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ON THE BEHAVIOR OF CHARACTERISTIC FUNCTIONS ON THE REAL LINE[1]

BY STEPHEN J. WOLFE

University of Delaware and Technological University, Eindhoven

This paper is concerned with the following question: if a characteristic function satisfies a certain property at the origin, what can be said about its behavior on the entire real line? If k is an even integer and $f(u)$ is a characteristic function, then the existence of $f^{(k)}(0)$ implies the existence of $f^{(k)}(u)$ for all u. If k is an odd integer, then it is possible to construct a characteristic function $f(u)$ such that $f^{(k)}(0)$ exists but $f^{(k)}(u)$ fails to exist for almost all u. However the existence of $f^{(k)}(0)$, when k is odd, implies that $f(u)$ satisfies a kth order smoothness condition uniformly on the real line and thus $f(u)$ has many of the properties of a characteristic function with a continuous kth derivative. Several other results are obtained that show that if a characteristic function has a property P at 0 then it either has the same property everywhere on the real line or comes close to having the property everywhere.

1. Introduction and summary. Let $g(u)$ be a function and let $\Delta_k{}^t g(u)$ be the kth symmetric difference of g at u, i.e., $\Delta_k{}^t g(u) = \sum_{i=0}^k (-1)^i \binom{k}{i} g[u + (k-2i)t]$. The function $g(u)$ is said to be smooth at u if $\Delta_2{}^t g(u) = o(t)$ as $t \to +0$. More generally, the function $g(u)$ is said to satisfy a smoothness condition s_k at u if $\Delta_{k+1}^t g(u) = o(t^k)$ as $t \to +0$. The function $g(u)$ is said to satisfy condition s_k uniformly on a subset D of the real line if $\Delta_{k+1}^t g(u) = o(t^k)$ as $t \to +0$ uniformly for $u \in D$.

The concept of a smooth function was first introduced by Riemann [6] in conjunction with a problem in trigonometric series. Properties of smooth functions were later studied by Rajchman and Zygmund. A detailed discussion of their work can be found in [10]. It is easy to see that if a function $g(u)$ is differentiable at u_0 then it is smooth at u_0. If $g(u)$ has a derivative of the kth order at u_0 then $g(u)$ satisfies a smoothness condition s_k at u_0. This result can be proved by using l'Hopital's rule and the fact that $\sum_{i=0}^k (-1)^i \binom{k}{i} i^s = 0$ if $1 \leqq s \leqq k-1$. However a function $g(u)$ can satisfy a smoothness condition s_k at u_0 and not have a kth derivative at u_0.

Relationships between the asymptotic behavior of a distribution function and the behavior of its characteristic function near the origin have been studied by the author and others. A description of this work can be found in [7]. This paper will be concerned with the following question: if a characteristic function satisfies a certain property at the origin, what can be said about its behavior

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on the rest of the real line? This work is of importance in probability theory because of its relationship with limit theorems. We give an example here. Let $\{X_i\}$ be a sequence of independent, identically distributed random variables with a distribution function $F(x)$ and a characteristic function $f(u)$. The random variables satisfy the strong law of large numbers if and only if $F(x)$ has an absolute moment of the first order. The random variables satisfy the weak law of large numbers if and only if $f(u)$ has a derivative of the first order at the origin. It follows that if u_0 is an arbitrary number, the random variables will not satisfy the strong law of large numbers if $f'(u_0)$ does not exist. It also follows from a result in this paper that the random variables will not satisfy the weak law of large numbers if either the real or imaginary part of $f(u)$ fails to be differentiable in a neighborhood of u_0.

