LIMIT THEOREMS FOR SEMI-MARKOV PROCESSES AND
RENEWAL THEORY FOR MARKOV CHAINS

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With any Harris-recurrent Markov chain one can associate a sequence
of random times at which the chain has the same distribution, and the
chain can thereby be shown to be equivalent to one having a recurrence
point. This idea makes available a regeneration scheme for such chains,
which is exploited in this paper to prove the ergodic theorem for semi-
Markov processes, and a renewal theorem for Markov chains on a general
state space.

1. Introduction. The circle of ideas covering the limit theory for recurrent
Markov chains, the renewal theorem, the theory of regenerative events and their
relations, has received much attention in the probability literature. When there
exists a single point in the state space of the Markov chain which is visited in-
finitely often, then much of the limit theory is simplified. One can then prove
the ergodic theorem for the chain via the renewal process of inter-return times
to this distinguished state. This idea also works nicely in the case of a semi-
Markov process. (See, e.g., Cinlar [4], also Kesten [10] for other references.)
Perhaps less well known is the converse direction, namely the proof of the renewal
theorem from the ergodic theorem. Such a result was demonstrated recently by
McDonald [12].

A useful technique which has more recently been applied to prove limit the-
ores for Markov chains is that of coupling a chain having an arbitrary initial
distribution to another stationary one, thereby drawing conclusions about the
limit behavior of the former. The use of this device to prove ergodic theorems
goes back to Doeblin [5], and has recently been developed by Griffiths [7], [8].
(These papers contain further references.) The method can also be used to give
simple proofs of the renewal theorem (Lindvall [11], and Athreya, McDonald
and Ney [1]).

When the chain in question returns i.o. to a point $x_0$, with finite mean recur-
rence time, then the “coupling” is easily accomplished by showing that the
stationary and the general processes will ultimately be at $x_0$ at the same time.
Even when no such recurrence point exists a clever construction shows that the
role of $x_0$ can effectively be played by a distinguished set, and a subsequence of
the sequence of hitting times of that set, at which the two processes are coupled
in such a way that their-marginal distributions are not changed.

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orem, regeneration.
This construction motivated us to observe an even simpler one, which enables us at certain random times to treat the distinguished set as a point, and directly bring to bear the tools of regeneration points and renewal theory. The purpose of this paper is to exploit this construction (introduced in Athreya and Ney [2]) to prove ergodic and renewal theorems for semi-Markov chains. Our hypotheses are in some ways better than presently known; and the proofs are strikingly simpler than current versions, some of which are quite technical.

The regeneration scheme for Markov chains developed in [2] is summarized in Section 2. The ergodic theorem for semi-Markov chains is proved in Section 3, and a general renewal theorem of H. Kesten [10] for functionals of Markov chains is in Section 4. Kesten’s paper contains a good list of references to earlier results of the kind considered here. Particularly relevant are the papers of Orey [13], [14] and Jacod [9]. In Section 5 we observe that the processes considered in this paper are equivalent to processes having a recurrence point, which are constructed by adjoining a point to the state space and suitably modifying the transition function.

2. Regeneration for Markov chains. In [2] we introduced a new approach to the limit theory of Harris-recurrent Markov chains. In this section we summarize the results needed here. Details are in [2].

Let \( \{X_n: n \geq 0\} \) be a Markov chain on a measurable space \((S, \mathcal{F})\) with transition function \(P(x, E)\). We shall say that \((X_n)\) is \((A, \lambda, \varphi, n_0)\)-recurrent if there exists a set \(A \in \mathcal{F}\), a probability measure \(\varphi\) on \(A\), a \(\lambda > 0\) and an integer \(n_0\) such that

\[
\begin{align*}
P_x(X_n \in A & \text{ for some } n \geq 1) \equiv 1 \\
P_x(X_{n_0} \in B) & \geq \lambda \varphi(B) \quad \text{for all } x \text{ in } A \text{ and } B \subseteq A.
\end{align*}
\]

This notion of recurrence is equivalent to the more standard definition of Harris recurrence but is more convenient for our purposes. We will limit ourselves here to the case \(n_0 = 1\). In the rest of the paper we shall merely write “recurrent” when we mean \((A, \lambda, \varphi, 1)\)-recurrent. The following is the key result.

