The Annals of Probability
1978, Vol. 6, No. 4, 523-531

UNIMODALITY OF INFINITELY DIVISIBLE DISTRIBUTION FUNCTIONS OF CLASS L

By Makoto Yamazato

University of Tsukuba

It is shown that all infinitely divisible distribution functions of class L are unimodal.

1. Introduction and results. A distribution function $F(x)$ is said to belong to the class L, or be an L distribution function, if there exists a sequence of independent random variables $\{X_n\}_{n\geqq 1}$ such that for suitably chosen constants $B_n > 0$ and A_n the distribution functions of the sums

$$Y_n = B_n^{-1} \sum_{k=1}^n X_k - A_n$$

converge to $F(x)$ and the random variables

$$X_{nk} = X_k/B_n \qquad 1 \leqq k \leqq n$$

are asymptotically constant (see Gnedenko–Kolmogorov (1954)). Obviously an L distribution function $F(x)$ is infinitely divisible and thus the logarithm of its characteristic function $\hat{F}(t)$ is written in the Lévy–Khintchine formula. Lévy (1937) found that, in order that a distribution function $F(x)$ belongs to the class L, it is necessary and sufficient that

$$\begin{aligned}\log \hat{F}(t) = i\gamma t - \sigma^2 t^2/2 \\ + \int_{-\infty}^{0-} (e^{itu} - 1 - itu/(1+u^2))|u|^{-1} l(u)\, du \\ + \int_{0+}^{\infty} (e^{itu} - 1 - itu/(1+u^2)) u^{-1} k(u)\, du\end{aligned} \tag{1}$$

where

$$k(u),\ l(u) \geqq 0\,, \qquad \int_{-1}^{0-} |u| l(u)\, du + \int_{0+}^{1} u k(u)\, du < \infty\,,$$
$$\int_1^\infty u^{-1} k(u)\, du + \int_{-\infty}^{-1} |u|^{-1} l(u)\, du < \infty$$

and $-l(u)$ and $k(u)$ are nonincreasing.

A distribution function $F(x)$ is said to be *unimodal with mode m* if $F(x)$ is convex for $x < m$ and concave for $x > m$. $F(x)$ is said to be *unimodal* if, for some m, it is unimodal with mode m.

The purpose of this paper is to prove the following

Theorem 1. *All distribution functions of the class L are unimodal.*

Gnedenko and Kolmogorov asserted this theorem in the original Russian edition (1949) of their book. Their proof of this assertion depended on a theorem of Lapin which stated that the convolution of two unimodal distribution

Received March 10, 1977.

AMS 1970 *subject classification.* Primary 60E05.

Key words and phrases. L distribution function, log concave function, unimodal distribution function.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Probability.* www.jstor.org

functions with mode zero is unimodal with mode zero. But Chung (1953) pointed out that Lapin's theorem is incorrect, and constructed a counterexample (see Chung's translation of the Gnedenko–Kolmogorov book). After this, Ibragimov (1957) asserted that there exist L distributions that are not unimodal. However, Sun (1967) pointed out that Ibragimov's examples are in fact unimodal. So it has been unknown whether all distribution functions of the class L are unimodal or not.

So far, some partial results have been obtained. Wintner (1956) showed that every symmetric L distribution function is unimodal. Ibragimov and Chernin (1959) asserted that the stable laws, which form a subclass of the class L, are unimodal, but their proof contained an error (Kanter (1976)). Zolotarev (1963) showed that an L distribution function is unimodal if $\sigma^2 = 0$ and $k(0+) + l(0-) \leqq 1$ in (1). Later, Wolfe (1971 b) showed that an L distribution function is unimodal if its Lévy measure is one-sided (that is, if either $k(u)$ or $l(u)$ identically vanishes). He showed also the unimodality in the case where $l(0-) \leqq 1$ and $k(0+) \leqq 1$. Yamazato (1975) showed the unimodality in the case where $l(0-) \leqq 1$ and $k(0+) \leqq 2$ (or $l(0-) \leqq 2$ and $k(0+) \leqq 1$).

On the other hand, Ibragimov (1956) introduced a concept of strong unimodality (he called a distribution function strongly unimodal if its convolution with every unimodal distribution function is unimodal) and found a necessary and sufficient condition for strong unimodality. His result is that a distribution function is strongly unimodal if and only if it has a log concave density (that is, the logarithm of the density is concave).

