasil'ev [10] for an analogous discrete time process. Holley and Stroock [6] have given an interesting example of a "one-sided" contact process in Z_2 which has a nontranslation invariant extreme equilibrium measure in addition to $\delta_\varnothing$ and ν. An obvious extension of the proposition applies to their example; so does the extended version of Harris' theorem with $\Lambda_R = B_R$.

3. Main results. Let (ξ_t) and $(\hat{\xi}_t)$ be local homogeneous associate processes on Ξ. Then $Z_d P^t \Rightarrow \nu$ and $Z_d \hat{P}^t \Rightarrow \hat{\nu}$ for some translation invariant equilibria ν and $\hat{\nu}$. In this section we identify some one-dimensional situations where (5) holds. *Assume from now on* that $\nu \neq \delta_\varnothing$, $\hat{\nu} \neq \delta_\varnothing$, and that

$$(9) \qquad P_{\{0\}}(x \in \xi_t) > 0 \qquad \text{for any} \quad x \in Z_d\,, \quad t > 0\,.$$

This rules out "one-sided" processes, but we have already seen that (5) does not hold in the simplest such case, i.e., in Example 2. Note that Examples 1, 3 and 4 do satisfy (9).

To begin, we give a general necessary and sufficient condition for (5), valid in any dimension. Unfortunately this criterion seems quite difficult to check directly. In the lemma which follows, let $(\xi_t, \hat{\xi}_t)$ be the process on $\Xi \times \Xi$ comprised of *independent* copies of (ξ_t) and $(\hat{\xi}_t)$, and let $\bar{P}_{\xi\hat{\xi}}$ govern this product process started at $(\xi, \hat{\xi})$.

LEMMA. *The limit laws* (5) *hold if and only if*

$$(10) \qquad \lim_{t\to\infty} \bar{P}_{\xi\hat{\xi}}(\xi_t \cap \hat{\xi}_t = \varnothing \mid \tau = \infty \ \textit{and} \ \hat{\tau} = \infty) = 0 \qquad \textit{for all} \quad \xi, \hat{\xi} \in \Xi_0 - \{\varnothing\}\,.$$

(The conditional probabilities in (10) are elementary, since $\nu \neq \delta_\varnothing$ and $\hat{\nu} \neq \delta_\varnothing$.)

PROOF. It suffices to show the equivalence of (5) and the condition:

$$(11) \qquad \lim_{t\to\infty} \bar{P}_{\xi\hat{\xi}}(\xi_t \cap \hat{\xi}_t = \varnothing, \tau > t, \hat{\tau} > t) = 0 \qquad \text{for all} \quad \xi, \hat{\xi} \in \Xi_0 - \{\varnothing\}\,.$$

Given $\xi, \hat{\xi} \in \Xi_0 - \{\varnothing\}$ and $t \geqq 0$, manipulate (1) to get

$$\begin{aligned}
P_\xi(\xi_{2t} \cap \hat{\xi} = \varnothing) &= \hat{P}_{\hat{\xi}}(\hat{\xi}_{2t} \cap \xi = \varnothing) \\
&= \textstyle\sum_{\eta\in\Xi_0} \hat{P}_{\hat{\xi}}(\hat{\xi}_t = \eta)\hat{P}_\eta(\hat{\xi}_t \cap \xi = \varnothing) \\
&= \textstyle\sum_{\eta\in\Xi_0} \hat{P}_{\hat{\xi}}(\hat{\xi}_t = \eta)P_\xi(\xi_t \cap \eta = \varnothing) \\
&= \hat{P}_{\hat{\xi}}(\hat{\xi}_t = \varnothing) + \textstyle\sum_{\eta\neq\varnothing} \hat{P}_{\hat{\xi}}(\hat{\xi}_t = \eta) \\
&\qquad \times [P_\xi(\xi_t = \varnothing) + P_\xi(\xi_t \cap \eta = \varnothing, \xi_t \neq \varnothing)] \\
\text{(A)} \qquad &= P_\xi(\xi_t = \varnothing) \\
\text{(B)} \qquad &\quad + P_\xi(\xi_t \neq \varnothing)\hat{P}_{\hat{\xi}}(\hat{\xi}_t = \varnothing) \\
\text{(C)} \qquad &\quad + \textstyle\sum_{\eta\neq\varnothing} \hat{P}_{\hat{\xi}}(\hat{\xi}_t = \eta)P_\xi(\xi_t \cap \eta = \varnothing, \xi_t \neq \varnothing)\,.
\end{aligned}$$

