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SOME PROBABILISTIC PROPERTIES OF BESSEL FUNCTIONS

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The Bessel function ratios $(b/a)^\nu K_\nu(as^{\frac{1}{2}})/K_\nu(bs^{\frac{1}{2}})$ $(a > b > 0, \nu \in R)$ and $(b/a)^\nu I_\nu(as^{\frac{1}{2}})/I_\nu(bs^{\frac{1}{2}})$ $(0 < a < b, \nu > -1)$ are infinitely divisible Laplace transforms in $s > 0$. These results are derived as hitting times of the Bessel diffusion process. The infinite divisibility of the t-distribution is deduced as a limiting result. A relationship with the von Mises-Fisher distribution is also demonstrated.

1. Introduction. In a recent paper Grosswald [6] proved the infinite divisibility (ID) of the t-distribution. The t-distribution of 2ν degrees of freedom ($\nu > 0$ real) has density

$$f(y) = \frac{\Gamma(\nu + \frac{1}{2})}{(2\pi\nu)^{\frac{1}{2}}\Gamma(\nu)} \left(1 + \frac{y^2}{2\nu}\right)^{-\nu-\frac{1}{2}},$$

and characteristic function

$$\hat{f}(s) = \frac{2^{1-\nu}}{\Gamma(\nu)} (|s|(2\nu)^{\frac{1}{2}})^\nu K_\nu(|s|(2\nu)^{\frac{1}{2}}) .$$

Grosswald's method was to show that $\hat{f}(s^{\frac{1}{2}})$ is an ID Laplace transform (LT). Hence $\hat{f}(s)$ is the characteristic function of a variance mixture of normal densities with ID mixing distribution. He used analytic methods to prove the ID of $\hat{f}(s^{\frac{1}{2}})$. We shall use probabilistic methods to deduce some related results and Grosswald's theorem will appear as a limiting case. See Corollary 3.1.

In later sections we explore a connection with the von Mises-Fisher distribution.

2. The Bessel process. Let q be any real number. Consider the *Bessel diffusion* process on $[0, \infty]$ with infinitesimal generator

$$A_q = \tfrac{1}{2}d^2/dx^2 + \tfrac{1}{2}x^{-1}(q-1)d/dx .$$

By a slight abuse of notation we shall call q the *dimension* of the process because if $q \geqq 1$ is an integer, then the Bessel process represents the radial motion of a standard Brownian motion (BM) in R^q.

Using the terminology for diffusions described in Mandl [12] (with that of Itô and McKean [8], page 108 in parentheses), here is the boundary behavior. For all real q, ∞ is a *natural* (not exit, not entrance) boundary. For $q \leqq 0$, 0 is an *exit* (exit, not entrance) boundary. We shall adopt the convention that once a

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TABLE 2.1

q	$\mathscr{D}(A_q)$	$\mathscr{B}_0$	$\mathscr{B}_\infty$
$q \leqq 0$	$\{f \in \mathscr{D}_q : A_q f(0+) = 0\}$	$f(0+)$ exists $A_q f(0+) = 0$	$f(\infty-)$ exists $A_q f(\infty-)$ exists
$0 < q < 2$	$\{f \in \mathscr{D}_q : x^{q-1}f'(x) \to 0$ as $x \to 0\}$	$f(0+)$ exists $A_q f(0+)$ exists $x^{q-1}f'(x) \to 0$ as $x \to 0$	$f(\infty-)$ exists $A_q f(\infty-)$ exists
$q \geqq 2$	$\mathscr{D}_q$	$f(0+)$ exists $A_q f(0+)$ exists	$f(\infty-)$ exists $A_q f(\infty-)$ exists

diffusing particle reaches an exit boundary, it stays there forever. For $0 < q < 2$, 0 is a *regular* (exit and entrance) boundary, which we make *instantaneously reflecting*. For $q \geqq 2$, 0 is an *entrance* (entrance, not exit) boundary. For a description and proof of these properties see Mandl [12], pages 13, 24–25, 67.

Associated with the q-dimensional Bessel process is a strongly continuous contractive semigroup of operators $\{T_t\}$ acting on the space of functions $C = C[0, \infty]$ = bounded continuous functions on $[0, \infty]$. The *infinitesimal operator* of this semigroup is A_q acting on a *domain* $\mathscr{D}(A_q)$. Table 2.1 describes $\mathscr{D}(A_q)$ for various values of q. In this table $\mathscr{D}_q = \{f \in C: A_q f \in C\}$. By $A_q f \in C$ we mean f is twice differentiable on $(0, \infty)$, and the limits $A_q f(0+)$, $A_q f(\infty-)$ exist and are finite. The boundary behavior listed under $\mathscr{B}_0$ and $\mathscr{B}_\infty$ reiterates the description under $\mathscr{D}(A_q)$.