The proof of our Theorem 1 consists of two parts. The first part is to relax the abovementioned condition of Ibragimov, under which the convolution of two unimodal distribution functions is unimodal. We say that a function $f(x)$ is log concave on an interval I if $0 < f(x) < \infty$ on I and $\log f(x)$ is concave on I. We consider two unimodal distributions, one of which is supported on $[0, \infty)$ and the other on $(-\infty, 0]$. We will prove that, if they have densities that are log concave on the intervals between the modes and 0, then their convolution is unimodal under weak additional conditions (Lemma 1). The second part is to check that a large class of one-sided L distribution functions satisfy the condition of Lemma 1, and that every L distribution function is the limit of convolutions of such distribution functions.

As an additional result, we show that all L distribution functions with one-sided Lévy measures have the above mentioned property.

THEOREM 2. *Let $f(x)$ be the density of an L distribution function $F(x)$. If $l \equiv 0$ and $\sigma^2 = 0$ in* (1), *then $F(x)$ has a mode a such that $f(x)$ is log concave on $(c, a]$ where $c = \inf\{x; f(x) > 0\}$. Similarly if $k \equiv 0$ and $\sigma^2 = 0$, then $F(x)$ has a mode a such that $f(x)$ is log concave on $[a, d)$ where $d = \sup\{x; f(x) > 0\}$.*

2. Proof of Theorem 1. A distribution function $F(x)$ is unimodal with mode a if and only if $F(x)$ is absolutely continuous on $(-\infty, a) \cup (a, \infty)$ and the density has a version nondecreasing in $(-\infty, a)$ and nonincreasing in (a, ∞). This follows easily from the definition. We always take this version of the density.

LEMMA 1. *Let $G(x)$ and $H(x)$ be unimodal distribution functions such that $G(0) = 0$ and $H(0) = 1$, and let*

$$F(x) = (G * H)(x) = \int G(x - y)\, dH(y)\,.$$

Let a and b be modes of $G(x)$ and $H(x)$, respectively. Suppose that $G(x)$ and $H(x)$ are absolutely continuous with respective densities $g(x)$ and $h(x)$. If $a > 0$, we assume that $g(x)$ is log *concave on $(0, a]$, $g(a) = g(a-) \geqq g(a+)$ and $g(0+) = 0$. If $b < 0$, we assume that $h(x)$ is* log *concave on $[b, 0)$, $h(b) = h(b+) \geqq h(b-)$ and $h(0-) = 0$. Then $F(x)$ is unimodal.*

PROOF. There are five cases: (1) $a = b = 0$, (2) $a > 0 > b$ and $a + b \geqq 0$, (3) $a > 0 > b$ and $a + b < 0$, (4) $a > 0 = b$, (5) $a = 0 > b$. Let $f(x) = \int_0^\infty h(x - y)g(y)\, dy$. This is a density of $F(x)$. It is easy to prove that $f(x)$ is nondecreasing on $(-\infty, b)$ and nonincreasing on (a, ∞). Thus, in Case 1, $F(x)$ is clearly unimodal.

CASE 2. We assume $g(x)$ and $h(x)$ are absolutely continuous on $(0, \infty)$ and $(-\infty, 0)$ respectively. Let $g'(x)$ and $h'(x)$ be their Radon–Nikodym derivatives. Then $f(x)$ has a continuous derivative $f'(x)$. We will use the following expression of it:

$$f'(x) = \int_{-\infty}^{b} h'(y)g(x - y)\, dy + \int_b^{0\wedge x} h'(y)g(x - y)\, dy\,, \qquad x > b\,. \tag{2}$$

We prove the following: (i) *If $f'(x) \leqq 0$ for some x in $[0, a + b)$, then $f'(y) \leqq 0$ for all y in $(x, a + b)$.* (ii) *If $f'(x) \geqq 0$ for some x in $(b, a + b]$, then $f'(y) \geqq 0$ for all y in $[b, x)$.*

Let

$$A_\varepsilon(x) = g(x + \varepsilon)/g(x) \quad \text{if } g(x) > 0$$
$$= 0 \quad \text{if } g(x) = 0\,.$$

Since $g(x)$ is continuous on $[0, \infty)$, $A_\varepsilon(x)$ is continuous on $(0, \infty)$. Since $g(x)$ is log concave on $(0, a]$, $A_\varepsilon(x)$ is nonincreasing for $0 \leqq x \leqq a - \varepsilon$. Since $g(x)$ is nondecreasing on $[0, a]$ and nonincreasing on $[a, \infty)$, we see that $A_\varepsilon(x)$ is nonincreasing also on $a - \varepsilon \leqq x \leqq a$ and $A_\varepsilon(x) \leqq 1$ for $x > a$ and $A_\varepsilon(x) \geqq 1$ for $x \leqq a - \varepsilon$.