**Regeneration Lemma.** If \(\{X_n\}\) is recurrent then there exists a random time \(N\) such that \(P_x(N < \infty) = 1\) and \(P_x(X_n \in B, N = n) = \varphi(A \cap B)P_x(N = n)\) for all \(B\) in \(\mathcal{F}\), \(x\) in \(S\), and nonnegative integers \(n\).

The idea of the proof is to define a transition function \(Q(x, E) = (P(x, E) - p\varphi(E \cap A))/(1 - p)\) for \(x \in A\). If \(X_n = x \notin A\), distribute \(X_{n+1}\) over \(S\) according to \(P(x, \cdot)\) just as before. However, if \(X_n = x \in A\), then with probability \(p < \lambda\) distribute it over \(A\) according to \(\varphi\), and with probability \((1 - p)\) distribute it over the entire state space \(S\) according to \(Q(x, \cdot)\). (Observe that due to the definition of \(Q\), the transition probabilities for the chain remain unchanged.) Since \(A\) is visited i.o., and each time there is (independent) probability \(p > 0\) that at the next step “\(A\) is entered according to \(\varphi\)”, this event will ultimately occur at some time \(N < \infty\) a.s. (See [2] for details.)
Corollary 2.1. Let \( \{X_n\} \) be recurrent. Then there exists a sequence of random times \( N_i, i = 1, 2, \ldots \) such that \( X_{N_i} \) has distribution \( \varphi \), and the random variables \( \{N_{i+1} - N_i, i = 1, 2, \ldots \} \) are i.i.d. and independent of \( N_i \).

The regeneration lemma can be used to show the existence and uniqueness of an invariant measure for \( \{X_n\} \). Define

\[
\nu(E) = E\varphi(\sum_{i=0}^{N-1} \chi_E(X_i)),
\]

where \( N \) is the first regeneration time as in the lemma. (\( P_x, P_\mu, E, E_\mu \) denote probability measures and expectations when \( X_0 = x \) or has distribution \( \mu \).)

Theorem 2.1. Let \( \{X_n\} \) be recurrent, and \( \nu(\cdot) \) be as in (2.2). Then \( \nu(\cdot) \) is an invariant measure for \( \{X_n\} \), and is unique up to a multiplicative constant. Furthermore \( \nu(\cdot) \) is finite if and only if \( E_\nu N < \infty \).

Corollary 2.2. An invariant probability measure \( \pi(\cdot) \) for \( \{X_n\} \) exists if and only if \( E_\nu N < \infty \), and in that case

\[
\pi(E) = \nu(E)/\nu(S).
\]

Proofs are in [2].

3. Ergodic theorem for semi-Markov processes on general state spaces. Let \( \{X_n: n = 0, 1, 2, \ldots\} \) be a Markov chain on a measurable space \( (S, \mathcal{S}) \) with transition function \( P(x, E) \). Conditioned on a realization \( \{X_n = x_n: n \geq 0\} \), one is given a sequence of independent, nonnegative random variables \( \{L_n\} \), such that the distribution of \( L_n \) depends only on \( x_n \). (The more general case when \( L_n \) also depends on \( x_{n+1} \) can be treated similarly by the methods of this paper if one is willing to add a minorization hypothesis of the form \( P(X_1 \in E, L_0 \in F | X_0 = x) \geq \lambda \varphi(E)G_x(F) \). We believe that this condition can in fact be removed, but this work is not yet complete.) Let \( V_i = L_0 + \cdots + L_{i-1}, i \geq 1, V_0 = 0 \), and consider a continuous time process in which the chain waits in each state a random length of time, defined as follows:

\[
W(t) = (X_n, t - V_n) \quad \text{if} \quad V_n \leq t < V_{n+1}, \quad n = 0, 1, \ldots.
\]