3. Hitting times for the Bessel process. Let $0 < a, b < \infty$. Start the Bessel process at a and let τ_{ab} denote the first time at which the process hits b. Let $\phi^\nu_{ab}(s)$ denote the LT of τ_{ab}. (We shall see that $\nu = (q - 2)/2$ is a more convenient index than q.)

Let $s > 0$. If $a < b$, then to calculate $\phi^\nu_{ab}(s)$ it is sufficient to find a twice differentiable function $f(x)$ such that

(3.1) $$f(b) = 1$$

(3.2) $$A_q f - sf = 0 \qquad 0 < x < \infty$$

(3.3) $$f \text{ satisfies } \mathscr{B}_0$$

where $\mathscr{B}_0$ is the relevant boundary condition at 0. Then $\phi^\nu_{ab}(s) = f(a)$. See for example, Mandl [12], page 62, Itô and McKean [8], page 129, or adapt the argument of Lemma 5.2 below. Similarly, if $a > b$, it is sufficient to find a twice differentiable function $f(x)$ satisfying

(3.4) $$f(b) = 1$$

(3.5) $$A_q f - sf = 0 \qquad 0 < x < \infty$$

(3.6) $$f \text{ satisfies } \mathscr{B}_\infty .$$

Then again $\phi^\nu_{ab}(s) = f(a)$.

Before displaying the formulae for $\phi_{ab}^{\nu}(s)$ let us note the following properties of the modified Bessel functions $I_\nu(x)$, $K_\nu(x)$. Both functions satisfy the differential equation

$$x^2u''(x) + xu'(x) - (x^2 + \nu^2)u = 0 \qquad 0 < x < \infty .$$

Also

$$d/dx[x^{-\nu}I_\nu(x)] = x^{-\nu}I_{\nu+1}(x) .$$

Their asymptotic behavior is given by

$$I_\nu(x) \sim \frac{(x/2)^\nu}{\Gamma(\nu+1)} \quad (x \to 0) \qquad \text{for} \quad \nu > -1 ,$$

$$K_{-\nu}(x) = K_\nu(x) \sim \frac{\Gamma(\nu)}{2}(x/2)^{-\nu} \quad (x \to 0) \qquad \text{for} \quad \nu > 0 ,$$

$$K_0(x) \sim -\log x \quad (x \to 0) ,$$

$$I_\nu(x) \sim (2\pi x)^{-\frac{1}{2}}e^x \quad (x \to \infty) \qquad \nu \in R ,$$

$$K_\nu(x) \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^x \quad (x \to \infty) \qquad \nu \in R .$$

See Watson [16] pages 77–80, 202–203.

THEOREM 3.1. *Write* $\nu = (q-2)/2$. *Then for all* $q \in R$ *and* $0 < b < a < \infty$,

$$\phi_{ab}^{\nu}(s) = \left(\frac{b}{a}\right)^\nu \frac{K_\nu(a(2s)^{\frac{1}{2}})}{K_\nu(b(2s)^{\frac{1}{2}})} \qquad s > 0 . \tag{3.7}$$

For $q > 0$ (*hence* $\nu > -1$) *and* $0 < a < b < \infty$,

$$\phi_{ab}^{\nu}(s) = \left(\frac{b}{a}\right)^\nu \frac{I_\nu(a(2s)^{\frac{1}{2}})}{I_\nu(b(2s)^{\frac{1}{2}})} \qquad s > 0 . \tag{3.8}$$

Further, these LT's *are* ID.

PROOF. Using the properties of Bessel functions described above, it is easy to see that these functions satisfy (3.1)—(3.3) and (3.4)—(3.6) respectively (as functions of a).

To show ID let c be between a and b. Then $\tau_{ab} = \tau_{ac} + \tau_{cb}$, and further τ_{ac} and τ_{cb} are independent by the strong Markov property. By putting more and more points between a and b, we may express τ_{ab} as the limit of a *null triangular array* (Feller [4], page 550). Hence τ_{ab} is ID.

REMARK. Notice that this reasoning tells us that *any* distribution which arises as the (one-sided) hitting time of a one-dimensional diffusion is ID.

COROLLARY 3.1. *For* $\nu > 0$ *and each* $a > 0$, $[\Gamma(\nu)]^{-1}2^{1-\nu}(a(2s)^{\frac{1}{2}})^\nu K_\nu(a(2s)^{\frac{1}{2}})$ *is an* IDLT.