Let $f'(x_0) \leqq 0$ for x_0 in $[0, a + b)$. From (2), we have

$$f'(x_0 + \varepsilon) = \int_{-\infty}^{b} h'(y)A_\varepsilon(x_0 - y)g(x_0 - y)\, dy + \int_b^0 h'(y)A_\varepsilon(x_0 - y)g(x_0 - y)\, dy\,. \tag{3}$$

Noting that $A_\varepsilon(x)$ is continuous and nonnegative on $(0, \infty)$, that $h'(x)$ has no change of sign on each of $(-\infty, b)$ and $(b, 0)$ and that $h'(y)g(x_0 - y)$ is integrable, we have, from (3)

$$f'(x_0 + \varepsilon) = A_\varepsilon(x_0 - \xi_1) \int_{-\infty}^{b} h'(y)g(x_0 - y)\, dy + A_\varepsilon(x_0 - \xi_2) \int_b^0 h'(y)g(x_0 - y)\, dy \tag{4}$$

where $x_0 \leqq x_0 - \xi_2 \leqq x_0 - b < a$ and $x_0 - b \leqq x_0 - \xi_1$. If we choose $\varepsilon > 0$ so that $a + b - x_0 > \varepsilon$, then $x_0 - \xi_2 < a - \varepsilon$ and thus $A_\varepsilon(x_0 - \xi_2) \geqq 1$. If $x_0 - \xi_1 < a$, we have

$$A_\varepsilon(x_0 - \xi_2) \geqq A_\varepsilon(x_0 - \xi_1),$$

since $A_\varepsilon(x)$ is nonincreasing on $(0, a)$. If $x_0 - \xi_1 \geqq a$, we also have

$$A_\varepsilon(x_0 - \xi_2) \geqq 1 \geqq A_\varepsilon(x_0 - \xi_1).$$

Therefore, comparing (4) with the expression (2) of $f'(x_0)$, we obtain $f'(x_0 + \varepsilon) \leqq 0$ for $0 < \varepsilon < a + b - x_0$. Thus (i) is true.

Let $f'(x_0) \geqq 0$ for some x_0 in $(b, a + b]$. For $\varepsilon < x_0 - b$, we have from (2)

$$f'(x_0 - \varepsilon) = A_\varepsilon(x_0 - \varepsilon - \xi_1)^{-1} \int_{-\infty}^{b} h'(y)g(x_0 - y)\, dy + A_\varepsilon(x_0 - \varepsilon - \xi_2)^{-1} \int_{b}^{x_0-\varepsilon} h'(y)g(x_0 - y)\, dy \tag{5}$$

where $0 \leqq x_0 - \xi_2 - \varepsilon \leqq x_0 - b - \varepsilon \leqq x_0 - \xi_1 - \varepsilon$ and $x_0 - b - \varepsilon < a - \varepsilon$. Thus, by a similar argument, we have $f'(x_0 - \varepsilon) \geqq 0$ for $0 < \varepsilon < x_0 - b$ and this proves (ii).

Since the assumptions are symmetric, we obtain from (ii) the following: (iii) *If* $f'(x_0) \leqq 0$ *for some* x_0 *in* $[a + b, a)$, *then* $f'(x_0 + \varepsilon) \leqq 0$ *for* $\varepsilon > 0$. (We did not use $a + b \geqq 0$ in proving (ii).) The unimodality of $F(x)$ is easily shown by (i), (ii) and (iii).