If \( W(t) \equiv (Z(t), A(t)) \) then \( \{Z(t); t \geq 0\} \) is called a semi-Markov process, the \( X_i \)'s the states of the process and the \( L_i \)'s the sojourn times. From its construction it is clear that \( \{W(t); t \geq 0\} \) is a Markov process, and we will now prove an ergodic theorem for this process. Such a result was first proved by Orey [14] using operator theory. More recently Jacod [9] gave a proof using the techniques of space-time harmonic functions (see also Çinlar [3] for background and other references). Kesten [10] proved a renewal theorem for the case when the \( L_n \)'s are not necessarily nonnegative by applying methods somewhat similar to Feller's [6] for the one-dimensional case. Orey [14] and Kesten [10] needed topological structure on \( (S, \mathcal{S}) \) and an invariant probability measure for \( \{X_n\} \), whereas we will need neither of these. Application of the regenerative scheme of the previous section makes possible a simpler proof, and somewhat more transparent hypotheses.
Recall that \( \{X_n; n \geq 0\} \) is assumed to be \((A, \lambda, \varphi, 1)\)-recurrent. The regeneration lemma (see Corollary 2.1) assures us that for any initial distribution of \(X_0\), there exists an infinite sequence of random times \(N_i\), such that \(X_{N_i}\) has distribution \(\varphi\). This in turn, ensures that the process \(\{W(t); t \geq 0\}\) is well defined for all \(t\) and that no explosions can occur in finite time (see Çinlar [3]). We may also take \(\{W(t); t \geq 0\}\) to be a strong Markov process.

Let \(f = S \times [0, \infty) \to \mathbb{R}^+\) be a bounded, measurable function, such that \(f(x, t)\) is continuous in \(t\) for each \(x \in A\), and let

\[
m(t) \equiv E_\varphi(f(W(t))).
\]

By the regeneration lemma it is clear that if we can show that \(\lim_t m(t)\) exists, then the same limit obtains for \(E_\mu(f(W(t)))\), for any distribution \(\mu\) of \(W(0)\). We will see that \(m(t)\) satisfies a one-dimensional renewal equation and apply Feller’s theorem ([6], page 363) to obtain the desired result.

Let

\[
T = \sum_{i=0}^{N-1} L_i,
\]

where \(N\) is as in the regeneration lemma of Section 2. Then by the regeneration property of \(N\) and the strong Markov property of \(\{W(t); t \geq 0\}\), we have,

\[
m(t) = a(t) + \int_0^t m(t - u) \, dG(u),
\]

where \(a(t) = E_\varphi(f(W(t); T > t))\), and \(G(u) = P_\varphi(T \leq u)\). If \(E_\varphi(T) < \infty\), \(G(\cdot)\) is nonlattice, and \(a(\cdot)\) is directly Riemann integrable (d.r.i.), then we can conclude from Feller’s theorem ([6], page 363) that

\[
\lim_{t \to \infty} m(t) = \frac{\int_0^\infty a(t) \, dt}{E_\varphi(T)}.
\]

It thus remains only to determine the hypotheses ensuring the above conditions and to identify the limit in (3.5). To that end note that by definition of \(W(\cdot)\),

\[
a(t) = E_\varphi\{\sum_{i=0}^{N-1} f(X_i, t - V_i); V_i \leq t < V_{i+1}\} = \sum_{i=0}^N E_\varphi\{f(X_i, t - V_i); V_i \leq t < V_{i+1}, N > i\}.
\]

Hence

\[
|a(t_0 + \varepsilon) - a(t_0)| \leq \sup f \cdot P_\varphi\{t_0 < T \leq t_0 + \varepsilon\} + \sum_{i=0}^N E_\varphi\{|f(X_i, t_0 + \varepsilon - V_i) - f(X_i, t_0 - V_i)|; V_i \leq t < V_{i+1}, N > i\}.
\]

If \(t_0\) is a continuity point of \(G(\cdot)\) then the first term on the right side of (3.6) \(\to 0\) as \(\varepsilon \to 0\); and the second term \(\to 0\) by the continuity of \(f(x, \cdot)\), and the dominated convergence theorem. Thus the set of discontinuities of \(a(t)\) is countable, and \(a(t)\) is \(R\)-integrable (in the ordinary sense) on any finite interval. Since furthermore \(a(t) \leq \sup f \cdot (1 - G(t))\), we see that \(a(\cdot)\) is d.r.i.
To evaluate the integral note that
\begin{equation}
\mathbb{E}_\nu f(W(t); T > t) dt = E_\nu(\int_0^\infty f(W(t)) dt) = E_\nu(\sum_{i=0}^{N-1} \int_0^\infty f(X_i, u) du) = E_\nu(\sum_{i=0}^{N-1} m_f(X_i)),
\end{equation}
where
\[ m_f(x) = E(\int_0^\infty f(x, u) du | X_0 = x). \]