Let $F(x)$ be a distribution function with a characteristic function $f(u)$. If k is a positive even integer, the existence of $f^{(k)}(0)$ implies the existence of the kth absolute moment of $F(x)$ and thus the existence and continuity of $f^{(k)}(u)$ for all real u. If k is a positive odd integer, the existence of $f^{(k)}(0)$ does not imply the existence of the kth absolute moment of $F(x)$. In a previous note [9] it was shown by the author that if k is a positive odd integer, then it is possible to construct a characteristic function $f(u)$ such that $f^{(k)}(0)$ exists but $f^{(k)}(u_m)$ does not exist for a sequence of numbers $\{u_m\}$ where $u_m \to 0$ as $m \to \infty$. The construction of this example depends on a theorem of Boas [2]. In Section 2 of this paper, a theorem concerning lacunary trigonometric series will be used to show that if k is a positive odd integer then it is possible to construct a characteristic function that has the property that $f^{(k)}(0)$ exists but $f^{(k)}(u)$ fails to exist almost everywhere.

Let k be an odd integer. Although the existence of $f^{(k)}(0)$ does not imply the existence of $f^{(k)}(u)$ for all u, quite a bit can be said about the behavior of such a characteristic function on the real line. It was shown in [7] that if l is a positive even integer and $0 < \lambda < l$ then $1 - F(x) + F(-x) = o(x^{-\lambda})$ as $x \to +\infty$ if and only if $\Delta_l^t f(0) = o(t^\lambda)$ as $t \to 0+$. By a similar argument, it can be shown that if $1 - F(x) + F(-x) = o(x^{-\lambda})$ as $x \to +\infty$ then $\Delta_l^t f(u) = o(t^\lambda)$ as $t \to 0+$ uniformly in u. It follows that if $f(u)$ satisfies the smoothness condition s_k at 0 then $f(u)$ satisfies the condition s_k uniformly on the real line. Thus the existence of $f^{(k)}(0)$ implies that $f(u)$ satisfies condition s_k uniformly on the real line. In other words characteristic functions that have a derivative of the kth order at 0 have many of the properties of characteristic functions that have a continuous derivative of the kth order on the entire real line.

A related result can be obtained using fractional derivatives. The derivative of the αth order of $f(u)$ will be defined to be

$$f^{(\alpha)}(u) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{u} \frac{f(u) - f(x)}{(u-x)^{\alpha+1}}\, dx \qquad \text{if } 0 < \alpha < 1 \text{ and}$$

$$f^{(\alpha)}(u) = \frac{d^n}{du^n}[f^{(\alpha-n)}(u)]$$

if $\alpha \geqq 1$ and n is the largest integer less than or equal to α. This definition of a fractional derivative was originally given by Marchaud [5]. The author has shown [8] that if $F(x)$ has an absolute moment of the λth order then $f^{(\lambda)}(u)$ exists and is continuous on the entire real line. He also gives a formula that expresses the λth absolute moment of $F(x)$ in terms of $f^{(\lambda)}(u)$.

Let k be an odd integer, let $0 < \lambda < k$, and let $F(x)$ be a distribution function with a characteristic function that satisfies the relationship $\Delta_{k+1}^t f(0) = o(t^k)$ as $t \to +0$. Then $1 - F(x) + F(-x) = o(x^{-k})$ as $x \to \infty$, $F(x)$ has an absolute moment of the λth order, and $f^{(\lambda)}(u)$ exists and is continuous on the entire real line. Thus if k is an odd integer and $f(u)$ is a characteristic function, the existence of $f^{(k)}(0)$ implies that $f^{(\lambda)}(u)$ exists and is continuous for $0 < \lambda < k$.