Now, drop the assumption of absolute continuity of $g(x)$ and $h(x)$. $g(x)$ is absolutely continuous on $(0, a]$, since $g(x)$ is log concave on $(0, a]$. Thus using $g(a-) \geqq g(a+)$, we can find a sequence $\{G_n(x)\}$ of absolutely continuous distribution functions with mode a and density $g_n(x)$ such that $G_n(x)$ converges to $G(x)$ as $n \to \infty$, $g_n(x)$ coincides with $g(x)$ on $(0, a]$, $g_n(x)$ is a nonincreasing step function on $[a, \infty)$ and $g_n(a-) = g_n(a) = g_n(a+)$. For each $G_n(x)$, we can choose a sequence $\{G_{nm}(x)\}$ of absolutely continuous unimodal distribution functions with mode a and density $g_{nm}(x)$ so that $G_{nm}(x)$ converges to $G_n(x)$ as $m \to \infty$ and $g_{nm}(x)$ is absolutely continuous on $(0, \infty)$ and coincides with $g(x)$ on $(0, a]$. Similarly we can choose a sequence of absolutely continuous unimodal distribution functions $\{H_{nm}(x)\}$ with mode b and density $h_{nm}(x)$ so that $\lim_{n\to\infty} \lim_{m\to\infty} H_{nm}(x) = H(x)$ and $h_{nm}(x)$ is absolutely continuous on $(-\infty, 0)$ and coincides with $h(x)$ on $[b, 0)$. We see that $(G_{nm} * H_{nm})(x)$ is unimodal and converges to $F(x) = (G * H)(x)$ as $m \to \infty$ and then $n \to \infty$. Since the limit distribution function of a sequence of unimodal distribution functions is unimodal, $F(x)$ is unimodal.

CASE 3. Obvious from the argument of Case 2.

CASE 4. We assume $g(x)$ and $h(x)$ are absolutely continuous on $(0, \infty)$ and $(-\infty, 0)$, respectively and $h(0-) < \infty$. Then $f(x)$ has a continuous derivative and

$$f'(x) = \int_{-\infty}^{0} h'(y)g(x - y)\, dy - h(0-)g(x).$$

If there is some x_0 in $[0, a)$ such that $f'(x_0) \leqq 0$, then we have for $\varepsilon > 0$

$$f'(x_0 + \varepsilon) = A_\varepsilon(x_0 - \xi) \int_{-\infty}^{0} h'(y)g(x_0 - y)\,dy - A_\varepsilon(x_0)h(0-)g(x_0)$$

where $x_0 \leqq x_0 - \xi$, and hence $f'(x_0 + \varepsilon) \leqq 0$ for $0 < \varepsilon < a - x_0$. Here we used $A_\varepsilon(x_0) \geqq A_\varepsilon(x_0 - \xi)$, which follows from the same argument as in Case 2. Hence $F(x)$ is unimodal.

When $g(x)$ and $h(x)$ do not satisfy the above assumption, we choose sequences $\{G_{nm}(x)\}$ and $\{H_{nm}(x)\}$ as in Case 2 with $h_{nm}(0-) < \infty$, and get the unimodality of $F(x)$.

CASE 5. Obvious from 4. This completes the proof of Lemma 1.

REMARK. We can drop the condition in Lemma 1 that $g(0+) = 0$ (if $a > 0$) and $h(0-) = 0$ (if $b < 0$). In this case, we use $f'(0+) = f'(0-) - h(0-)g(0+) \leqq f'(0-)$. The reason why we assumed the above condition is that one-sided L distribution functions that we will consider satisfy the condition, and if we drop it, then the proof of Lemma 1 becomes more complicated.

To prove Theorem 1, we need the following results on L distribution functions.

Let $k(u)$ be a function of the form

$$(6)\qquad \begin{aligned} k(u) &= \lambda_1 + \cdots + \lambda_n && 0 \leqq u < p_1 \\ &= \lambda_2 + \cdots + \lambda_n && p_1 \leqq u < p_2 \\ &\cdots && \cdots \\ &= \lambda_n && p_{n-1} \leqq u < p_n \\ &= 0 && p_n \leqq u \end{aligned}$$

where $\lambda_1, \cdots, \lambda_n > 0$, and let $G(x)$ be the distribution function of class L for which

$$\log \hat{G}(t) = \int_{0+}^{\infty} (e^{itu} - 1)u^{-1}k(u)\,du .$$

Let $\lambda = \lambda_1 + \cdots + \lambda_n$.

(a) *$G(x)$ is absolutely continuous. $G(x)$ has a density $g(x)$ which is continuous except at $x = 0$. $g(x)$ is 0 for $x < 0$ and positive for $x > 0$. It satisfies the equation*

$$(7)\qquad xg'(x) = (\lambda - 1)g(x) - \lambda_1 g(x - p_1) - \cdots - \lambda_n g(x - p_n)$$

except at $x = 0, p_1, \cdots, p_n$ (Wolfe (1971 b), page 913).

