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THE LAW OF THE ITERATED LOGARITHM AND UPPER-LOWER CLASS TESTS FOR PARTIAL SUMS OF STATIONARY GAUSSIAN SEQUENCES

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Herein, laws of the iterated logarithm and various upper-lower class refinements are established for partial sums of stationary Gaussian random variables. These stationary Gaussian random variables are not necessarily in any sense weakly dependent. For example, if the random variables are nonnegatively correlated, then the upper half of the law of the iterated logarithm holds. Under more restrictive, but still quite general hypotheses, an upper-lower class test which classifies all monotone sequences $\{\phi(n)\}$ is established.

1. Introduction and summary. Let $\{X_n, n \geqq 1\}$ be a stationary sequence of zero-mean random variables with finite variances and let $S_n = \sum_{i=1}^n X_i$, $n \geqq 1$. Most of the laws of the iterated logarithm and related strong limit theorems for $\{S_n\}$ in the literature apply only to the case where the sequence $\{X_n\}$ is weakly dependent (see Philipp and Stout (1975)). By weak dependence, we mean that

$$(1.1)\qquad E|E(X_{n+k}|X_1, \cdots, X_n)| \to 0 \quad \text{as} \quad k \to \infty \qquad \text{for each} \quad n = 1, 2, \cdots,$$

and

$$(1.2)\qquad ES_n^2 \sim \sigma^2 n \quad \text{as} \quad n \to \infty \qquad \text{for some} \quad \sigma > 0 .$$

This includes independent, mixing, lacunary trigonometric and martingale difference sequences. It is known (cf. Philipp and Stout (1975)) that for a large class of weakly dependent random variables, the almost sure invariance principle holds, i.e., for some $\lambda > 0$,

$$(1.3)\qquad |S_n - \sigma W(n)| \ll n^{\frac{1}{2}-\lambda} \quad \text{a.s.}$$

Here $\{W(t), t \geqq 0\}$ is a standard Wiener process and we use Vinogradov's symbol $\ll$ instead of Landau's 0 notation. From (1.3) and the law of the iterated logarithm for Brownian motion it follows immediately that

$$(1.4)\qquad \limsup_{n\to\infty} (2\sigma^2 n \log_2 n)^{-\frac{1}{2}} S_n = 1 \quad \text{a.s.}$$

(Throughout this paper, we denote $\log \log n$ by $\log_2 n$, and more generally, we let $\log_k n = \log(\log_{k-1} n)$ for $k \geqq 2$.) Furthermore, in view of (1.3) and the Kolmogorov–Petrovski–Erdös test for Brownian motion, if $\phi(t)$ is a positive

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nondecreasing function then

$$(1.5) \qquad P[S_n > \sigma n^{\frac{1}{2}}\phi(n) \text{ i.o.}] = 0 \text{ or } 1 \quad \text{according as} \quad \int_1^\infty t^{-1}\,\phi(t)\exp(-\tfrac{1}{2}\phi^2(t))\,dt < \infty \text{ or } = \infty\,.$$

When $\{X_n\}$ is not weakly dependent, the asymptotic fluctuation behavior of the sequence $\{S_n\}$ may be very different from that of Brownian motion. In the first place, (1.2) may no longer be true. Set $V(n) = \operatorname{Var} S_n$. It is natural to ask under what conditions (other than weak dependence) (1.4) can be modified as

$$(1.6) \qquad \limsup_{n\to\infty} (2V(n)\log_2 V(n))^{-\frac{1}{2}} S_n = 1 \quad \text{a.s.}$$

A natural first step in such an inquiry is to assume that the sequence $\{X_n\}$ is stationary Gaussian since iterated logarithm behavior in (1.4) is closely related to the tail probabilities of the normal distribution and stationarity at least guarantees the ergodic theorem. In this case, Taqqu (1977) has recently shown that (1.6) indeed holds if for some $0 < \alpha < 2$,

$$(1.7) \qquad V(n) = n^\alpha L(n)\,,$$

where the function L is slowly varying and for some positive constants C and δ,

$$(1.8) \qquad |L(m+n)/L(n) - 1| \leqq Cm/n \quad \text{if} \quad \delta n \geqq m \geqq C\,.$$

Throughout the rest of this paper, we shall assume that $\{X_n\}$ is a zero-mean stationary Gaussian sequence. In Section 4 below, we study the problem of sharpening the law of the iterated logarithm (1.6) into an integral test like (1.5). We shall show that if $V(n)$ satisfies a condition similar to (1.7) and if $\phi(t)$ is a positive nondecreasing function, then

$$(1.9) \qquad P[S_n > V^{\frac{1}{2}}(n)\phi(n) \text{ i.o.}] = 0 \text{ or } 1 \quad \text{according as} \quad \int_1^\infty t^{-1}(\phi(t))^{(2/\alpha)-1}\exp(-\tfrac{1}{2}\phi^2(t))\,dt < \infty \text{ or } = \infty\,.$$

The case $\alpha = 1$ in (1.9) corresponds to the classical integral test (1.5) and so (1.9) can be regarded as a generalization of (1.5).