In Sections 3 and 4 the polynomial approximation of characteristic functions will be discussed. It is well known that if k is an even integer and if the characteristic function $f(u)$ of the distribution function $F(x)$ admits the expansion $f(t) = 1 + \sum_{j=1}^{k-1} C_j t^j/j! + O(t^k)$ as $t \to 0$ then $F(x)$ has an absolute moment of the kth order. A proof of this statement can be found in [4]. The author showed [7] that if k is a positive integer and $k < \lambda < k + 1$ then the characteristic function $f(u)$ admits the expansion $f(t) = 1 + \sum_{j=1}^{k} C_j t^j/j! + o(|t|^\lambda)$ as $t \to 0$ if and only if $1 - F(x) + F(-x) = o(x^{-\lambda})$ as $x \to \infty$. This statement remains true if o is replaced by 0. Boas [2] proved that a characteristic function $f(u)$ has the property that $\int_0^\varepsilon t^{-1}|f(t) - 1|\, dt < \infty$ for some $\varepsilon > 0$ if and only if its distribution function $F(x)$ has the property that $\int_{-\infty}^{\infty} \ln^+ |x|\, dF(x) < \infty$ and that a characteristic function $f(u)$ has the property that $\int_0^\varepsilon t^{-\lambda-1}|f(t) - 1|\, dt < \infty$ for some $\varepsilon > 0$, where $0 < \lambda < 1$ if and only if its distribution function $F(x)$ has the property that $\int_{-\infty}^{\infty} |x|^\lambda\, dF(x) < \infty$. This theorem can be generalized. If k is a nonnegative integer and if $k < \lambda < k + 1$ then $f(u)$ admits an expansion $f(t) = 1 + \sum_{j=1}^{k} C_j t^j/j! + R(t)$ where $\int_0^\varepsilon t^{-\lambda-1}|R(t)|\, dt < \infty$ for some $\varepsilon > 0$ if and only if $\int_{-\infty}^{\infty} |x|^\lambda\, dF(x) < \infty$. If k is a nonnegative even integer then $f(u)$ admits an expansion $f(t) = 1 + \sum_{j=1}^{k} C_j t^j/j! + R(t)$ where $\int_0^\varepsilon t^{-k-1}|R(t)|\, dt < \infty$ for some $\varepsilon > 0$ if and only if $\int_{-\infty}^{\infty} x^k \ln^+ |x|\, dF(x) < \infty$. If k is a positive odd integer then $f(u)$ admits an expansion $f(t) = 1 + \sum_{j=1}^{k} C_j t^j/j! + R(t)$ where $\int_0^\varepsilon t^{-k-1}|R(t)|\, dt < \infty$ if $\int_{-\infty}^{\infty} |x|^k \ln^+ |x|\, dF(x) < \infty$. The converse of the previous statement is true if $F(x) = 0$ for $x < 0$ but is not true in general. By using arguments similar to those used in [7], it is possible to prove the following three theorems:

THEOREM 1. *Let k be a positive even integer. If the characteristic function $f(u)$ admits the expansion $f(t) = 1 + \sum_{j=1}^{k-1} C_j t^j/j! + O(t^k)$ as $t \to 0$ then*

$$f(u + t) = \sum_{j=0}^{k} f^{(j)}(u)t^j/j! + o(t^k) \qquad as \quad t \to 0$$

uniformly for all real u.

THEOREM 2. *Let k be a positive integer and let $k < \lambda < k + 1$. If the characteristic function $f(u)$ admits an expansion*

$$f(t) = 1 + \sum_{j=1}^{k} C_j t_j/j! + o(|t|^\lambda) \qquad as \quad t \to 0$$

then

$$f(u+t) = \sum_{j=0}^{k} f^{(j)}(u)t^j/j! + o(|t|^\lambda) \qquad as \quad t \to 0$$

uniformly for all real u. This statement remains true if o is replaced by 0.

THEOREM 3. *Let λ be a positive real number that is not an odd integer and let k be the largest integer that is less than or equal to λ. If the characteristic function $f(u)$ admits the expansion*

$$f(t) = 1 + \sum_{j=1}^{k} C_j t^j/j! + R(t)$$

where $\int_0^\varepsilon t^{-\lambda-1}|R(t)|\,dt < \infty$ for some $\varepsilon > 0$ then

$$f(u+t) = \sum_{j=0}^{k} f^{(j)}(u)t^j/j! + S(u, t)$$

where corresponding to every $C>0$ there exists an $\varepsilon'>0$ such that $\int_0^{\varepsilon'} t^{-\lambda-1}|S(u,t)|dt<C$ for all u.

The author has studied the relationship between the asymptotic behavior of distribution functions and the behavior of symmetric differences of their characteristic functions at the origin. It was shown in [7] that if $0 < \lambda < l$ where l is a positive even integer then $\Delta_l^t f(0) = o(t^\lambda)$ as $t \to +0$ if and only if $1 - F(x) + F(-x) = o(x^{-\lambda})$ as $x \to +\infty$ and $\int_0^\varepsilon t^{-\lambda-1}|\Delta_l^t f(0)|\,dt < \infty$ for some $\varepsilon > 0$ if and only if $\int_{-\infty}^{\infty} |x|^\lambda\,dF(x) < \infty$. The first relationship is true if o is replaced by 0. By using a similar method of proof, it is possible to obtain the following result.