PROOF. Let $a > b > 0$. Note that for $\nu > 0$ $\phi_{ab}^{\nu}(s) \to (b/a)^{2\nu} < 1$ as $s \to 0$, so there is positive probability that $\tau_{ab} = \infty$. However, $(a/b)^{2\nu}\phi_{ab}^{\nu}(s)$ is the IDLT of a *proper* probability distribution on $[0, \infty]$ (that is, no mass is assigned to ∞).

Now a pointwise limit of IDLT's is itself an IDLT. Thus

$$\lim_{b\to\infty}\left(\frac{a}{b}\right)^{\nu}\frac{K_\nu(a(2s)^{\frac{1}{2}})}{K_\nu(b(2s)^{\frac{1}{2}})}=\frac{2^{1-\nu}}{\Gamma(\nu)}(a(2s)^{\frac{1}{2}})^{\nu}K_\nu(a(2s)^{\frac{1}{2}}) \tag{3.9}$$

is an IDLT (which is easily seen to be proper also). Alternatively we may regard (3.9) as $\lim_{b\to 0}\phi_{ab}^{-\nu}(s)$.

COROLLARY 3.2. *$K_0(a(2s)^{\frac{1}{2}})$ is an* IDLT *of a σ-finite measure on* $(0, \infty)$.

PROOF. Let $b\to 0$ in (3.7) after multiplying $\phi_{ab}^{0}(s)$ by $-\log b$. Note that this corollary is the one place in the paper where we refer to a LT which is *not* the LT of a *probability* distribution.

COROLLARY 3.3. *For $\nu > -1$ and each $b > 0$, $(\frac{1}{2}b(2s)^{\frac{1}{2}})^{\nu}/[I_\nu(b(2s)^{\frac{1}{2}})\Gamma(\nu+1)]$ is an* IDLT.

PROOF. Let $a\to 0$ in (3.8).

4. The BMB$_q$ process. Consider the following 2-dimensional diffusion defined on $(-\infty, \infty)\times[0, \infty)$. In the first component is a 1-dimensional BM $X_1(t)$ with infinitesimal variance 1 and drift $\mu \geqq 0$. In the second component is an independent $(q-1)$-dimensional Bessel process $X_2(t)$. We shall call this process $\mathbf{X}(t) = (X_1(t), X_2(t))$, the q-dimensional *Brownian motion–Bessel* (BMB$_q$) process. *Throughout Sections* 4—6 *we shall require* $q > 1$.

Let $\kappa \geqq 0$ and $q > 1$. Call the distribution on $[-1, 1]$ given by the density

$$\alpha(q)c(\kappa, q)e^{\kappa u}(1-u^2)^{(q-3)/2} \qquad -1<u<1 \tag{4.1}$$

the q-dimensional *projected von Mises–Fisher* distribution. Here

$$\alpha(q)=\pi^{-\frac{1}{2}}\Gamma(q/2)/\Gamma(\tfrac{1}{2}(q-1))$$

is a normalization constant associated with the weight function, $(\alpha(q)\int_{-1}^{1}(1-u^2)^{(q-3)/2}\,du = 1)$, and

$$c(\kappa, q)=(\kappa/2)^{\nu}/[\Gamma(q/2)I_\nu(\kappa)]$$

is a normalization constant associated with this particular distribution (Magnus et al. [11], page 221). Note $\nu = (q-2)/2$ and interpret $c(0, q) = c(0+, q) = 1$.

Our terminology is motivated by the von Mises–Fisher (VMF) distribution defined on the unit sphere Ω_q in R^q (integer $q \geqq 2$). Let $\mathbf{e}_1$ denote a unit vector in R^q pointing in the x_1 direction. Then the VMF distribution oriented about $\mathbf{e}_1$ has a density, with respect to Lebesgue measure on Ω_q, proportional to $\exp(\kappa\mathbf{x}\cdot\mathbf{e}_1)$, $\mathbf{x}\in\Omega_q$. If this distribution is projected into the x_1-axis, one gets the density (4.1). See Mardia [13] for more details about this distribution.

Let S denote the semicircle $S = \{\mathbf{x}\in R^2\colon x_1^2 + x_2^2 = 1, x_2 \geqq 0\}$. Our interest in the BMB$_q$ process comes from the following theorem.