The proof of the upper-lower class test (1.9) will be presented in Section 4. It involves two main steps. The first is an estimate of the first exit probability

$$(1.10) \qquad P[S_n \geqq V^{\frac{1}{2}}(n)\phi(n) \quad \text{for some} \quad c^k \leqq n < c^{k+1}] = P(E_k)\,, \quad \text{say,}$$

where c is a positive integer > 1. To do this, we make use of the Gaussian structure of the sequence $\{X_n\}$ together with Slepian's lemma which enables us to compare $P(E_k)$ with known results on certain first exit probabilities for stationary Gaussian processes. The second step is an adaptation of the Borel–Cantelli lemma so that the probability in (1.9) is 0 or 1 according as the series $\sum P(E_k)$ converges or diverges. The key reason why this can be done lies in the fact that our choice of the geometric subsequence $\{c^k\}$ in (1.10) yields the asymptotic independence of the events E_k. One may recall that in most standard proofs of the classical upper-lower class test (1.5) for the i.i.d. case (and therefore $\alpha = 1$), the subsequence chosen is $\{n_k\}$ instead of $\{c^k\}$, where $n_k = [\exp(k/\log k)]$ (see

Feller (1943), for example). As will be indicated in Section 4, it turns out that to estimate $P(E_k)$ closely, we need to introduce smaller blocks into the large blocks $[c^k, c^{k+1}]$ by using finer subsequences like $\{n_k\}$. However, the geometric subsequence $\{c^k\}$ turns out to be the "right" subsequence to give asymptotic independence and an easy adaptation of the Borel–Cantelli lemma, as we shall show in Section 4.

Since $V(n) = nEX_1^2 + 2\sum_{i=1}^{n-1}(n-i)EX_1X_{i+1}$, the condition (1.7) on the growth of $V(n)$ is actually a condition on the autocorrelations of the stationary Gaussian sequence $\{X_n\}$. The condition (1.7) covers a wide spectrum of dependence situations: the independent and weakly dependent case (with $\alpha = 1$), the highly positively correlated case (with α close to 2) and the highly negatively correlated case (with α near 0). We shall discuss more about this in the corollaries of the upper-lower class test (1.9) in Section 4.

In our proof of the upper-lower class test (1.9), we need to use the fact that S_n at least satisfies the upper half of the law of the iterated logarithm (1.6), i.e.,

$$\limsup_{n\to\infty} (2V(n)\log_2 n)^{-\frac{1}{2}} S_n \leqq 1 \quad \text{a.s.} \tag{1.11}$$

Since stationarity implies that $V(n) \leqq n^2EX_1^2$, we can replace $\log_2 V(n)$ in (1.6) by $\log_2 n$. It is interesting to find weak sufficient conditions under which $\{S_n\}$ satisfies (1.11). In Section 2 we shall show that (1.11) holds under very weak restrictions (much weaker than (1.7)) on the growth rate of $V(n)$. These results are proved via a mixture of the classical approach involving subsequences, the manipulation of the maximal inequalities of Marcus and Shepp for Gaussian sequences, and the reduction to known results of stationary Gaussian sequences by means of Slepian's lemma. Somewhat similar in spirit to these results is the law of the iterated logarithm established by Orey (1971) for continuous-parameter Gaussian processes with stationary increments and Hölder continuous covariance kernels.

Throughout the rest of this paper, in order to avoid "fussy" details, we shall redefine the function $\log x$ (and therefore $\log_2 x$ as well) for $0 < x \leqq e^e$ by setting $\log x = e$ in this interval.

2. Upper half of the law of the iterated logarithm.

As the following theorem shows, the upper half of the law of the iterated logarithm can be established under quite unrestrictive hypotheses, hypotheses which in no way imply that the X_i's are weakly dependent.

THEOREM 1. *Suppose that*

$$\liminf_{n\to\infty} V(Kn)/V(n) = A > 1 \quad \textit{for some integer} \quad K \geqq 2 \tag{2.1}$$

and that for each $\varepsilon > 0$, *there exists* $\rho = \rho(\varepsilon) < 1$ *such that*

$$\limsup_{n\to\infty} \{\max_{\rho n \leqq i \leqq n} V(i)/V(n)\} < 1 + \varepsilon\,. \tag{2.2}$$

Then

$$\limsup_{n\to\infty} |S_n|/(2V(n)\log_2 n)^{\frac{1}{2}} \leqq 1 \quad \text{a.s.} \tag{2.3}$$

REMARK. Note that if either $V(n)$ is nondecreasing or $\max_{i \leqq n} V(i) \sim V(n)$, then (2.2) holds. It will be shown in Section 4 that if $V(n)$ satisfies the assumptions of Theorem 4 for the integral test (1.9), then (2.1) and (2.2) hold.

Before proving Theorem 1, we state and prove two corollaries.