If \( \nu(A) \) is as defined in (2.2), then
\begin{equation}
\mathbb{E}_\nu(f(W(t)); T > t) dt = \int_0^\infty m_f(x) \nu(dx) = \int_0^\infty f(x, u) P(L_0 > u | X_0 = x) d\nu(dx).
\end{equation}

Setting \( f \equiv 1 \), (3.8) yields
\begin{equation}
E_\nu(T) = \int_0^\infty m_f(x) \nu(dx) = \int_0^\infty \int_0^\infty P(L_0 > u | X_0 = x) d\nu(dx).
\end{equation}

Since \( \nu(\cdot) \) is an invariant measure for \( \{X_n\} \), \( E_\nu(T) < \infty \) if and only if \( \int m_f(x) \lambda(dx) < \infty \) for any nontrivial invariant measure \( \lambda(\cdot) \). Thus we have proved

**Theorem 3.1.** (Orey–Jacod–Kesten). Assume that:

(i) \( \{X_n\} \) is recurrent;

(ii) \( \int m_f(x) \nu(dx) < \infty \);

(iii) \( P_\nu(T \leq u) \) is nonlattice.

Then for any bounded measurable \( f : S \times [0, \infty) \to R \) such that \( f(x, t) \) is continuous in \( t \) for each \( x \in A \) and any initial measure \( \mu \) on \( S \)
\begin{equation}
\lim_{t \to \infty} E_\nu f(W(t)) = \int_0^\infty \frac{m_f(x) \nu(dx)}{\int_0^\infty m_f(x) \nu(dx)}.
\end{equation}

**Remarks.**

1. A more standard form of the above limit is
\[ \frac{\int_0^\infty \int_0^\infty f(x, u) P(L_0 > u | X_0 = x) d\nu(dx)}{\int_0^\infty \int_0^\infty P(L_0 > u | X_0 = x) d\nu(dx)}. \]

This is clear since \( m_f(x) = \int_0^\infty f(x, u) P(L_0 > u | X_0 = x) d\nu(dx) \).

2. As observed in Section 2 we note that under recurrence, \( \nu(\cdot) \) is a nontrivial invariant measure and is unique up to a multiplicative constant. Thus in the above theorem \( \nu(\cdot) \) can be replaced by any nontrivial invariant measure \( \lambda(\cdot) \).

3. The result can easily be extended to functions \( f \) such that for all \( x \in A \), \( f(x, t) \) is discontinuous in \( t \) on at most a fixed (independent of \( x \)) countable set \( C \).

4. **Kesten's renewal theorem for Markov chains.** The situation here remains the same as in Section 3, except that the requirement that the \( L_n \)'s be nonnegative is dropped. Let \( g(x, t) : S \times R \to R^+ \) be bounded, and continuous in \( t \) for each \( x \in A \). Kesten's theorem [10] is concerned with the limiting behavior of
\begin{equation}
m(x, t) \equiv E_\nu(\sum_0^t g(X_k, t - V_k))
\end{equation}
where \( V_0 = 0, V_k = \sum_{i=0}^{k-1} L_i, k = 1, 2, \ldots \). Let \( N_0 = 0 \), and \( N_1, N_2, \ldots \) be the
sequence of regeneration times for the chain \( \{X_n\} \). Then,

\[
(4.2) \quad m(x, t) = E_p\left( \sum_{j=0}^{N_1-1} g(X_k, t - V_k) \right) + E_p\left( \sum_{j=1}^{N_j+1} g(X_k, t - V_k) \right).
\]

Since the \( N_i \)'s are regeneration times we get

\[
(4.3) \quad m(x, t) = E_p\left( \sum_{j=0}^{N_1-1} g(X_k, t - V_k) \right) + E_p\left( \sum_{j=0}^\infty K(t - V_{N_1} - S_k) \right),
\]

where

\[
K(t) = E_p\left( \sum_{j=0}^{N_1-1} g(X_k, t - V_k) \right), \quad S_0 = 0, \quad S_k = \sum_{j=1}^k (V_{N_{j+1}} - V_{N_j}).
\]

Notice that the random variables

\[
\{V_{N_{j+1}} - V_{N_j}, j = 1, 2, \ldots \}
\]

are i.i.d. and independent of \( V_{N_1} \).