As $t \to \infty$, (A) tends to $P_\xi(\tau < \infty)\delta_\varnothing\{\eta: \eta \cap \hat{\xi} = \varnothing\}$ and (B) tends to $P_\xi(\tau = \infty)\nu\{\eta: \eta \cap \hat{\xi} = \varnothing\}$. The term (C) is precisely the $\bar{P}$-probability in (11). The prescriptions of a measure on the cylinders $(\{\eta: \eta \cap \hat{\xi} = \varnothing\}; \hat{\xi} \in \Xi_0)$ uniquely determine that measure. Thus (5) holds whenever $\mu = \delta_\xi$, $\xi \in \Xi_0$, if and only if (11) holds. To obtain (5) for arbitrary $\xi \in \Xi$, we use the fact that $P_\xi(\tau < \infty) \to 0$ as $|\xi| \to \infty$. This follows from Lemma (9.14) of [2] applied to $\mu = \hat{\nu}$ and the process $(\hat{\xi}_t)$. A simple monotone approximation argument now shows that $\xi P^t \Rightarrow \nu$ whenever $|\xi| = \infty$. Finally, integrate to deduce (5) for arbitrary μ.

REMARK. Under our assumptions, the symmetries of association imply that (5) holds for (ξ_t) if and only if

$$(\hat{5}) \qquad \mu\hat{P}^t \Rightarrow \hat{P}_\mu(\hat{\tau} < \infty)\delta_\varnothing + \hat{P}_\mu(\hat{\tau} = \infty)\hat{\nu} \qquad \text{for all} \quad \mu \in \mathscr{M}.$$

The conditions (5), $(\hat{5})$ and (10) are equivalent because of the identity

$$\hat{P}_{\hat{\xi}}(\hat{\xi}_t = \varnothing) + \hat{P}_{\hat{\xi}}(\hat{\xi}_t \neq \varnothing)P_\xi(\xi_t = \varnothing) = P_\xi(\xi_t = \varnothing) + P_\xi(\xi_t \neq \varnothing)\hat{P}_{\hat{\xi}}(\hat{\xi}_t = \varnothing),$$

which was used in the proof of the lemma.

Let us now specialize to $d = 1$, and prove (5) for a class of interactions which includes Example 1 when $\lambda > \lambda_0^*$, and Examples 3 and 4 over the parameter ranges mentioned above. We assume that (ξ_t) is additive, in order to exploit the graphical representation formulated by Harris in [3]. In this setting $(\hat{\xi}_t)$ can be viewed as a certain reverse time process, and for $0 \leqq t \leqq 2T$, T any fixed time, (ξ_t) and $(\hat{\xi}_t)$ can be constructed simultaneously from the same Poisson flows. In the space-time diagram, time runs "up" from 0 to $2T$ for (ξ_t) and "down" from $2T$ to 0 for $(\hat{\xi}_t)$. See [3] for details. In using a closely related space-time scheme for certain discrete time processes, Vasil'ev [10] observed that the forward process (ξ_t) and the backward process $(\hat{\xi}_t)$ are independent until time T. From T to $2T$, however, there is a complicated dependence induced by the common Poisson flows. By exploiting this *coupling* $\tilde{P}^{2T}_{\xi\hat{\xi}}$ of the associate processes, (10) can be checked in some simple cases. For additional applications of the graphical representation, see [3]. Say that an additive process (ξ_t) on Z is *nearest neighbor* if $\mathscr{A}_0$, as given by (8.2) of [3], lives on $\{-1, 0, 1\}$.

THEOREM. *Let (ξ_t) be a nearest neighbor additive process on Z, with associate $(\hat{\xi}_t)$. Assume that $\nu \neq \delta_\varnothing$, $\hat{\nu} \neq \delta_\varnothing$. For $\hat{\xi}_t \in \Xi_0 - \{\varnothing\}$, write $\hat{M}_t = \max\{x \in Z: x \in \hat{\xi}_t\}$, $\hat{m}_t = \min\{x \in Z: x \in \hat{\xi}_t\}$. If, for every $\xi, \hat{\xi} \in \Xi_0 - \{\varnothing\}$, $0 < K < \infty$,*

$$(12) \qquad \lim_{t\to\infty} P_\xi(\xi_t \subset [-K, K] \,|\, \tau = \infty) = 0$$

and

$$(13) \qquad \lim_{t\to\infty} \hat{P}_{\hat{\xi}}(\hat{M}_t \leqq K \,|\, \hat{\tau} = \infty) = \lim_{t\to\infty} \hat{P}_{\hat{\xi}}(\hat{m}_t \geqq -K \,|\, \hat{\tau} = \infty) = 0,$$