(b) *If $\lambda \leqq 1$, then $g(x)$ is nonincreasing for $x > 0$* (Wolfe (1971 b), page 913).

(c) *If $\lambda > 1$, then $G(x)$ is unimodal with mode $a > 0$ and $g(0+) = 0$* (Wolfe (1971 b), pages 914–915).

(d) *If $1 < \lambda \leqq 2$, then $g(x)$ is concave on* $(0, a]$ (Yamazato (1975), page 133).

We prove (d) for completeness. By (7) and $g(0+) = 0$, $g'(x)$ is continuous for $x > 0$. Hence $g''(x)$ exists except at $x = 0, p_1, \cdots, p_n$ and satisfies

$$(8)\qquad xg''(x) = (\lambda - 2)g'(x) - \lambda_1 g'(x - p_1) - \cdots - \lambda_n g'(x - p_n) .$$

Since $g'(x) \geqq 0$ for x in $(0, a)$, (8) shows that $g''(x) \leqq 0$ for $x \leqq a$ except possibly at $x = 0, p_1, \cdots, p_n$. Therefore $g(x)$ is concave on $(0, a]$.

PROOF OF THEOREM 1. Let $G(x)$ be as above. We claim that $G(x)$ has all the properties required in Lemma 1. By virtue of (a), (b), (c), it remains only to check that $g(x)$ is log concave on $(0, a]$ in case $\lambda > 1$. If $1 < \lambda \leqq 2$, this log concavity follows from (d) since concave functions are log concave. Suppose $\lambda > 2$. Since $g(x) = cx^{\lambda-1}$ $(c > 0)$ for $0 \leqq x \leqq p_1$, $g'(x)$ is continuous on the whole line. Hence we have (8) for all $x > 0$, and $g''(x)$ is continuous for $x > 0$. From (7) and (8), we have

$$
\begin{aligned}
&x(g'(x)^2 - g(x)g''(x)) \\
&\quad = g(x)g'(x) + \lambda_1(g(x)g'(x-p_1) - g'(x)g(x-p_1)) + \cdots \\
&\qquad + \lambda_n(g(x)g'(x-p_n) - g'(x)g(x-p_n)) \,.
\end{aligned} \tag{9}
$$

Let $B(x) = g'(x)^2 - g(x)g''(x)$. Obviously $B(x)$ is continuous on $(0, \infty)$ and $B(x) > 0$ for $0 < x \leqq p_1$. Let us prove $B(x) > 0$ for $p_1 < x < a$. Suppose $B(x) \leqq 0$ for some x in $(0, a)$. Let $B(x_0) = 0$ and $B(x) > 0$ for $0 < x < x_0$. We have $p_1 < x_0 < a$ and $g'(x_0)g(x_0) \geqq 0$. Let us consider two cases.

CASE 1. $g(x_0)g'(x_0 - p_i) - g'(x_0)g(x_0 - p_i) < 0$ for some i.

CASE 2. $g(x_0)g'(x_0 - p_i) - g'(x_0)g(x_0 - p_i) \geqq 0$ for all i.

In Case 1, we have $x_0 > p_i$ and $g(x_0 - p_i) > 0$ and hence

$$g'(x_0 - p_i)/g(x_0 - p_i) - g'(x_0)/g(x_0) < 0 \,.$$

By the mean value theorem, we have for some x_1 in $(x_0 - p_i, x_0)$

$$\left(\frac{g'}{g}\right)'(x_1) > 0 \,,$$

that is, $B(x_1) < 0$. This contradicts the choice of x_0. In Case 2, it follows from (9) that

$$g(x_0)g'(x_0 - p_i) - g'(x_0)g(x_0 - p_i) = 0 \qquad \text{for all } i \,.$$

Since $p_1 < x_0$, we have $g(x_0 - p_1) > 0$ and thus

$$g'(x_0 - p_1)/g(x_0 - p_1) - g'(x_0)/g(x_0) = 0 \,.$$

Using the mean value theorem again, we get $B(x_1) = 0$ for some x_1 in $(x_0 - p_1, x_0)$, which contradicts the choice of x_0. This completes the proof that $g(x)$ is log concave on $(0, a]$.