If \( \{S_k\} \) is a nonlattice random walk with finite, nonzero mean and if \( K(\cdot) \) is directly Riemann integrable, then by applying the one-dimensional key renewal theorem (see Remark 4 below) to the random walk \( \{S_k\} \), we could conclude that

\[
E_p\left( \sum_{j=0}^\infty K(t - S_k) \right) \text{ and hence } E_p\left( \sum_{j=0}^\infty K(t - V_{N_1} - S_k) \right) \text{ converge to } (E(S_1))^{-1} \int_0^\infty K(u) \, du.
\]

The nonlattice and moment requirements can be stated as

(i) \( P_p(T \leq u) \) is nonlattice in \( u \), \( T = \sum_{j=0}^{N_1-1} L_i \);

(ii) \( E_p(|T|) < \infty \);

(iii) \( E_p(T) > 0 \).

Sufficient conditions for these are:

(i)' Same as (i);

(ii)' \( \int A E(p|L_0| \mid X_0 = x) \nu(dx) < \infty \);

(iii)' \( \int E(L_0 \mid X_0 = x) \nu(dx) > 0 \), where \( \nu(E) \equiv \sum_{j=0}^{N_1-1} \chi_E(X_i) \) as in Section 2.

Let us examine what is needed for the direct Riemann integrability of \( K(\cdot) \). Observe first that from its definition

\[
K(t) = \sum_{i=0}^\infty E_p\left[ g(X_i, t - V_i); N > i \right],
\]

and arguing as in the proof of Theorem 3.1, the continuity hypotheses on \( g \) implies that \( K(\cdot) \) is continuous, and thus \( R \)-integrable on \([0, t_0]\) for \( t_0 < \infty \). Furthermore for any \( h > 0 \)

\[
(4.4) \quad \sum_{n=-\infty}^{+\infty} \sup_{n k \leq t \leq (n+1) h} K(t) \leq E_p\left( \sum_{n=0}^{+\infty} \sum_{j=-\infty}^{+\infty} \sup_{n k \leq t \leq (n+1) h} g(X_k, t) \right)
= \int_R \left( \sum_{n=-\infty}^{+\infty} \sup_{n k \leq t \leq (n+1) h} g(x, t) \right) \nu(dx).
\]

Thus, \( K(\cdot) \) is directly Riemann integrable if \( g(x, \cdot) \) is continuous for \( x \in A \), and for some \( h > 0 \)

\[
(4.5) \quad \int_S \left( \sum_{n=-\infty}^{+\infty} \sup_{n k \leq t \leq (n+1) h} g(x, t) \right) \nu(dx) < \infty.
\]

Finally we identify the limit by observing that if \( g \geq 0 \), then

\[
\int_R K(t) \, dt = E_p\left( \sum_{j=0}^{+\infty} \int_R g(X_k, t - V_k) \, dt \right)
= E_p\left( \sum_{j=0}^{+\infty} \int_R g(X_k, t) \, dt \right) = \int_S \left( \int_R g(x, t) \, dt \right) \nu(dx).
\]
Applying the above discussion to the positive and negative parts of a measurable function $g$ we have the following

**Theorem 4.1.** (Kesten [10]). Assume that:

(i) $\{X_n: n \geq 0\}$ is recurrent;
(ii) $\int S E(|L_0| \mid X_0 = x) \nu(dx) < \infty$, $\int S E(L_0 \mid X_0 = x) \nu(dx) > 0$, $P_\psi(\sum_{i=0}^{n-1} L_i \leq u)$ is nonlattice in $u$;
(iii) $g(x, t): S \times R \to R$ is bounded, measurable, continuous in $t$ for $x \in A$, and satisfies (4.5).

Then,

(a) $E_\psi(\sum_0^\infty g(X_k, t - V_k)) \to \int S (\int_R g(x, t) dt) \nu(dx) / \int S E(L_0 \mid X_0 = x) \nu(dx),$
(b) $E_\psi(\sum_0^\infty g(X_k, t - V_k))$ converges to the same limit as above, provided

$$E_\psi(\sum_0^{N-1} g(X_k, t - V_k)) \to 0 \quad \text{as} \quad t \to \infty.$$  

A set of sufficient conditions for (4.6) is given by the following

**Proposition 4.1.** Let $g: S \times R \to R$ and $\bar{g}(x) = \sup_t |g(x, t)|$ be measurable and satisfy

(i) $\int S \bar{g}(x) \nu(dx) < \infty$, and
(ii) $g(x, t) \to 0$ as $t \to \infty$ a.e. $\nu$.