COROLLARY 1. *Suppose that for all* i, j,

$$\operatorname{Cov}(X_i, X_j) \geqq 0 . \tag{2.4}$$

Then (2.3) *holds.*

PROOF. Clearly $V(n)$ is increasing and hence, by the remark above, (2.2) holds. Trivially, $V(2n) \geqq 2V(n)$ and hence (2.1) holds. □

REMARK. Note that in the particular case of total dependence given by $X_i = X_1$ a.s. for $i \geqq 1$,

$$|S_n|/(2V(n)\log_2 n)^{\frac{1}{2}} = |X_1|/(2(EX_1^2)\log_2 n)^{\frac{1}{2}} \to 0 \quad \text{a.s.}$$

This helps explain the apparently surprising fact that assumption (2.4) in Corollary 1 places no restriction on the amount of dependence between the X_i's.

COROLLARY 2. *Suppose that for some slowly varying function* $L(\cdot)$ *and some* $0 < \alpha \leqq 2$,

$$V(n) \sim n^\alpha L(n) .$$

Then (2.3) *holds.*

PROOF. According to the uniform convergence theorem for slowly varying functions (see Seneta (1976), page 2), $L(tx)/L(t) \to 1$ as $t \to \infty$, uniformly for x in any specified finite interval. Thus (2.1) and (2.2) follow trivially. □

The proof of Theorem 1 depends on the following lemmas.

LEMMA 1. (i) *If* (2.1) *and* (2.2) *hold, then*

$$V_*(n) = \max_{i \leqq n} V(i) \ll V(n) . \tag{2.5}$$

(ii) *If* (2.1) *and* (2.5) *hold, then* $\lim_{n\to\infty} V(n) = \infty$ *and*

$$V(n+1)/V(n) \to 1 \quad \text{as} \quad n \to \infty . \tag{2.6}$$

PROOF. Assume (2.1). Then there exist n_0 and $B > 1$ such that

$$V(Kn) > BV(n) \qquad \text{for all} \quad n \geqq n_0 . \tag{2.7}$$

Therefore $V(n) \to \infty$ as $n \to \infty$ along the sequence $\{n_0 K^j\}$ and so

$$\max_{i \leqq n} V(i) \to \infty \qquad \text{as} \quad n \to \infty . \tag{2.8}$$

Choose $0 < \theta < K^{-1}$. By (2.7), we have for $n > n_0$

$$\max_{n_0 \leqq i \leqq n} V(i) = \max_{\theta n \leqq i \leqq n} V(i) . \tag{2.9}$$

Suppose (2.1) and (2.5) both hold. Then $\lim_{n\to\infty} V(n) = \infty$ by (2.5) and (2.8).

Moreover, since

$$|V(n+h) - V(n)| \leqq ES_h^2 + 2(V(n)ES_h^2)^{\frac{1}{2}},$$

it is easy to see that (2.6) holds. Therefore we have proved (ii).

To prove (i), suppose (2.1) and (2.2) both hold. Take $\varepsilon > 0$. By (2.2) there exists $0 < \rho < 1$ such that for all $h = 1, 2, \cdots$

$$\max_{\rho^h n \leqq i \leqq n} V(i) \leqq (1+\varepsilon)^h (1 + o(1)) V(n) . \tag{2.10}$$

From (2.9) and (2.10) (with h sufficiently large), (2.5) follows. □

LEMMA 2. *Assume* (2.1) *and* (2.5). *Define* $V(t)$ *for real* $t \geqq 0$ *by linear interpolation of* $\{V(n), n \geqq 0\}$ (*setting* $V(0) = 0$). *Let* $0 < \gamma < (\log A)/\log K$, *where* A *and* K *are defined in* (2.1). *Then there exist* $C > 0$ *and* $t_0 > 0$ *such that for all* $s > 1$, $t \geqq t_0$,

$$V(st)/V(t) \geqq Cs^\gamma \tag{2.11}$$

and so

$$V(r) \gg r^\gamma . \tag{2.12}$$

PROOF. By Lemma 1, (2.1) and (2.5) imply (2.6). Now, note that $V([Kt])/V(K[t]) \to 1$. This follows from (2.6) and the fact that $|[Kt] - K[t]| \leqq |[Kt] - Kt| + |Kt - K[t]| \leqq 1 + K$. Second, note that $\min\{V([t]), V([t]+1)\} \leqq V(t) \leqq \max\{V([t]), V([t]+1)\}$ and $\max\{V([Kt]), V([Kt]+1)\} \geqq V(Kt) \geqq \min\{V([Kt]), V([Kt]+1)\}$. Combining, it follows by (2.1) and (2.6) that for each t sufficiently large

$$V(Kt)/V(t) \sim V([Kt])/V([t]) \sim V(K[t])/V([t]) > B \tag{2.13}$$

where $A > B > 1$.

Let $B = K^\gamma$ define γ. By (2.13), for t sufficiently large and $j \geqq 1$

$$V(K^j t)/V(t) \geqq B^j = K^{j\gamma} . \tag{2.14}$$

For $K^j \leqq s < K^{j+1}$, $j \geqq 0$, t large (say $t \geqq t_0$), we have

$$V(st)/V(K^j t) \geqq V(st)/V_*(st) \geqq \delta \tag{2.15}$$

for some positive constant δ in view of (2.5) and so

$$V(st)/V(t) \geqq \delta V(K^j t)/V(t) \geqq K^{j\gamma}\delta \geqq (\delta/K^\gamma)s^\gamma \tag{2.16}$$

by (2.15) and (2.14). □

REMARK. It should be noted that (2.1) and Lemma 2 are closely related to the concept of dominated variation (Feller (1969)).