THEOREM 4.1. *Consider the* BMB$_q$ *process* ($q > 1$ *real*) *started at the origin* $\mathbf{x}^0 = 0$. *Let τ denote the first hitting time of the process on the semicircle S. Then*

(i) $X_1(\tau)$ *has a q-dimensional projected* VMF *distribution with parameter* $\kappa = \mu$.
(ii) *The* LT *of* τ *is given by*

$$\psi_\nu(s) = \psi_\nu(s; \mu) = \left[\frac{(\mu^2 + 2s)^{\frac{1}{2}}}{\mu}\right]^\nu \frac{I_\nu(\mu)}{I_\nu((\mu^2 + 2s)^{\frac{1}{2}})} . \tag{4.2}$$

(*Here* $\nu = (q - 2)/2$. *Interpret* $\psi_\nu(s; 0) = \psi_\nu(s; 0+)$.)
(iii) $\mathbf{X}(\tau)$ *and* τ *are independent.*
(iv) τ *is* ID.

PROOF. This theorem was first proved for $q = 2$ by G.E.H. Reuter (unpublished). Our proof is based on his method. The bulk of the proof will occupy Sections 5—6. Here we lay the groundwork.

We shall use the *Legendre* polynomials (sometimes known as the *Gegenbauer* or *ultraspherical* polynomials when renormalized and indexed by $\nu = (q - 2)/2$). The Legendre polynomial of degree $m \geqq 0$ (integer) in $q > 1$ (real) dimensions is given by the Rodriguez formula

$$P_{qm}(u) = (-\tfrac{1}{2})^m \frac{\Gamma(\frac{1}{2}(q-1))}{\Gamma(\frac{1}{2}(q-1)+m)} (1 - u^2)^{(3-q)/2} \frac{d^m}{du^m} [(1 - u^2)^{(2m+q-3)/2}] .$$

See Muller [14], page 17 or Magnus et al. [11], pages 218–221. Note $P_{q0}(u) = 1$. For $q = 2$, there is the identity $P_{2m}(\cos\theta) = \cos m\theta$. These polynomials are useful because they are orthogonal on $(-1, 1)$ with respect to the density $\alpha(q)(1 - u^2)^{(q-3)/2}\, du$. They are normed by $P_{qm}(1) = 1$ and satisfy the differential equation

$$\frac{d^2}{d\theta^2} P_{qm}(\cos\theta) + (q - 2)\cot\theta \frac{d}{d\theta} P_{qm}(\cos\theta) + m(m + q - 2)P_{qm}(\cos\theta) = 0 .$$

To prove Theorem 4.1 we shall set the problem up in a bit more generality. Let D denote the semidisc $D = \{\mathbf{x} \in R^2 : x_1^2 + x_2^2 \leqq 1,\ x_2 \geqq 0\}$, let $\mathbf{x}$ be an arbitrary starting point in D, and let $\tau = \tau_\mathbf{x}$ denote the first hitting time on S. Write $U = \mathbf{X}(\tau)\cdot \mathbf{e}_1 = X_1(\tau)$. We shall calculate the quantities

$$G_{sm}(\mathbf{x}) = E_\mathbf{x}[\exp(-s\tau_\mathbf{x} - \mu U)P_{qm}(U)] \qquad s > 0,\ m = 0, 1, 2, \cdots ,$$

which determine the joint distribution of $\mathbf{X}(\tau)$ and τ. Notice it does not matter that U is undefined on the set $\{\tau = \infty\}$ since $e^{-s\infty} = 0$. Using the properties of Legendre polynomials described above, we see that the theorem will follow if and only if

$$\begin{aligned} G_{sm}(\mathbf{0}) &= c(\mu, q)\psi(s) && m = 0 \\ &= 0 && m > 0 . \end{aligned} \tag{4.3}$$

We shall deduce the ID of $\psi(s)$ later. To calculate $G_{sm}(\mathbf{x})$ we turn to semigroup theory.

5. Semigroup theory for the BMB$_q$ process. Let $\{T_t^{(1)}\}$ denote the strongly continuous contractive semigroup of the BM $X_1(t)$ acting on $C_1 = C[-\infty, \infty]$

with infinitesimal operator

$$A_1 = \tfrac{1}{2}\, d^2/dx_1^2 + \mu\, d/dx_1$$

acting on the domain $\mathscr{D}(A_1) = \{f \in C_1 : A_1 f \in C_1\}$. Similarly, let $\{T_t^{(2)}\}$ be the strongly continuous contractive semigroup of the $(q-1)$-dimensional Bessel process $X_2(t)$ acting on $C_2 = C[0, \infty]$ with infinitesimal operator

$$A_2 = \tfrac{1}{2}\, d^2/dx_2^2 + \tfrac{1}{2}x_2^{-1}(q-2)\, d/dx_2$$

acting on the domain given in Table 2.1.