THEOREM 4. *Let l be a positive even integer, let k be an integer such that $0 < k < l$, and let λ be a real number such that $k \leqq \lambda < l$. If the characteristic function $f(u)$ admits the expansion*

$$f(t) = 1 + \sum_{j=1}^{k} C_j t^j + R(t) \tag{1}$$

where $\Delta_l^t R(0) = o(t^\lambda)$ as $t \to +0$ then

$$f(u+t) = \sum_{j=0}^{k} f^{(j)}(u)t^j/j! + S(u, t) \tag{2}$$

where $\Delta_l^t S(u, 0) = o(t^\lambda)$ as $t \to +0$ uniformly in u. This result remains true if o is replaced by 0. *If $f(u)$ admits an expansion of the form* (1) *where $\int_0^\varepsilon t^{-\lambda-1}|\Delta_l^t R(0)|\,dt < \infty$ then* (2) *holds where corresponding to every $C > 0$ there exists an $\varepsilon' > 0$ such that $\int_0^{\varepsilon'} t^{-\lambda-1}|\Delta_l^t S(u, 0)|\,dt < C$ for all u.*

In the above theorem the symmetric difference of $S(u, t)$ is taken with respect to the second variable. Theorems 1 to 4 will be proved in Section 3.

Let k be an odd integer. In view of the example in Section 2, if a characteristic function $f(u)$ admits the expansion $f(t) = 1 + \sum_{j=1}^{k} C_j t^j/j! + o(t^k)$ as $t \to 0$ then it need not follow that $f(u+t) = 1 + \sum_{j=1}^{k} f^{(j)}(u)t^j/j! + o(t^k)$ as $t \to 0$ for all real u. However it is possible to use theorems concerning smooth functions to obtain weaker results. In Section 4 the following theorems will be proved.

THEOREM 5. *Let k be a positive odd integer and let $f(u)$ be a characteristic function with real part $\phi(u)$ and imaginary part $\psi(u)$. If $f(u)$ satisfies a smoothness condition*

s_k *at* 0 *then there exists everywhere dense subsets* D_1 *and* D_2 *of the real line that both have the power of the continuum such that*

$$\phi(u+t) = \sum_{j=0}^{k} \phi^{(j)}(u)t^k/k! + o(t^k) \qquad as \quad t \to 0$$

uniformly in D_1 *and*

$$\psi(u+t) = \sum_{j=0}^{k} \psi^{(j)}(u)t^k/k! + o(t^k) \qquad as \quad t \to 0$$

uniformly in D_2.

COROLLARY 5. *The conclusion of Theorem* 5 *holds if the characteristic function* $f(u)$ *admits an expansion*

$$f(t) = 1 + \sum_{j=0}^{k} C_j t^j/j! + o(t^k) \qquad as \quad t \to 0\,.$$

THEOREM 6. *Let k be a positive odd integer. If* $f(u)$ *satisfies a smoothness condition* s_k *at the origin* [i.e., *if* $\Delta_{k+1}^t f(u) = o(t^k)$ *as* $t \to 0$] *then*

$$f(u+t) = \sum_{j=0}^{k-1} f^{(j)}(u)t^j/j! + o(t^k \ln t) \qquad as \quad t \to 0+$$

uniformly in u. This theorem remains true if o is replaced by 0.

COROLLARY 6. *Let k be a positive odd integer. If* $f(u)$ *admits an expansion*

$$f(t) = 1 + \sum_{j=1}^{k} C_j t^k + o(t^k) \qquad as \quad t \to 0+$$

then the conclusion of Theorem 6 *holds.*

It should be noted that in Theorem 1 the hypothesis that $f(t) = 1 + \sum_{j=1}^{k-1} C_j t^j/j! + O(t^k)$ as $t \to 0$ can be replaced by the hypothesis that the real part of $f(u)$ satisfies the same requirement. Similar results hold for the other theorems and their corollaries.