LEMMA 3 (Marcus and Shepp (1971)). *Let* $\{Y_i, i \geqq 1\}$ *be a sequence of jointly Gaussian random variables with* $P[\sup_{i\geqq 1} |Y_i| < \infty] = 1$. *Then, letting* $\sigma^2 = \sup_{i \geqq 1} \operatorname{Var}(Y_i)$, *for* $\rho > 0$ *and all sufficiently large* t, *we have*

$$P[\sup_{i\geqq 1} |Y_i| > t] < \exp(-(1-\rho)t^2/(2\sigma^2)) . \tag{2.17}$$

PROOF OF THEOREM 1. We first assert that for each $\varepsilon > 0$,

$$\limsup_{n\to\infty} |S_n|/\{V^{\frac{1}{2}}(n)(\log n)^\varepsilon\} < \infty \quad \text{a.s.} \tag{2.18}$$

Before proving this assertion, we now show how (2.18) allows us to apply the Marcus–Shepp inequality (2.17) to produce the desired result (2.3). Let

$$n_k = [\exp(k^\alpha)]\,, \tag{2.19}$$

where $0 < \alpha < 1$ is to be chosen below. Fix $\delta > 0$. Then by (2.19),

$$P[|S_{n_k}| > (1+\delta)\{2V(n_k)\log_2 n_k\}^{\frac{1}{2}}] \ll \exp\{-(1+\delta)^2(1-\delta)\alpha \log k\}\,. \tag{2.20}$$

But choosing S close to 0 and α close to 1 makes the above expression a term of a convergent series. Hence

$$P[|S_{n_k}| > (1+\delta)(2V(n_k)\log_2 n_k)^{\frac{1}{2}} \text{ i.o.}] = 0\,. \tag{2.21}$$

Now, for $n_k < n \leqq n_{k+1}$, letting $M_k = \max_{n_k<n\leqq n_{k+1}} |S_n - S_{n_k}|$,

$$|S_n| \leqq |S_{n_k}| + M_k\,. \tag{2.22}$$

Let $V_* = V_*(n_{k+1} - n_k)$ and $c_k = (2V(n_k)\log_2 n_k)^{\frac{1}{2}}$. Fixing $\varepsilon > 0$ and using the stationarity of $\{X_i\}$ and Lemma 1, we have for all k sufficiently large

$$\begin{aligned} P[M_k > \delta c_k] &\leqq P[\sup_{n>n_k} |S_n - S_{n_k}|/\{V^{\frac{1}{2}}(n-n_k)\log^\varepsilon (n-n_k)\} \\ &\qquad > \delta c_k/\{V_*^{\frac{1}{2}} \log^\varepsilon (n_{k+1}-n_k)\}] \\ &\leqq P[\sup_{n\geqq 1} |S_n|/\{V^{\frac{1}{2}}(n)(\log n)^\varepsilon\} \\ &\qquad > \theta c_k/\{V^{\frac{1}{2}}(n_{k+1}-n_k)\log^\varepsilon (n_{k+1}-n_k)\}] \end{aligned} \tag{2.23}$$

for some $\theta > 0$. By (2.18), we can apply Lemma 3 to obtain that for some $\xi > 0$ and all k sufficiently large,

$$P[M_k > \delta c_k] \leqq \exp\{-\xi V(n_k)(\log_2 n_k)/(V(n_{k+1}-n_k)\log^{2\varepsilon}(n_{k+1}-n_k))\}\,. \tag{2.24}$$

By Lemma 2 and the mean value theorem,

$$V(n_k)/V(n_{k+1}-n_k) \gg k^{(1-\alpha)\gamma}\,. \tag{2.25}$$

Moreover,

$$\log^{2\varepsilon}(n_{k+1}-n_k) \ll k^{2\alpha\varepsilon}\,. \tag{2.26}$$

Thus, applying (2.25) and (2.26) to (2.24), we obtain that for all large k

$$P[M_k > \delta c_k] \leqq \exp\{-k^{(1-\alpha)\gamma-2\alpha\varepsilon}\}\,. \tag{2.27}$$

Noting that $\alpha < 1$, we choose ε small enough so that the right-hand side of (2.27) is a term of a convergent series. Hence

$$P[M_k > \delta c_k \text{ i.o.}] = 0 \tag{2.28}$$

for each $\delta > 0$. Combining this with (2.21) and (2.22) yields for each $\delta > 0$ with probability one that for all large k

$$\max_{n_k<n\leqq n_{k+1}} |S_n| \leqq (1+2\delta)\{2V(n_k)\log_2 n_k\}^{\frac{1}{2}}\,. \tag{2.29}$$

But, noting that $n_{k+1}/n_k \to 1$, it follows from (2.2) that for $n_k < n \leqq n_{k+1}$,

$$V(n_k)/V(n) \leqq \max_{n_k \leqq i \leqq n} V(i)/V(n) \to 1 . \tag{2.30}$$

Thus the desired conclusion (2.3) follows from (2.29) and (2.30).