Suppose $f_2^*(x_2) \in C^2(R)$ is an *even* function of compact support. Let $f_2(x_2) = f_2^*(x_2)|_{x_2 \geqq 0}$. Checking that f_2 has the appropriate boundary behavior near 0 (in particular $x_2^{-1}f_2'(x_2)$ tends to a finite limit as $x_2 \to 0$) we see that $f_2 \in \mathscr{D}(A_2)$. (Remember $q - 1 > 0$.) More easily we see that if $f_1(x_1) \in C^2(R)$ has compact support, then $f_1 \in \mathscr{D}(A_1)$.

Let $\{T_t\}$ denote the transition semigroup of the BMB_q process. Define $C = C([-\infty, \infty] \times [0, \infty])$. By approximating any function $f \in C$ uniformly by a finite sum of the form $\sum_{j=1}^n f_1^j(x_1) f_2^j(x_2)$ where $f_i^j \in C_i$, $i = 1, 2$, one can easily show that $\{T_t\}$ forms a strongly continuous contractive semigroup on C. Denote its infinitesimal operator by A acting on a domain $\mathscr{D}(A)$.

Using our knowledge of $\mathscr{D}(A_1)$ and $\mathscr{D}(A_2)$, we can describe a sufficiently large class of functions in $\mathscr{D}(A)$ to prove Theorem 3.1. Let A' denote the differential operator

$$A' = \tfrac{1}{2}[\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + x_2^{-1}(q-2)\, \partial/\partial x_2] + \mu\, \partial/\partial x_1$$

acting on $C^2(R^2)$.

We are careful to distinguish between A and A' because they act on different classes of functions. (We did not emphasize this distinction in Section 2 because the infinitesimal operator of a 1-dimensional diffusion is defined precisely on a *subset* of twice differentiable functions.)

LEMMA 5.1. *Let $f^*(x_1, x_2) \in C^2(R^2)$ be a function of compact support, even in x_2. Set $f = f^*|_{x_2 \geqq 0}$. Then $f \in \mathscr{D}(A)$ and $Af = A'f$.*

PROOF. First, it is not difficult to show that if $f_1(x_1) \in \mathscr{D}(A_1)$ and $f_2(x_2) \in \mathscr{D}(A_2)$, then $f_1 f_2 \in \mathscr{D}(A)$ and $Af_1 f_2 = A'f_1 f_2 = f_1 A_2 f_2 + f_2 A_1 f_1$.

Secondly, by using standard approximation theorems it is possible to approximate f and all its partial derivatives up to order 2 uniformly by functions of the sort $f_n(x) = \sum_1^n f_{1n}^j(x_1) f_{2n}^j(x_2)$ where f_{in}^j, $i = 1, 2$, have the properties described in the second paragraph of this section, for $j = 1, \cdots, n$ and $n = 1, 2, \cdots$ (Treves [15], page 409). Then because $\partial/\partial x_2 f_n(x_1, 0) = 0$ and because $\partial^2/\partial x_2^2 f_n(x_1, x_2)$ is bounded as $n \to \infty$ uniformly in $\mathbf{x}$, it follows that $Af_n = A'f_n \to A'f$ uniformly for $x \in (-\infty, \infty) \times [0, \infty)$.

Since A is a *closed* linear operator (Dynkin [3], page 23), we see that $f \in \mathscr{D}(A)$ and $Af = A'f$.

The next lemma tells us how to find $G_{sm}(\mathbf{x})$. Let $R_s = \int_0^\infty e^{-st}T_t\,dt$ denote the resolvent operator of $\{T_t\}$. Recall that the resolvent is a bijection $R_s\colon C \to \mathscr{D}(A)$ with inverse $(sI - A)$.

LEMMA 5.2. *Let $v(\mathbf{x})$ be a function defined on S and let $g^* \in C^2(R^2)$ be a function even in x_2. Set $g = g^*|_{x_2 \geq 0}$. Suppose g satisfies*

(i) $A'g - sg = 0 \quad \mathbf{x} \in (-\infty, \infty) \times (0, \infty)$;

(ii) $f(\mathbf{x}) = v(\mathbf{x}) \quad \mathbf{x} \in S$.

Then $g(\mathbf{x}) = E_\mathbf{x}(e^{-s\tau}v(\mathbf{X}(\tau)))$ for $\mathbf{x} \in D$.