then (5) *holds.*

PROOF. Fix $\varepsilon > 0$, $\xi, \hat{\xi} \in \Xi_0 - \{\varnothing\}$. Select $K > 0$ so that $\xi \cup \hat{\xi} \subset [-K, K]$. By the hypotheses we can choose T_0 so that for any $T \geqq T_0$,

$$(14) \qquad P_\xi(T < \tau \leqq 2T) < \varepsilon, \qquad \hat{P}_{\hat{\xi}}(T < \hat{\tau} \leqq 2T) < \varepsilon,$$

and

$$(15) \qquad P_\xi(\xi_{2T} \subset [-K, K], \tau > 2T) < \varepsilon,$$

and

$$(16) \qquad \hat{P}_{\hat{\xi}}(\hat{\tau} > 2T, \hat{M}_{2T} \leqq K) < \varepsilon, \qquad \hat{P}_{\hat{\xi}}(\tau > 2T, \hat{m}_{2T} \geqq -K) < \varepsilon.$$

Let $\tilde{P}^{2T}_{\xi\hat{\xi}}$ be the joint probability law constructed in Section 9 of [3], which governs $(\xi_t)_{0\leqq t\leqq 2T}$ started at ξ and $(\hat{\xi}_t)_{0\leqq t\leqq 2T}$ started at $\hat{\xi}$. Noting that under $\tilde{P}^{2T}_{\xi\hat{\xi}}$ the two processes are independent up to time t, and using (14)—(16), we have

$$\begin{aligned}\bar{P}_{\xi\hat{\xi}}(\xi_T \cap \hat{\xi}_T = \emptyset, \tau > T, \hat{\tau} > T)&\\ &= \tilde{P}^{2T}_{\xi\hat{\xi}}(\xi_T \cap \hat{\xi}_T = \emptyset, \xi_T \neq \emptyset, \hat{\xi}_T \neq \emptyset)\\ &\leqq \tilde{P}^{2T}_{\xi\hat{\xi}}(\xi_T \cap \hat{\xi}_T = \emptyset, \xi_{2T} \neq \emptyset, \hat{\xi}_{2T} \neq \emptyset, \hat{M}_{2T} > K,\\ &\qquad \hat{m}_{2T} < -K, \{M_{2T} > K \ \text{or} \ m_{2T} < -K\}) + 5\varepsilon,\end{aligned}$$

where $M_t = \max\{x \in Z: x \in \xi_t\}$, $m_t = \min\{x \in Z: x \in \xi_t\}$. Now there are *active paths* (cf. [3]) "up" in the space-time diagram from $\xi \times \{0\}$ to $(M_{2T}, 2T)$ and $(m_{2T}, 2T)$, and active paths "down" from $\hat{\xi} \times \{0\}$ to $(\hat{M}_{2T}, 2T)$ and $(\hat{m}_{2T}, 2T)$. If $\hat{M}_{2T} > K$, $\hat{m}_{2T} < -K$ and either $M_{2T} > K$ or $m_{2T} < -K$, then an up and a down path intersect, and so there is a path connecting some point in ξ at the bottom to a point in $\hat{\xi}$ at the top. In particular there are active paths from ξ and $\hat{\xi}$ to some common point (x, T), so that $\xi_T \cap \hat{\xi}_T \neq \emptyset$. We conclude that the last probability above is 0. Since $\xi, \hat{\xi} \in \Xi_0 - \{\emptyset\}$ and $\varepsilon > 0$ are arbitrary, (10) holds. The lemma yields (5), completing the proof.

To apply the theorem to Examples 1, 3 and 4, we need to check $\nu \neq \delta_\emptyset$, $\hat{\nu} \neq \delta_\emptyset$, (12) and (13). Let us do this now in each case.

EXAMPLE 1. The basic contact process is *self-associate* (cf. [2]), so $\nu = \hat{\nu}$. For $\lambda > \lambda^*$, $\nu \neq \delta_\emptyset$ and (12) holds as a consequence of Lemma (9.3) in [2]. By self-association and symmetry, it remains only to show

(17) $\qquad \lim_{t\to\infty} P_\xi(M_t \leqq K \mid \tau = \infty) = 0 \qquad$ for any $\xi \in \Xi_0 - \{\emptyset\}$, $K < \infty$.