Then,

$$E_\psi(\sum_0^{N-1} g(X_k, t - V_k)) \to 0 \quad \text{as} \quad t \to \infty \quad \text{a.e.} \quad \nu.$$  

**Proof.** Let $\phi(y) \equiv E_\psi(\sum_0^{N-1} \bar{g}(X_k))$. By (i)

$$\infty > \int S \bar{g}(x) \nu(dx) = E_\psi(\sum_0^{N-1} \bar{g}(X_k)) \geq E_\psi(\sum_0^{N-1} \bar{g}(X_k); N > r) \geq E_\psi(\sum_0^{N-1} \bar{g}(X_k); N > r) = \int S \phi(y) \nu_r(dy), \quad \nu_r(E) = P_\psi(X_r \in E, N > r).$$

Thus, $\phi(y) < \infty$ a.e. $\nu_r(\cdot)$ for each $r$ and hence a.e. $\nu(\cdot)$. Now note that if $\nu(F) = 0$ then, because $\nu(\cdot)$ is invariant for $P(\cdot)$ and satisfies $\nu(F) = \int_S P^k(y, F) \nu(dy)$ for each $k$,

$$\nu \{ y: P_\psi(X_k \in F \text{ for some } k \geq 1) > 0 \} = 0.$$  

Taking $F^c = \{ y: \phi(y) < \infty, g(y, t) \to 0 \text{ as } t \to \infty \}$, we see that for $y$ in $F^c$

$$\sum_0^{N-1} g(X_k, t - V_k) \to 0 \quad \text{a.e.} \quad P_\psi;$$

and $|\sum_0^{N-1} g(X_k, t - V_k)| \leq \sum_0^{N-1} \bar{g}(X_k)$, which is integrable with respect to $P_\psi$. By the dominated convergence theorem, $E_\psi(\sum_0^{N-1} g(X_k, t - V_k)) \to 0$ for all $y$ in $F^c$.

**Remarks.** 1. If $g(\cdot)$ is of the form $\chi_+(x)\chi_+(t)$ where $\nu(A) < \infty$, and $I$ is a bounded interval then the hypothesis (iii) of the theorem as well as those of the above proposition are automatically satisfied, thus yielding Blackwell's theorem.
for Markov chains: Under hypotheses (i) and (ii) of Theorem 4.1,
\[ E_x[\# \text{ visits to } A \times (t - I) \text{ by } (X_n; V_n)_{n=0}^\infty] \to c|I|\nu(A) \quad \text{a.e. } \nu(\cdot), \]
where
\[ c^{-1} = \int_0^\infty E(L_0|X_0 = x)\nu(dx). \]
A special case of the above, when \( L_t \)'s are nonnegative, is in Jacod [9], in a slightly different form.

2. As in Section 3, we may replace \( \nu(\cdot) \) in Theorem 4.1 and Proposition 4.1 by any nontrivial invariant measure \( \lambda(\cdot) \).

3. As in Remark 4 of Section 3, \( g(x, \cdot) \) can be allowed to have discontinuities on a fixed (independent of \( x \)) countable set.

4. For completeness we state and sketch a proof of the key renewal theorem for the two sided random walk on the line, in the form in which it is used above.

**Theorem 4.2.** Let \( \{S_n : n = 0, 1, 2, \ldots \} \) be a nonlattice random walk with finite positive mean. Let \( K : R \to R \) be a directly Riemann integrable function. Then,
\[ M(t) = E \sum_{n=0}^\infty K(t - S_n) \to \frac{1}{E(S_1)} \int_0^\infty K(u) \, du. \]