If the characteristic function $f(u)$ admits an expansion of the form $f(t) = 1 + \sum_{j=0}^{k-1} C_j t^j/j! + O(t^k)$ as $t \to 0$ where k is a positive even integer then $f(u+t) = \sum_{j=0}^{k} f^{(j)}(u)t^j/j! + o(t^k)$ as $t \to 0$ uniformly in u. In Section 5, an example will be constructed to show that if k is a positive odd integer then it is possible to construct a characteristic function $f(u)$ that has the property that it admits the expansion $f(t) = 1 + \sum_{j=1}^{k-1} C_j t^j/j! + O(t^k)$ as $t \to 0$ but it is not true that $f(u+t) = \sum_{j=0}^{k} f^{(j)}(u)t^j/j! + o(t^k)$ as $t \to 0$ for any value of u.

2. An example. A lacunary trigonometric series is a series of the form $\sum_{k=1}^{\infty} (a_k \cos n_k x + b_k \sin n_k x)$ where the n_k satisfy the inequality $n_{k+1}/n_k > q > 1$ for all k. It can be shown (see [1], Volume II, page 241 or [12], Volume I, page 203) that a lacunary series converges almost everywhere if $\sum_{k=1}^{\infty} (a_k{}^2 + b_k{}^2) < \infty$ and diverges almost everywhere if $\sum_{k=1}^{\infty} (a_k{}^2 + b_k{}^2) = \infty$. It can also be shown that if $A_k \to 0$ and $B_k \to 0$ then the set of points where the series converges is identical to the set of points where the integrated series possesses a derivative and that the integrated series is uniformly smooth on $[0, 2\pi]$. These results will be used to show that if k is a positive odd integer then it is possible to construct a characteristic function $f(u)$ such that $f^{(k)}(0)$ exists but $f^{(k)}(u)$ fails to exist for almost all values of u. Only the case $k = 1$ will be considered as the result can be generalized easily.

Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of numbers such that $n_{k+1}/n_k > q > 1$ for all k. Let $\{P_k\}_{k=-\infty}^{\infty}$ be a sequence of numbers such that $kP_k \to 0$ and $kP_{-k} \to 0$ as $k \to \infty$, $P_k \geqq 0$ for all k, $\sum_{k=-\infty}^{\infty} P_k = 1$, and $P_0 = 0$. Then $f(u) = \sum_{k=1}^{\infty}[(P_k + P_{-k}) \cos n_k u + i((P_k - P_{-k}) \sin n_k u]$ is a characteristic function that is uniformly smooth on the real line. If $\sum_{k=1}^{\infty} n_k^2(P_k - P_{-k})^2 = \infty$ both the real part and imaginary part of $f(u)$ are not differentiable almost everywhere. If $\sum_{k=1}^{\infty} n_k^2(P_k - P_{-k})^2 < \infty$ but $\sum_{k=1}^{\infty} n_k^2(P_k + P_{-k})^2 = \infty$ then the imaginary part of $f(u)$ is differentiable almost everywhere but the real part of $f(u)$ is not differentiable almost everywhere. If $\sum_{k=1}^{\infty} n_k^2(P_k + P_{-k})^2 < \infty$ then both the real part of $f(u)$ and the imaginary part of $f(u)$ are differentiable almost everywhere. Also $f'(0)$ exists if and only if $\lim_{K\to\infty} \sum_{k=1}^{K} n_k(P_k - P_{-k})$ exists.

For example, let $n_k = 2^k$ and let $P_k = P_{-k} = C/2^k k^{\frac{1}{2}}$ where C is properly chosen. Then $f(u)$ is a uniformly smooth real characteristic function that is differentiable at the origin but not differentiable except on a set of measure zero.