To complete the proof of Theorem 1, we now show that (2.18) indeed holds for every $\varepsilon > 0$. First an easy application of the Borel–Cantelli lemma shows that (2.18) holds for $\varepsilon = \frac{1}{2}$. Now fix $0 < \varepsilon < \frac{1}{2}$. We shall apply the Marcus–Shepp inequality as before to show that (2.18) holds for the fixed ε. Define n_k by (2.19) where $0 < \alpha < 1$ is chosen below, and set

$$b_k = V^{\frac{1}{2}}(n_k)(\log n_k)^{\varepsilon} . \tag{2.31}$$

In place of (2.20), we now have

$$P[|S_{n_k}| > b_k] \ll \exp(-\tfrac{1}{2}k^{\alpha\varepsilon}) . \tag{2.32}$$

Defining M_k as before, we now replace (2.23) by

$$\begin{aligned} P[M_k > b_k] \leqq P[&\sup_{n \geqq 1} |S_n|/\{V(n)\log n\}^{\frac{1}{2}} \\ &> \theta b_k/\{V(n_{k+1} - n_k)\log(n_{k+1} - n_k)\}^{\frac{1}{2}}] , \end{aligned} \tag{2.33}$$

where θ is a positive constant. Using the Marcus–Shepp inequality (which is applicable since (2.18) holds for $\varepsilon = \frac{1}{2}$) and arguing as in (2.24)—(2.26) with the obvious modifications, we obtain in place of (2.27) that

$$P[M_k > b_k] \leqq \exp\{-k^{(1-\alpha)\gamma-\alpha}\} \tag{2.34}$$

for all large k. Choose α close to 0 so that the right-hand side of (2.34) is a term of a convergent series. Therefore (2.18) follows from (2.30), (2.32), (2.34) and the Borel–Cantelli lemma. □

The preceding proof therefore consists of two stages, the first of which is to check (2.18) involving the cruder boundary $V^{\frac{1}{2}}(n)(\log n)^{\varepsilon}$ so that the Marcus–Shepp inequality can be applied with this boundary. Now that we have established the upper half (2.3) of the law of the iterated logarithm, it is natural to ask whether we can apply the Marcus–Shepp inequality to the boundary $(2V(n)\log_2 n)^{\frac{1}{2}}$ instead to obtain a sharper result than (2.3). Such manipulation of the Marcus–Shepp inequality gives the following refinement of Theorem 1 in the case where $V(n)$ is nondecreasing.

THEOREM 2. *Suppose that $V(n)$ is nondecreasing and* (2.1) *holds. Let $0 < \gamma <$ $(\log A)/(\log K)$, where A and K are as in* (2.1). *Let $\phi(t)$ be a positive nondecreasing function on $[1, \infty)$ such that*

$$\int_1^\infty t^{-1}(\phi(t))^{(6/\gamma)-1}\exp(-\tfrac{1}{2}\phi^2(t))\,dt < \infty . \tag{2.35}$$

Then

$$P[|S_n| > V^{\frac{1}{2}}(n)\phi(n) \text{ i.o.}] = 0 . \tag{2.36}$$

REMARK. Let $V(n) = n^{\alpha}L(n)$ for some $0 < \alpha \leqq 2$ and some nondecreasing slowly varying function $L(n)$. Then $\lim_{n\to\infty} V(Kn)/V(n) = K^\alpha$. Therefore Theorem 2 holds for $0 < \gamma < \alpha$, and the condition (2.35) already looks quite like the best possible condition $\int_1^\infty t^{-1}(\phi(t))^{(2/\alpha)-1} \exp(-\frac{1}{2}\phi^2(t))\, dt < \infty$ as given by (1.9). Although Theorem 2 does not give the sharpest upper-class test, it yields a much more delicate result than the upper half (2.3) of the law of the iterated logarithm. For example,

$$\phi(t) = \{2 \log_2 t + (1 + \varepsilon + 6\gamma^{-1}) \log_3 t\}^{\frac{1}{2}}$$

satisfies (2.35) for each $\varepsilon > 0$. The only additional assumption we have to add to those in Theorem 1 for this sharper result is that $V(n)$ is nondecreasing. As pointed out before, this simple condition implies condition (2.2) of Theorem 1. When $\mathrm{Cov}\,(X_i, X_j) \geqq 0$ for all i, j, the conditions of Theorem 2 are satisfied, as has been shown in the proof of Corollary 1.