Let $l(u)$ be a function of the form

$$
\begin{aligned}
l(u) &= \mu_1 + \cdots + \mu_m && q_1 < u \leqq 0 \\
&= \mu_2 + \cdots + \mu_m && q_2 < u \leqq q_1 \\
&\cdots && \cdots \\
&= \mu_m && q_m < u \leqq q_{m-1} \\
&= 0 && u \leqq q_m
\end{aligned}
$$

where $\mu_1, \cdots, \mu_m > 0$ and let $H(x)$ be the distribution function for which

$$\log \hat{H}(t) = \int_{-\infty}^{0-} (e^{itu} - 1)|u|^{-1}l(u)\, du .$$

Then, by an analogous argument, we see that $H(x)$ has all the properties required in Lemma 1. Hence, by Lemma 1, $(G * H)(x)$ is unimodal. It follows that, if $F(x)$ is the L distribution function for which

$$\log \hat{F}(t) = \int_{-\infty}^{0-} (e^{itu} - 1 - itu/(1 + u^2))|u|^{-1}l(u)\, du + \int_{0+}^{\infty} (e^{itu} - 1 - itu/(1 + u^2)u^{-1}k(u)\, du ,$$

then $F(x)$ is unimodal. If $F(x)$ is a general L distribution function with $\gamma = 0$ and $\sigma^2 = 0$ in (1), we can choose sequences of monotone step functions $k_n(u)$ and $l_n(u)$ in such a way that $F_n(x)$ with

$$\log \hat{F}_n(t) = \int_{-\infty}^{0-} (e^{itu} - 1 - itu/(1 + u^2))|u|^{-1}l_n(u)\, du + \int_{0+}^{\infty} (e^{itu} - 1 - itu/(1 + u^2))u^{-1}k_n(u)\, du$$

converges to $F(x)$ as $n \to \infty$. It follows that $F(x)$ is unimodal. Since the normal distribution is strongly unimodal, this proves Theorem 1.

3. Proof of Theorem 2. For simplicity we use the notation

$$\Delta_\delta f(x) = f(x + \delta) - f(x) .$$

LEMMA 2. *Let $F(x)$ be unimodal distribution function with mode a. We assume that $F(x)$ is twice continuously differentiable on (c, d) where $c = \inf\{x; F(x) > 0\}$ and $d = \sup\{x; F(x) < 1\}$. Then, $F'(x)$ is* log *concave on (c, a) if and only if*

$$\Delta_\varepsilon \Delta_\delta F(x) \int_y^{y+\varepsilon} \Delta_\delta F(t)\, dt \geqq \Delta_\varepsilon \Delta_\delta F(y) \int_x^{x+\varepsilon} \Delta_\delta F(t)\, dt \tag{10}$$

for all x, y and $\varepsilon, \delta > 0$ which satisfy

$$c + \theta < x + \theta < y < a - \theta$$

where $\theta = \varepsilon + \delta$.

PROOF. If $F'(x)$ is log concave on (c, a), then $F''(x)/F'(x)$ is nonincreasing on (c, a). Thus, we have

$$F''(s)F'(t) \geqq F''(t)F'(s) \qquad \text{for} \quad c < s < t < a . \tag{11}$$

Let $c + \theta < x + \theta < y < a - \theta$ and let $u + \delta < v < a - \delta$. Integrating both sides of (11) with respect to s and t on $[u, u + \delta]$ and $[v, v + \delta]$ respectively, we get

$$\Delta_\delta F'(u)\Delta_\delta F(v) \geqq \Delta_\delta F'(v)\Delta_\delta F(u) . \tag{12}$$

Also integrating both sides of (12) with respect to u and v on $[x, x + \varepsilon]$ and $[y, y + \varepsilon]$ respectively, we have (10).

Conversely, let (10) hold for $c + \theta < x + \theta < y < a - \theta$. Multiplying both sides of (10) by $(\varepsilon\delta)^{-2}$ and letting $\varepsilon \to 0$ and then $\delta \to 0$, we have (11) for $s = x$ and $t = y$. This completes the proof.