PROOF. Let $h^*(\mathbf{x}) \in C^2(R^2)$ be a function even in x_2 with compact support satisfying $h^*(\mathbf{x}) = 1$ for $\|\mathbf{x}\| \leqq 1$. Set $\hat{g} = g^*h^*|_{x_2 \geq 0} \in \mathscr{D}(A)$. Then $A\hat{g} = A'\hat{g}$ and $\hat{g}(\mathbf{x}) = g(\mathbf{x})$ for $\mathbf{x} \in D$. Set $f(\mathbf{x}) = (sI - A)\hat{g}(\mathbf{x})$. Noting that $E_\mathbf{x}$ denotes integration with respect to the underlying probability measure $P_\mathbf{x}(d\omega)$, we see by Dynkin's ([3], page 132) formula that for $\mathbf{x} \in D$,

$$\begin{aligned}\hat{g}(\mathbf{x}) &= R_s f(\mathbf{x}) = \int_0^\infty e^{-st}T_t f(\mathbf{x})\,dt \\ &= E_\mathbf{x} \int_0^\infty e^{-st}f(\mathbf{X}(t))\,dt \\ &= \int_{\{\tau<\infty\}} \{\int_0^\tau e^{-st}f(\mathbf{X}(t))\,dt + \int_\tau^\infty e^{-st}f(\mathbf{X}(t))\,dt\}P_\mathbf{x}(d\omega) \\ &\quad + \int_{\{\tau=\infty\}} \{\int_0^\infty e^{-st}f(\mathbf{X}(t))\,dt\}P_\mathbf{x}(d\omega)\,.\end{aligned}$$

Since $\mathbf{X}(t) \in D$ for $t < \tau$, we see that $f(\mathbf{X}(t)) = (sI - A)\hat{g}(\mathbf{X}(t)) = 0$ for $t < \tau$ and hence the first and last integrals vanish. Letting $\mathscr{F}_\tau$ denote the σ-field of events generated by BMB_q process up to time τ,

$$\begin{aligned}\hat{g}(\mathbf{x}) &= \int_{\{\tau<\infty\}} E_\mathbf{x}[\int_0^\infty e^{-s(\tau+t)}f(\mathbf{X}(\tau + t))\,dt \mid \mathscr{F}_\tau]P_\mathbf{x}(d\omega) \\ &= \int_{\{\tau<\infty\}} \{e^{-s\tau} \int_0^\infty e^{-st}E_{\mathbf{X}(\tau)} f(\mathbf{X}(\tau + t))\,dt\}P_\mathbf{x}(d\omega) \quad \text{(strong Markov property)} \\ &= \int e^{-s\tau}R_s f(\mathbf{X}(\tau))P_\mathbf{x}(d\omega) \\ &= E_\mathbf{x}[e^{-s\tau}\hat{g}(\mathbf{X}(\tau))] = E_\mathbf{x}[e^{-s\tau}v(\mathbf{X}(\tau))]\,.\end{aligned}$$

6. The formula for $G_{sm}(\mathbf{x})$. We are now ready to solve for $G_{sm}(\mathbf{x})$. We wish to find a function f satisfying the following conditions:

(i) $A'f - sf = 0 \quad \mathbf{x} \in (-\infty, \infty) \times (0, \infty)$,

(ii) $f(\mathbf{x}) = P_{qm}(x_1)\exp(-\mu x_1) \quad \mathbf{x} \in S$,

(iii) f can be extended to all of R^2 to be twice-continuously differentiable and even in x_2.

It will then follow from Lemma 5.2 that $G_{sm}(\mathbf{x}) = f(\mathbf{x})$ for $\mathbf{x} \in D$.

To solve these equations introduce polar coordinates ($r\cos\theta = x_1$, $r\sin\theta = x_2$) and solve (i) for $g(r, \theta) = \exp(\mu r\cos\theta)f$, using separation of variables. Noting the boundary conditions (ii) and (iii) we reach the solution

$$G_{sm}(\mathbf{x}) = \frac{I_{m+\nu}(r(\mu^2 + 2s)^{\frac{1}{2}})}{I_{m+\nu}((\mu^2 + 2s)^{\frac{1}{2}})}\, r^{-\nu}P_{qm}(\cos\theta)\exp(-\mu r\cos\theta)\,. \tag{6.1}$$

The power series formula

$$\left(\frac{r}{2}\right)^{-m-\nu} I_{\nu+m}(r) = \sum_{k=0}^\infty \frac{(r^2/4)^k}{\Gamma(\nu + m + k + 1)!\,k!}$$

in $r^2 = x_1^2 + x_2^2$ and the fact that $r^m P_{qm}(\cos\theta)$ is a polynomial in x_1 and x_2 with only even powers of x_2 ensure that the boundary condition (iii) is satisfied.

Letting $r \to 0$ gives us (4.3) and hence parts (i)—(iii) of Theorem 4.1 follow.