Proof of Theorem 2. By Theorem 1, the upper half (2.3) of the law of the iterated logarithm holds. Therefore without loss of generality, we may assume that for all large t,

$$\phi(t) \leqq 2(\log_2 t)^{\frac{1}{2}}\,. \tag{2.37}$$

Let $\tilde{\phi}(t) = \phi(t) - b/\phi(t)$, where $b > 0$ is to be chosen later. Now (2.35) implies that $\phi(t) \geqq (\log_2 t)^{\frac{1}{2}}$ for all large t (cf. Jain, Jogdeo and Stout (1975), Lemma 2.3), so

$$\int_e^\infty (t\tilde{\phi}(t))^{-1}(\log_2 t)^{3/\gamma} \exp(-\tfrac{1}{2}\tilde{\phi}^2(t))\, dt < \infty\,. \tag{2.38}$$

Let $n_k = [\exp(k(\log k)^{-3/\gamma})]$. We note that

$$P[|S_{n_k}| > V^{\frac{1}{2}}(n_k)\tilde{\phi}(n_k)] \ll (\tilde{\phi}(n_k))^{-1} \exp(-\tfrac{1}{2}\tilde{\phi}^2(n_k))\,. \tag{2.39}$$

Applying a change of variable $t = \exp(u(\log u)^{-3/\gamma})$ to (2.38), we obtain using the integral-comparison test for series that

$$\sum (\tilde{\phi}(n_k))^{-1} \exp(-\tfrac{1}{2}\tilde{\phi}^2(n_k)) < \infty\,.$$

Hence by the Borel–Cantelli lemma,

$$P[|S_{n_k}| > V^{\frac{1}{2}}(n_k)\phi(n_k) - bV^{\frac{1}{2}}(n_k)/\phi(n_k) \text{ i.o.}] = 0\,. \tag{2.40}$$

Let $M_k = \max_{n_k < n \leqq n_{k+1}} |S_n - S_{n_k}|$. Since $V(n)$ and $\phi(n)$ are both nondecreasing and (2.40) holds, to prove (2.36), it suffices to show that

$$P[M_k > bV^{\frac{1}{2}}(n_k)/\phi(n_k) \text{ i.o.}] = 0\,. \tag{2.41}$$

Let $x_k = V^{\frac{1}{2}}(n_k)/\{\phi(n_k)V^{\frac{1}{2}}(n_{k+1} - n_k) \log_2{}^{\frac{1}{2}}(n_{k+1} - n_k)\}$. As in (2.23), we obtain that

$$P[M_k > bV^{\frac{1}{2}}(n_k)/\phi(n_k)] \leqq P[\sup_{n\geqq 1} (V(n) \log_2 n)^{-\frac{1}{2}}|S_n| > bx_k]\,. \tag{2.42}$$

(Recall that $\log_2 x$ is redefined as 1 for $x \leqq e^e$ and is monotone, as stated Section 1.) Since (2.3) holds by Theorem 1, we can apply Lemma 3 to the right-hand side of (2.42) and obtain that

$$P[M_k > bV^{\frac{1}{2}}(n_k)/\phi(n_k)] \leqq \exp(-\tfrac{1}{4}b^2 x_k^2) \tag{2.43}$$

for all large k. By the mean value theorem,

$$n_{k+1} - n_k \sim (\log k)^{-3/\gamma} \exp\{k(\log k)^{-3/\gamma}\} . \tag{2.44}$$

Therefore by Lemma 2, $V(n_k)/V(n_{k+1} - n_k) \gg (\log k)^3$. This together with (2.37) and (2.44) implies that there exists $c > 0$ for which $x_k^2 \geqq c^2 \log k$ for all large k. Hence choosing $b > 2/c$ makes the right-hand side of (2.43) a term of a convergent series, thus proving (2.41). □

3. Lower half of the law of the iterated logarithm. In this section, we consider the problem of finding restrictions on $\{V(n)\}$ which imply the lower half of the law of the iterated logarithm. The following theorem presents a lower-class test and the remarks which follow give some important special cases and corollaries of the theorem.

THEOREM 3. *Suppose that* (2.1) *and* (2.5) *hold and there exist* $M \geqq 1$ *and* $\delta > 0$ *such that for all* $m \geqq M$ *and* $m/n \leqq \delta$,

$$V(n + m)/V(n) - 1 \leqq M\{V(m)/V(n)\}^{\frac{1}{2}}/\log_2(n/m) . \tag{3.1}$$

If $\phi(t)$ *is a positive nondecreasing function on* $[1, \infty)$ *such that*

$$\int_1^\infty (t\phi(t))^{-1} \exp(-\tfrac{1}{2}\phi^2(t))\, dt = \infty , \tag{3.2}$$

then $P[S_n > V^{\frac{1}{2}}(n)\phi(n) \text{ i.o.}] = 1$.

REMARKS. (i) The function $\phi(t) = (2 \log_2 t + \log_3 t)^{\frac{1}{2}}$ clearly satisfies (3.2). Thus Theorem 3 implies the lower half of the law of the iterated logarithm. Although Theorem 3 does not give the sharpest lower-class test, the condition (3.2) already closely resembles the best possible condition $\int_1^\infty (t\phi(t))^{-1}(\phi(t))^{2/\alpha} \exp(-\frac{1}{2}\phi^2(t))\, dt = \infty$ given by (1.9) under much more stringent conditions on $V(n)$.

(ii) Suppose (2.1) and (2.5) hold and

$$\operatorname{Cov}(X_i, X_j) \leqq 0 \qquad \text{for} \quad i \neq j . \tag{3.3}$$

Then (3.1) is satisfied. To see this, we note that

$$V(n + m)/V(n) - 1 \leqq V(m)/V(n) = (V(m)/V(n))^{\frac{1}{2}}/(V(n)/V(m))^{\frac{1}{2}} ,$$

and that by Lemma 2, $V(n)/V(m) \geqq C(n/m)^\gamma > C \log_2{}^2(n/m)$ if $m \geqq h$ and $n/m \geqq h$, for some positive constants C, γ, and h.