**Proof.** Assume without loss of generality that \( K(\cdot) \) is nonnegative. Then,
\[ M(t) = E(\sum_{j=0}^\infty \sum_{M_j+1}^{M_{j+1}} K(t - S_n)) \]
where \( M_0 = 0, M_1, M_2, \ldots \) are successive ladder epochs for \( \{S_n\} \) i.e., \( M_1 = \inf\{n : n > 0, S_n > 0\} \),
\[ M_{k+1} = \inf\{n : n > M_k, S_n > S_{M_k}\} \quad k = 1, 2, \ldots. \]
Let \( f(t) = E(\sum_{n=1}^\infty K(t - S_n)) \). The (ordinary) \( R \)-integrability of \( K \) implies that its set of discontinuities in \([0, t_0)\), for any \( t_0 < \infty \), is a set of measure zero. This implies the same property for \( f \), and hence its \( R \)-integrability in \([0, t_0)\). Furthermore, since \( EM_1 < \infty \) and \( K(\cdot) \) is d.r.i.
\[ \sum_{n=1}^\infty h \sup_{n \leq t < (n+1)h} f(t) \leq \left( \sum_{n=1}^\infty h \sup_{n \leq t < (n+1)h} K(t) \right) E(M_1) < \infty, \]
and thus \( f(\cdot) \) is d.r.i.

We can write \( M(t) = E \sum_{j=0}^{M_j} f(t - S_{M_j}) = \int_0^\infty f(t - y)U(dy) \), where \( U(E) = E(\sum_{j=0}^{M_j} \chi_{0,S_{M_j}}) \). Since \( S_{M_0} = 0, S_{M_j} = \sum_{r=1}^j Z_r \), where \( Z_r = S_{M_r} - S_{M_{r-1}} \), \( r = 1, 2, \ldots \) are i.i.d. nonnegative nonlattice random variables, we see by Blackwell's renewal theorem for nonnegative random variables that
\[ U(t, t + h] \to \frac{h}{E(Z_1)}. \]
Now arguing as in Feller [7]
\[ \int_0^\infty f(t - y)U(dy) \to \frac{1}{E(Z_1)} \int_0^\infty f(u) \, du. \]
But,

\[ \sum_{t=0}^{\infty} f(u) \, du = E \sum_{t=0}^{\infty} K(t - S_n) \, dt = E(N) \sum_{t=0}^{\infty} K(t) \, dt \]

and \( E(Z) = E(N)E(S) \), implying the theorem.

5. Equivalence to processes with recurrence points. In [2] we showed that given any \((A, \lambda, \varphi)\)-recurrent chain \( \{X_n\} \) with stationary measure \( \pi \), and any bounded measurable \( f : S \to R \), one can adjoin a point \( \Delta \) to \( S \), extend \( f \) to \( f' \) on \( S \cup \Delta \), and define a Markov chain \( \{\hat{X}_n\} \) with transition function \( \tilde{P} \) on \( S \cup \Delta \), and stationary measure \( \tilde{\pi}(\cdot) \) such that

\[ E_\mu f(X_n) = E_\mu f'(\hat{X}_n) \]

for any initial distribution \( \mu \) on \( S \), and

\[ E_{\tilde{\pi}} f(X_n) = E_{\tilde{\pi}} f(\hat{X}_n) \, . \]

Similarly for the semi-Markov process \( X(t) \), and bounded, measurable \( f : S \times R^+ \to R \), one can extend \( f \) to \( f' : (S \cup \Delta) \times R^+ \to R \) by defining

\[ f(\Delta, u) = \int_{\Delta} f(x, u) \varphi(dx) \, , \]

and define a corresponding process \( \hat{X}(t) \) on \( S \cup \Delta \) for which

\[ E_\mu f(W(t)) = E_\mu f(\hat{W}(t)) \, . \]

An analogous construction for the Kesten process yields

\[ E_{\tilde{\pi}} \sum g(X_n, t - S_n) = E_{\tilde{\pi}} \sum \tilde{g}(\hat{X}_n, t - \hat{S}_n) \, , \quad x \in S \]

with \( \hat{X}_n \) having a recurrent point.

These equivalences can be proved from the same renewal equations as appeared in the proofs in Sections 3 and 4 above. They tell us, in effect, that for the semi-Markov processes under consideration, it is no loss of generality to assume that the underlying Markov chain has a recurrence point.

ADDENDUM. Between the times of submission and revision of this paper we have learned of concurrent work by E. Nummelin along the lines of [2] and the present paper, contained in reports of the Helsinki University of Technology, and currently also submitted for publication. He demonstrates and utilizes the existence of an atom whose role is similar to our regeneration sets, but his approach and results differ considerably in detail from ours.

REFERENCES


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