The ID of $\psi_\nu(s)$ can be deduced probabilistically. Let τ denote the first hitting time of the BMB_q process, starting at $\mathbf{0}$, on S. Pick a, $0 < a < 1$. Let τ_1 denote the first hitting time on aS and set $\tau_2 = \tau - \tau_1$. By the strong Markov property, τ_2 depends on $\{\mathbf{X}(t) : t \leqq \tau_1\}$ only through the value $\mathbf{X}(\tau_1)$. Since part (ii) of Theorem 4.1 tells us that τ_1 is independent of $\mathbf{X}(\tau_1)$, we see that τ_1 and τ_2 are independent. By inserting more and more semicircles with radii between 0 and 1, we can express τ as the limit of a null triangular array, and hence τ is ID.

REMARKS.

(1) If $\mu = 0$, then as we expect, $\psi(s; 0)$ equals the LT in Corollary 3.3 (with $b = 1$).

(2) If A' is replaced by the operator

$$\frac{\sigma^2}{2}[\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + x_2^{-1}(q-2)\,\partial/\partial x_2] + \mu\,\partial/\partial x_2\,,$$

and S by aS $(a > 0)$, then Theorem 4.1 remains valid with changes in the constants. More specifically, $a^{-1}X_1(\tau)$ has a projected VMF distribution with parameter $\kappa = \mu a/\sigma^2$, and is independent of τ, which has IDLT

$$\psi(s) = \left(\frac{\gamma}{\kappa}\right)^\nu \frac{I_\nu(\kappa)}{I_\nu(\gamma)}\,,$$

where $\gamma = \sigma^{-1}a(\mu^2/\sigma^2 + 2s)^{\frac{1}{2}}$.

7. The case $q = 1$. Since 0 is an exit boundary of the 0-dimensional Bessel process, a BMB_1 process started at $\mathbf{0}$ is equivalent to its first component. The results of the last sections have their counterpart in this simpler situation.

THEOREM 7.1. *Let $X(t)$ be a one-dimensional standard* BM *with drift $\mu \geqq 0$ started at* 0 *and let τ denote the first hitting time on $\{\pm 1\}$. Then,*

(i) $P(X(\tau) = 1) = \frac{1}{2}e^{\mu}/\cosh\mu$, $P(X(\tau) = -1) = \frac{1}{2}e^{-\mu}/\cosh\mu$,

(ii) *τ has* IDLT

$$\psi(s) = \frac{\cosh\mu}{\cosh((\mu^2 + 2s)^{\frac{1}{2}})}\,,$$

(iii) *$X(\tau)$ and τ are independent.*

PROOF. The proof follows the same lines as the proof of Theorem 4.1, but is simpler because the boundary contains only two points. Note that Bessel functions do not appear explicitly in the solution because of the identity

$$I_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cosh x$$

(Watson [16], page 55).

8. Some further probabilistic properties of Bessel functions. So far we have considered LT's in which the Laplace variable enters the *argument* of Bessel functions. It is also possible to prove results in which the Laplace variable enters the *order* of Bessel functions.

THEOREM 8.1. *Let* $0 < a < b < \infty$ *and write* $I_\nu(\kappa) = I(\nu; \kappa)$, $K_\nu(\kappa) = K(\nu; \kappa)$. *Then the following functions are* IDLT's (*in* $s \geqq 0$).

(i) $(I(s^{\frac{1}{2}}; a)/I(s^{\frac{1}{2}}; b))(I(0; b)/I(0; a))$,
(ii) $(K(s^{\frac{1}{2}}; b)/K(s^{\frac{1}{2}}; a))(K(0; a)/K(0; b))$,
(iii) $I(s^{\frac{1}{2}}; a)/I(0; a)$,
(iv) $K(0; a)/K(s^{\frac{1}{2}}; a)$.

For a proof see Hartman [7]. Parts (iii) and (iv) are limiting results as $b \to \infty$. A direct proof of (iii) can also be given. See Kent [10].

A notion of convolution (and hence ID) may be defined for distributions on $[-1, 1]$ for any real $q \geqq 2$ dimensions (though not for $1 < q < 2$). For $q = 2$ this notion corresponds to the usual addition mod 2π of axially symmetric random angles on the circle. It can then be shown (Kent [10]) that the projected VMF distribution of (4.1) is ID for all real $q \geqq 2$.

9. Bessel functions in densities. So far, we have looked at Bessel functions appearing in LT's. However, Bessel functions also appear in *densities*. We limit ourselves to an example arising from the Bessel process. See Feller [4] for others. Define

$$p^*(t, x, y) = t^{-1}x^{-\nu}I_\nu\left(\frac{xy}{t}\right)\exp\left\{-\frac{1}{2t}(x^2 + y^2)\right\}y^{\nu+1} \tag{9.1}$$

where $\nu = (q-2)/2 > -1$.