(iii) Suppose $V(n) = n^\alpha L(n) + O(n^{\alpha\beta})$ where $0 < \alpha \leqq 2$, $\beta < \frac{1}{2}$ and the function L is slowly varying, monotone increasing, and satisfies (1.8) for some positive constants C and δ. Then (3.1) holds. Moreover, (2.1) and (2.2) are clearly satisfied, and so (2.5) holds by Lemma 1.

The proof of Theorem 3 depends on the following two lemmas. Lemma 4 was obtained by Pathak and Qualls (1973) and independently by Lai (1973) under a slightly stronger assumption but as a corollary of a more general result. Lemma 5 was obtained by Slepian (1962).

LEMMA 4. *Let* $\{Y_k, k \geqq 1\}$ *be a stationary zero-mean Gaussian sequence such that* Var $Y_1 = 1$ *and*

$$EY_1 Y_k = O(1/\log k)\,. \tag{3.4}$$

Let $\psi(k)$ *be a positive nondecreasing function on* $\{1, 2, \cdots\}$. *Then*

$$P[Y_k > \psi(k) \text{ i.o.}] = 0 \quad or \quad 1 \qquad according\ as \tag{3.5}$$
$$\textstyle\sum_1^\infty (\psi(k))^{-1} \exp(-\frac{1}{2}\psi^2(k)) < \infty \quad or \quad = \infty\,.$$

LEMMA 5 (Slepian's lemma). *Let* $(Z_1', \cdots, Z_N')$, $(Z_1'', \cdots, Z_N'')$ *be two zero-mean Gaussian random vectors such that*

$$\operatorname{Var} Z_i' = \operatorname{Var} Z_i'' \ (i = 1, \cdots, N) \qquad and \tag{3.6}$$
$$\operatorname{Cov}(Z_i', Z_j') \leqq \operatorname{Cov}(Z_i'', Z_j'') \qquad for \quad 1 \leqq i < j \leqq N\,.$$

Then for all constant vectors $(a_1, \cdots, a_N)$,

$$P(\textstyle\bigcup_{i=1}^N [Z_i' > a_i]) \geqq P(\bigcup_{i=1}^N [Z_i'' > a_i])\,. \tag{3.7}$$

PROOF OF THEOREM 3. Set $n_k = [e^{\lambda k}]$ where $\lambda > 0$ will be chosen later. Let $Z_k = S_{n_k}/V^{\frac{1}{2}}(n_k)$. Noting that $2ES_m S_n = ES_m^2 + \{ES_n^2 - E(S_n - S_m)^2\}$, we obtain that for $h \leqq k$,

$$2\operatorname{Cov}(Z_h, Z_k) = (V(n_h)/V(n_k))^{\frac{1}{2}} + (V(n_k) - V(n_k - n_h))/\{V(n_h)V(n_k)\}^{\frac{1}{2}}\,. \tag{3.8}$$

Therefore by Lemma 2, (2.5) and (3.1), there exist positive constants ρ, C, and Δ (which do not depend on λ) such that for all $h < k$ with $\lambda h \geqq C$ and $\lambda(k - h) \geqq \Delta$,

$$\operatorname{Cov}(Z_h, Z_k) \leqq Ce^{-\lambda\rho(k-h)} + C/\log(\lambda(k - h))\,. \tag{3.9}$$

Take $\lambda = \max(\Delta, C, e^e)$. Then (3.9) holds for all $k > h \geqq 1$. Let $r(t)$ be a continuous convex (on $[0, \infty)$) even function on the real line such that $r(0) = 1$ and $r(t) = Ce^{-\lambda\rho t} + C/(\log \lambda t)$ for $t \geqq 1$. By Pólya's criterion (cf. Lukacs (1970), page 83), $r(t)$ is the covariance kernel of a zero-mean stationary Gaussian process $\{Y(t), t \geqq 0\}$. By (3.9), $\operatorname{Cov}(Z_h, Z_k) \leqq \operatorname{Cov}(Y(h), Y(k))$ for all $k > h \geqq 1$. Moreover, $EZ_k^2 = EY^2(k) = 1$ for all k. Therefore by Lemma 5,

$$P[Z_k > \phi(n_k) \text{ i.o.}] \geqq P[Y(k) > \phi(n_k) \text{ i.o.}]\,. \tag{3.10}$$

Applying a change of variable $t = e^{\lambda u}$ to the integral in (3.2), we obtain by the integral-comparison test for series that $\sum_1^\infty (\phi(n_k))^{-1} \exp(-\frac{1}{2}\phi^2(n_k)) = \infty$. Therefore by Lemma 4, $P[Y(k) > \phi(n_k) \text{ i.o.}] = 1$. In view of (3.10), this implies that $P[S_n > V^{\frac{1}{2}}(n)\phi(n) \text{ i.o.}] = 1$. □

4. Characterization of upper and lower functions. In this section, we regulate more precisely the rate of growth of $V(n)$ than in Sections 2 and 3. The following theorem gives an integral criterion for classifying monotone functions $\phi(t)$ into upper and lower classes associated with $\{S_n\}$ sequence.