Unfortunately, we have only succeeded in proving (17) when $\lambda > \lambda_0^*$. Let $\sigma_N = \min\{t \geqq 0: |\xi_t| = N\}$ $(N \geqq 1)$. Then $\sigma_N < \infty$ P_ξ-a.s. on $\{\tau = \infty\}$ when $N \geqq |\xi|$ by Lemma (9.3) of [2] again. For such N,

$$\begin{aligned}&P_\xi(\liminf_{t\to\infty} M_t < \infty, \tau = \infty)\\ &\quad = \int_{s\in(0,\infty)} \sum_{\eta:|\eta|=N} P_\xi(\sigma_N \in ds, \xi_{\sigma_N} = \eta) P_\eta(\liminf_{t\to\infty} M_t < \infty, \tau = \infty).\end{aligned}$$

Condition (17) is therefore implied by

$$\lim_{N\to\infty} \sup_{\eta:|\eta|=N} P_\eta(\liminf_{t\to\infty} M_t < \infty, \tau = \infty) = 0.$$

Now the basic contact process with parameter λ *dominates* the one-sided process with the same λ, in the sense of [3]. Moreover, in the one-sided process $M_t \to \infty$ a.s. on $\{\tau = \infty\}$. Hence it suffices to prove for the one-sided process that

$$\lim_{N\to\infty} \sup_{\eta:|\eta|=N} P_\eta(\tau < \infty) = 0.$$

For $\lambda > \lambda_0^*$ this is precisely the content of Lemma (9.14) of [2] when applied to the measure ν of Example 2. (Harris' condition 9.2(b) must be avoided by means of the argument in the proof of Theorem (9.17) of [2], or by using the extension described in Section 2 above.) For additional applications of the one-sided process to the study of additive processes, the reader is referred to [3].

EXAMPLE 2. The hypotheses of the theorem hold for any $\lambda > 1$. $\nu = \delta_Z$, and $\hat{\nu} \neq \delta_\varnothing$ because $P_{\{0\}}(\tau = \infty) = (\lambda - 1)/\lambda > 0$. In fact, starting from $\xi_0 = \{0\}$, $|\xi_t|$ is a random walk with absorption at 0 and positive mean. To verify (12), apply Lemma (9.3) of [2] to (ξ_t). Now, $(\hat{\xi}_t)$ consists of simple symmetric random walks with interference, but modified so that a "birth" can occur at sites which neighbor an occupied one. If a walk is situated at x, it displaces to $x + 1$ with rate 1, to $x - 1$ with rate 1, and in addition a new walk is born at $x + 1$ or $x - 1$, each with rate $\lambda - 1$. Walks which try to occupy the same site coalesce. Note that $\hat{P}_{\hat{\xi}}(\hat{\tau} = \infty) = 1$ for all $\hat{\xi} \neq \varnothing$, that $\hat{M}_t$ is a random walk with positive mean, and that $\hat{m}_t$ is a random walk with negative mean. Thus (13) holds and (5) is proved. By the remark after the lemma, we also obtain $(\hat{5})$ for the random walks with interference and births.

EXAMPLE 3. $(\hat{\xi}_t)$ is also a simple exclusion with births. Thus $\nu = \hat{\nu} = \delta_Z$, and $P_\xi(\tau < \infty) = \hat{P}_{\hat{\xi}}(\hat{\tau} < \infty) = 1$ for all $\xi; \hat{\xi} \neq \varnothing$. If $p + r > q$ and $q + s > p$, then (ξ_t) and $(\hat{\xi}_t)$ both contain a random walk with positive mean and a random walk with negative mean. (12) and (13) follow, so the theorem applies. It is interesting to note that Schwartz [8] used the *recurrence* of certain random walks embedded in processes which generalize Examples 2 and 3 as a means to study their ergodic properties.

4. Open problems. We conclude the discussion by mentioning some unresolved questions connected with our results.

PROBLEM 1. Does (5) hold whenever the basic contact process of Example 1 is nonergodic? We conjecture that the answer is yes. Perhaps the easiest way to prove this would be to establish (17). Routine considerations yield

$$P_\xi(\{\liminf_{t\to\infty} M_t = +\infty\} \cup \{\liminf_{t\to\infty} M_t = -\infty, \limsup_{t\to\infty} M_t = +\infty\} \mid \tau = \infty) = 1,$$

but we have been unable to rule out the second possibility.

PROBLEM 2. For Example 2, are $\delta_\varnothing$ and ν the only extreme equilibria, or is there a nontranslation invariant equilibrium? We conjecture that the first alternative holds.

PROBLEM 3. Is (5) satisfied by the d-dimensional versions of Examples 1, 3 and 4, $d > 1$? For this problem we have no compelling intuition one way or the other.

Added in Proof. T. Liggett has solved some of the open problems. Namely, he has confirmed the conjecture in Problem 2, and obtained the same result for the basic contact process. See [7].

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