THEOREM 9.1. *The probability transition density of the q-dimensional Bessel process is given by* (9.1).

PROOF. Let $p(t, x, y)$ denote the ptd of the Bessel process. We wish to show $p = p^*$.

Let $\phi_1(x) = (x(2s)^{\frac{1}{2}})^{-\nu}I_\nu(x(2s)^{\frac{1}{2}})$ and $\phi_2(x) = (x(2s)^{\frac{1}{2}})^{-\nu}K_\nu(x(2s)^{\frac{1}{2}})$ denote the solutions of (3.2)—(3.3) and (3.5)—(3.6), which are unique up to a constant factor. Then from Itô and McKean [8], page 150, the resolvent density $r_s(x, y) = \int_0^\infty e^{-st}p(t, x, y)\,dt$ is given by the formula

$$\begin{aligned} r_s(x, y) &= B^{-1}\phi_1(x)\phi_2(y)m'(y) \qquad & x \leqq y \\ &= B^{-1}\phi_1(y)\phi_2(x)m'(y) \qquad & x \geqq y\,. \end{aligned}$$

Here $m'(x) = 2x^{2\nu+1}$ is (the density of) the *speed measure*, $\rho'(x) = x^{-2\nu-1}$ defines the *natural scale*, and B is the Wronskian

$$\begin{aligned} B &= \phi_2(x)\frac{\phi_1'(x)}{\rho'(x)} - \phi_1(x)\frac{\phi_2'(x)}{\rho'(x)} \\ &= (2s)^{-\nu}\,. \end{aligned}$$

(See Watson [16], pages 79–80.) Thus,

$$(9.2)\qquad r_s(x, y) = 2x^{-\nu}y^{\nu+1}I_\nu(x(2s)^{\frac{1}{2}})K_\nu(y(2s)^{\frac{1}{2}}) \qquad x \leqq y$$
$$= 2x^{-\nu}y^{\nu+1}I_\nu(y(2s)^{\frac{1}{2}})K_\nu(x(2s)^{\frac{1}{2}}) \qquad y \leqq x .$$

Standard formulae for Bessel functions (Bateman [2], page 200) tell us that $r_s^*(x, y) = \int_0^\infty e^{-st}p^*(t, x, y)\, dt$ also has the form (9.2). Thus by the uniqueness of the Laplace transform, $p = p^*$.

It is easy to derive an ID density from (9.1). Setting $\frac{1}{2}y^2 = v$, $\frac{1}{2}x^2 = c$ and $t = 1$, we obtain the density

$$(9.3)\qquad f(v) = (v/c)^{\nu/2}I_\nu(2(cv)^{\frac{1}{2}})e^{-c-v} \qquad v > 0$$

with LT

$$(9.4)\qquad \phi(s) = (s + 1)^{-\nu-1}\exp[-sc/(s + 1)]$$

(Bateman [2], page 197). Since $\phi^{1/m}(s)$ has the same functional form (and hence is a LT) for all $m \geqq 1$, we see that (9.3) is an ID density. For an interpretation of (9.3) as a compound Poisson distribution, see Feller [4], page 415.

Note that for $c = 0$, (9.3) becomes a gamma density.

10. Final notes.

(1) Theorem 4.1 can also be approached using likelihood ratio martingales. The underlying idea behind this method is that the sample paths of 1-dimensional BM *with* drift have an absolutely continuous density with respect to BM *without* drift. Suppose that parts (i)—(iii) of Theorem 4.1 are known for $\mu = 0$. Then using this likelihood ratio approach, one may easily prove (i)—(iii) for $\mu > 0$.

Note that for *integer* $q \geqq 2$, parts (i) and (iii) are obviously true with $\mu = 0$ because of the symmetry of q-dimensional BM. Also, we can suppose the formula for $\psi_\nu(s; 0)$ is known from Section 3. Thus, for integer $q \geqq 2$, the likelihood ratio approach leads to a simple proof of Theorem 4.1.

For the details of the likelihood ratio approach applied to the proof of part (i) of Theorem 4.1 for $q = 2$, see Gordon and Hudson [5]. In passing we comment on a point of priority concerning the main result of [5]. The result is *not* new, but was proved several years ago by G. E. H. Reuter (see the proof of Theorem 4.1 above). Indeed, Reuter's work has been acknowledged by D. G. Kendall in the author's reply to the discussion of [9], page 416. Also, another proof of parts (i)—(iii) of Theorem 4.1 for $q = 2$, using the same approach as Gordon and Hudson has been given by D. Williams (unpublished).

(2) Hitting time results similar to those of Section 3 have also been obtained by Barndorff-Nielsen [1].

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