3. Proof of Theorems 1—4. Let $F(x)$ be a distribution function with characteristic function $f(t)$, let $G(x) = F(x) - F(-x)$ and let

$$S(u, t) = f(u + t) - \sum_{j=0}^{k} f^{(i)}(u)t^j/j!\ .$$

If

$$f(t) = 1 + \sum_{j=1}^{k-1} C_j t^j/j! + O(t^k) \qquad \text{as} \quad t \to 0\,,$$

where k is an even integer then $F(x)$ has an absolute moment of the kth order. Since

$$|e^{i\theta} - \sum_{j=0}^{n} (i\theta)^j/j!| \leqq \theta^{n+1}/(n + 1)!$$

(see [3], page 512) it follows that

$$\begin{aligned} |S(u, t)| &\leqq \int_0^\infty |e^{ixu}[e^{ixt} - \sum_{j=0}^{k} (ixt)^j/j!]|\, dG(x) \\ &\leqq \int_0^\infty |e^{ixt} - \sum_{j=0}^{k-1} (ixt)^j/j!|\, dG(x) + \int_0^\infty (xt)^k/k!\, dG(x) \\ &= o(t^k) \qquad \text{as} \quad t \to 0 \end{aligned}$$

and Theorem 1 is true.

If k is a positive integer, $k < \lambda < k + 1$, and $f(u)$ admits an expansion $f(t) = 1 + \sum_{j=1}^{k} C_j t^j/j! + o(|t^\lambda|)$ as $t \to 0$. Then it follows from Theorem 3 of [7] that $1 - F(x) + F(-x) = o(x^{-\lambda})$ as $x \to \infty$. An argument similar to that used in the proof of Theorem 1 can be used to prove Theorem 2.

Let l be a positive even integer, let k be an integer such that $0 < k < l$, and let λ be a real number such that $k \leqq \lambda < l$. If

$$f(t) = 1 + \sum_{j=1}^{k} C_j t^j/j! + R(t) \tag{3}$$

where $\int_0^\varepsilon t^{-\lambda-1}|R(t)|\, dt < \infty$ for some $\varepsilon > 0$ then it follows easily that $\int_0^\varepsilon t^{-\lambda-1}|\Delta_t{}^t R(0)|\, dt < \infty$ and thus $F(x)$ has an absolute moment of the λth order by Theorem 2 of [7]. If $f(u)$ has an expansion of the form (3) where $\Delta_l{}^t R(0) = o(t^\lambda)$ as $t \to +0$ then it follows from Theorem 1 of [7] that $1 - F(x) + F(-x) = o(x^{-\lambda})$ as $x \to \infty$.

Assume that the hypothesis of Theorem 3 holds. Two cases must be considered. If λ is not an integer and k is the largest integer less than λ then

$$\int_0^{\varepsilon'} t^{-\lambda-1}|S(u,t)|\,dt \leqq \int_0^{\varepsilon'} t^{-\lambda-1}\Big[\int_0^{\infty} \Big|e^{itx} - \sum_{j=0}^k ((itx)^j/j!)\Big|\,dt\Big]\,dG(x)$$
$$= \int_0^{\infty} x^{\lambda}\Big[\int_0^{\varepsilon' x} \Big|\Big(e^{iy} - \sum_{j=0}^k ((iy)^j/j!)/y^{\lambda+1}\Big)\Big|\,dy\Big]\,dG(x)\,.$$

If $h(y) = (e^{iy} - \sum_{j=0}^k ((iy)^j/j!)/y^{\lambda+1}$, it follows from Feller's lemma that $|h(y)| < y^{k-\lambda}/k!$ for small y, Thus $\int_0^\infty |h(y)|\,dy < \infty$ and the conclusion of Theorem 3 follows.

If the hypothesis of Theorem 3 holds, where k is an even integer and $\lambda = k$ then

$$\int_0^{\varepsilon} t^{-k-1}|R(t)|\,dt = \int_0^{\varepsilon} t^{-k-1}\Big|\int_{-\infty}^{\infty} e^{itx} - \sum_{j=0}^k ((itx)^j/j!)\,dF(x)\Big|\,dt$$
$$= \int_0^{\infty} x^k\Big|\int_0^{\varepsilon x} \Big(e^{iy} - \sum_{j=0}^k ((iy)^j/j!)y^{k+1}\Big)\Big|\,dy\,dG(x)\,.$$

But there exists a constant $A > 0$ such that

$$\Big|\Big(e^{iy} - \sum_{j=0}^k (iy)^j\Big)/y^{k+1}\Big| > A/y \qquad \text{for large} \quad y\,.$$

Thus it follows that

$$\int_{-\infty}^{\infty} x^k \ln^+ x\,dF(x) < \infty\,.$$

The conclusion of Theorem 3 now follows from an argument similar to that used in the proof of the first case.