PROOF OF THEOREM 2. Note that all one-sided L distribution functions have the properties (b) and (d) in 2. Thus we have nothing to prove if $\lambda = k(0+) \leqq 2$. Let $\lambda > 2$. Then $F(x)$ is twice continuously differentiable (Wolfe (1971 a), page 2070). Let $k_n(x)$ be a function of the form (6) and let $F_n(x)$ be an L distribution function for which

$$\log \hat{F}_n(t) = \int_{0+}^{\infty} (e^{itu} - 1 - itu/(1+u^2))u^{-1}k_n(u)du .$$

We choose $k_n(u)$ so that $F_n(x)$ converges to $F(x)$ as $n \to \infty$. $F_n(x)$ is unimodal with mode a_n and has a density $f_n(x)$ log concave on (c_n, a_n) where $c_n = \inf\{x; F_n(x) > 0\}$. We have $\limsup_{n\to\infty} c_n \leqq c$. Let $\liminf_{n\to\infty} a_n = a$ and let $\{n(k)\}$ be a subsequence such that $a_{n(k)} \to a$ as $k \to \infty$. Obviously, a is a mode of $F(x)$. Let x, y and ε, $\delta > 0$ be numbers for which

$$c + \theta < x + \theta < y < a - \theta$$

where $\theta = \varepsilon + \delta$. We can choose N large enough so that $c_{n(k)} < x$ and $y < a_{n(k)} - \theta$ for $n(k) > N$. Then, $F_{n(k)}(x)$ satisfies (10) if $n(k) > N$. Since $F_{n(k)}(x)$ converges to $F(x)$ at every point, we have by the bounded convergence theorem

$$\int_y^{y+\varepsilon} \Delta_\delta F_{n(k)}(t)\, dt \to \int_y^{y+\varepsilon} \Delta_\delta F(t)\, dt$$

and

$$\int_x^{x+\varepsilon} \Delta_\delta F_{n(k)}(t)\, dt \to \int_x^{x+\varepsilon} \Delta_\delta F(t)\, dt$$

as $k \to \infty$. Thus, $F(x)$ satisfies (10) whenever $c + \theta < x + \theta < y < a - \theta$. By Lemma 2 we have the log concavity of $f(x)$ on (c, a).

EXAMPLE. Let $f(x) = C(2\pi)^{-\frac{1}{2}}x^{-\frac{3}{2}} \exp(-C^2/2x)$ for $x > 0$ where $C > 0$. Then $f(x)$ is the density of a one-sided stable distribution function of exponent $\frac{1}{2}$. The distribution function has a mode at $x = C^2/3$ and $f(x)$ is log concave on $(0, 2C^2/3)$.

Acknowledgment. I express my hearty thanks to Professor K. Sato who gave helpful suggestions and advice to me in preparing this article. He pointed out and filled a gap in my argument.

REFERENCES

[1] CHUNG, K. L. (1953). Sur les lois de probabilités unimodales. *C. R. Acad. Sci. Paris* 236: 6 583–584.

[2] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, Cambridge.

[3] IBRAGIMOV, I. A. (1956). On the composition of unimodal distributions. *Theor. Probability Appl.* **1** 255–260.

[4] IBRAGIMOV, I. A. (1957). A remark on probability distributions of class *L*. *Theor. Probability Appl.* **2** 117–119.

[5] IBRAGIMOV, I. A. and CHERNIN, K. E. (1959). On the unimodality of stable laws. *Theor. Probability Appl.* **4** 417–419.

[6] KANTER, M. (1976). On the unimodality of stable densities. *Ann. Probability* **4** 1006–1008.

[7] LÉVY, P. (1937). *Théorie de l'addition des variables aléatoires*. Gauthier-Villars, Paris.

[8] SUN, T. C. (1967). A note on the unimodality of distributions of class *L*. *Ann. Math. Statist.* **38** 1296–1299.

[9] WINTNER, A. (1956). Cauchy's stable distributions and an "explicit formula" of Mellin. *Amer. J. Math.* **78** 819–861.

[10] WOLFE, S. J. (1971 a). On the continuity properties of L functions. *Ann. Math. Statist.* **42** 2064–2073.

[11] WOLFE, S. J. (1971 b). On the unimodality of L functions. *Ann. Math. Statist.* **42** 912–918.

[12] YAMAZATO, M. (1975). Some results on infinitely divisible distributions of class L with applications to branching processes. *Sci. Rep. Tokyo Kyoiku Daigaku Sect. A* **13** 133–139.

[13] ZOLOTAREV, V. M. (1963). The analytic structure of infinitely divisible laws of class L. *Litovsk. Math. Sb.* **3** 123–140.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF THUKUBA
SAKURA-MURA IBARAKI, 300-31
JAPAN