Assume that l is a positive even integer, k is an integer such that $0 < k < l$, and λ is a real number such that $k \leqq \lambda < l$. If the characteristic function $f(u)$ admits the expansion

$$f(t) = 1 + \sum_{j=1}^k C_j t^j + R(t)$$

where $\Delta_l{}^t R(0) = o(t^\lambda)$ as $t \to +0$ then

$$|\Delta_l{}^t S(u,0)| = |\Delta_l{}^t f(u)| = 2^n\Big|\int_{-\infty}^{\infty} e^{ixu}(\sin xt)^l\,dF(x)\Big| \leqq 2^n \int_{-\infty}^{\infty} (\sin xt)^l\,dF(x)$$
$$= \Delta_l{}^t f(0) = \Delta_l{}^t R(0)\,.$$

Thus it follows that $\Delta_l{}^t S(u,0) = o(t^\lambda)$ as $t \to +0$. The proof of the second part of Theorem 4 is similar.

4. Proof of Theorems 5 and 6. Rajchman proved that if a function $g(x)$ is continuous and smooth in an interval (a, b) then the derivative $g'(x)$ exists and is finite on an everywhere dense set of points of that interval. This result was strengthened by Zalcwasser who showed that if $g(x)$ is continuous and smooth on (a, b) then the set of points of differentiability of $g(x)$ is of the power of the continuum in every subinterval of (a, b) This proof, that can be found on page 45 of Volume I of [10], can be altered to show that if $g(x)$ is continuous and uniformly smooth in an interval (a, b) then there exists an everywhere dense subset D that is of the power of the continuum in every subinterval of (a, b) such that $g(u + t) = g(u) + g'(u) + o(t)$ as $t \to 0$ uniformly for all u in D. The result can be used to prove Theorem 5.

Assume that $f(u)$ is smooth at the origin. Then $1 - F(x) + F(-x) = o(x^{-1})$

as $x \to +\infty$ and $f(u)$ is uniformly smooth on the real line. It follows that both the real and imaginary parts of $f(u)$ are uniformly smooth on the real line and that the conclusion of Theorem 5 holds.

Suppose that $f(u)$ satisfies a smoothness condition s_k at the origin where k is an odd integer greater than one. Let $g(u) = f^{(k-1)}(u)/f^{(k-1)}(0)$ and let $G(x)$ be the distribution function that has characteristic function $g(u)$. The fact that $f(u)$ satisfies a smoothness condition s_k at 0 implies that $1 - F(x) + F(-x) = o(x^{-k})$ as $x \to +\infty$ and thus $1 - G(x) + G(-x) = o(x^{-1})$ as $x \to +\infty$ and $g(u)$ is uniformly smooth on the real line. Thus there exists an everywhere dense subset D_1 of the real line that is of the power of the continuum in every subinterval such that $\phi^{(k-1)}(u + t) = \phi^{(k-1)}(u) + \phi^{(k)}(u)t + o(t)$ as $t \to 0$ uniformly for u in D_1 where $\phi(u)$ is the real part of $f(u)$. It follows that $\phi(u + t) = \sum_{j=0}^{k} \phi^{(j)}(u)t^j/j! + o(t^k)$ as $t \to 0$ uniformly for u in D_1. A similar result holds for the imaginary part of $f(u)$.

If the characteristic function $f(u)$ admits an expansion $f(t) = 1 + \sum_{j=1}^{k} C_j t^j/j! + o(t^k)$ as $t \to 0$ then it follows that $f(u)$ satisfies the smoothness condition s_k at 0. Thus Corollary 5 follows immediately.

It is shown on page 44 of Volume 1 of [12] that if $g(x)$ is continuous and $\Delta_2^t g(u) = o(t)$ as $t \to 0$ uniformly for u in $[a, b]$ then $\omega(h, g) = o(h \ln h)$ as $h \to 0$ where $\omega(h, g) = \sup |g(x_1) - g(x_2)|$ for $x_1 \in [a, b]$, $x_2 \in [a, b]$ and $|x_2 - x_1| \leqq h$. This result also is true if o is replaced by 0. The theorem can be generalized to the case where g is continuous, bounded, and uniformly smooth on $(-\infty, \infty)$. This result yields Theorem 6 when $k = 1$. An argument similar to that used in the proof of Theorem 5 can be used to obtain Theorem 6 when k is an odd integer greater than 1.

5. Another example. In this section it will be shown that it is possible to construct a characteristic function $f(u)$ such that $f(u)$ admits an expansion of the form $f(t) = 1 + \sum_{j=1}^{k-1} C_j t^j/j! + O(t^k)$ as $t \to 0$ but it is not true that $f(u + t) = \sum_{j=0}^{k} f^{(j)}(u)t^j/j! + o(t^k)$ as $t \to 0$ for any value of t. Only the case $k = 1$ need be considered.

Zygmund [11] proved that if $f(u)$ is a characteristic function of a distribution function $F(x)$ such that $f(u)$ is smooth at 0, then $f'(0)$ exists if and only if $\lim_{t\to\infty} \int_{-t}^{t} x\, dF(x)$ exists. A similar method of proof can be used to show that if $f(t) + f(-t) - 2 = O(t)$ as $t \to 0$ then $\limsup_{t\to 0} |(f(t) - 1)/t| < \infty$ if and only if $\limsup_{t\to\infty} |\int_{-t}^{t} x\, dF(x)| < \infty$. It follows that a characteristic function $f(u)$ admits an expansion of the form $f(t) = 1 + O(t)$ as $t \to 0$ if and only if $f(t) + f(-t) - 2 = O(t)$ as $t \to 0$ and $\limsup_{t\to\infty} |\int_{-t}^{t} x\, dF(x)| < \infty$.

If $0 < \alpha \leqq 1$ and b is an integer greater than one the Weierstrass function $g(x) = \sum_{n=1}^{\infty} b^{-n\alpha} \cos b^n x$ is nowhere differentiable on the real line and has the property that $g(t) + g(-t) - 2g(0) = O(t)$ as $t \to 0$. The function $f(u) = g(u)/g(0)$ is the characteristic function of a symmetric distribution function and thus admits an expansion of the form $f(t) = 1 + O(t)$ as $t \to 0$ but does not have the property that $f(u + t) = f(u) + f'(u)t + o(t)$ as $t \to 0$ for any value of u.

If $f(u)$ admits an expansion of the form $f(t) = 1 + \sum_{j=1}^{k-1} C_j t^j/j! + O(t^k)$ as $t \to 0$ where k is odd then $\Delta_{k+1}^t f(0) = O(t^k)$ as $t \to 0$ and thus $\Delta_{k+1}^t f(u) = O(t^k)$ as $t \to 0$ uniformly in u. It follows that even though $f(u)$ does not have a kth derivative at any point each of the four Dini derivatives of $f^{(k-1)}(u)$ exist on nowhere dense subsets of the real line that are of the power of the continuum in any subinterval.

6. An unanswered question. Let $f(u) = \phi(u) + i\psi(u)$ be a characteristic function, let D_1 be the set of points at which $\phi(u)$ is differentiable, and let D_2 be the set of points at which $\psi(u)$ is differentiable. It has been shown that if $f(u)$ has a derivative at the origin then both D_1 and D_2 are everywhere dense sets that have the power of the continuum. Is it possible to construct a characteristic function that has the property that the intersection of D_1 and D_2 consists only of the origin?

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DEPARTMENT OF MATHEMATICS
223 SHARP LABORATORY
UNIVERSITY OF DELAWARE
NEWARK, DELAWARE 19711