THEOREM 4. *Suppose that there exist* $0 < \alpha \leqq 2$, $\gamma > \alpha/2$, $M \geqq 1$, $\delta > 0$, $\beta > 0$ *and a positive sequence* $\{L(n)\}$ *satisfying the following conditions*:

$$n^\alpha L(n) \ll V(n) \ll n^\alpha L(n) \quad \text{as} \quad n \to \infty\,; \tag{4.1}$$

$$|V(n+m)/V(n) - 1| \leqq M(m/n)^\gamma \quad \text{if} \quad \delta n \geqq m \geqq M\,; \tag{4.2}$$

$$\begin{aligned} &\limsup_{n\to\infty} \{\max_{n(\log_2 n)^{-\beta} \leqq j \leqq n} L(j)/L(n)\} < \infty \quad \text{and} \\ &\liminf_{n\to\infty} \{\min_{n(\log_2 n)^{-\beta} \leqq j \leqq n} L(j)/L(n)\} > 0\,; \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\forall \rho > 0\,, \ \exists m_\rho \ \text{and} \ 1 \geqq \lambda_\rho > 0 \ \text{such that if} \ \lambda_\rho n \geqq m \geqq m_\rho\,, \\ &\text{then} \ (m/n)^\rho \leqq L(n)/L(m) \leqq (n/m)^\rho\,. \end{aligned} \tag{4.4}$$

Let $\phi(t)$ *be a positive nondecreasing function on* $[1, \infty)$. *Then*

$$\begin{aligned} &P[S_n > V^{\frac{1}{2}}(n)\phi(n) \ \text{i.o.}] = 0 \ \text{or} \ 1 \quad \text{according as} \\ &\int_1^\infty t^{-1}(\phi(t))^{(2/\alpha)-1} \exp(-\tfrac{1}{2}\phi^2(t))\,dt < \infty \quad \text{or} \quad = \infty\,. \end{aligned} \tag{4.5}$$

REMARKS. (i) If a function $V(\cdot)$ is regularly varying (with exponent α) and satisfies (4.2) with $\gamma = 1$, then it is called smoothly α-varying by Taqqu (1977). Thus $V(n) = n^\alpha L(n)$ in (1.7) is smoothly α-varying since $L(n)$ is a slowly varying function satisfying (1.8). In Theorem 4, for $\alpha < 2$, since we only require $\gamma > \alpha/2$ in (4.2), our condition is weaker than in the case of smooth variation. Taqqu (1977, Lemma A2) has obtained the following useful connection between regular variation and smooth variation: if $v(\cdot)$ is $(\rho - 1)$-varying (at infinity) with $\rho > 0$, then $V(n) = \sum_{k=1}^n v(k)$ is smoothly ρ-varying.

(ii) The conditions (4.3) and (4.4) closely resemble certain well-known properties of slowly varying functions. Of course they are not strong enough to imply that $L(\cdot)$ is slowly varying. On the other hand, (4.3) is a little too strong for functions which are barely slowly varying. While it is clearly satisfied by slowly varying functions like $\log_k x$ ($k \geqq 1$), it is not hard to construct slowly varying functions (though of a somewhat pathological nature) which violate (4.3). Using the integral representation for slowly varying functions (cf. Feller (1966), page 274), it is easy to see that condition (4.4) is satisfied by slowly varying functions.

COROLLARY 3. *Let* $r(n) = EX_1X_{n+1}$, $n = 0, 1, \cdots$. *Let* $L(\cdot)$ *be a positive slowly varying function satisfying conditions* (4.3) (*for some* $\beta > 0$).

(a) *Suppose* $1 < \alpha < 2$ *and*

$$r(n) \sim n^{\alpha-2}L(n)\,. \tag{4.6}$$

Then the upper-lower class test (4.5) *holds for* $0 < \phi(\cdot) \uparrow$, *and*

$$V(n) \sim 2\{\alpha(\alpha-1)\}^{-1} n^\alpha L(n)\,. \tag{4.7}$$

(b) *Suppose* $0 < \alpha < 1$ *and*

$$r(n) \sim -n^{\alpha-2}L(n)\,; \tag{4.8a}$$

$$r(0) + 2\sum_{n=1}^\infty r(n) = 0\,. \tag{4.8b}$$

Then the upper-lower class test (4.5) *holds for* $0 < \phi(\cdot)\uparrow$, *and*

$$V(n) \sim 2\{\alpha(1-\alpha)\}^{-1} n^{\alpha} L(n) . \tag{4.9}$$

REMARK. In Corollary 3(b), since $0 < \alpha < 1$ and $|r(n)| \sim n^{\alpha-2}L(n)$, $\sum_1^\infty |r(n)| < \infty$. Hence $\sigma^2 = r(0) + 2\sum_{n=1}^\infty r(n)$ is a finite nonnegative number. (The fact that it is nonnegative is well known; see the proof of Corollary 4 below.) Corollary 3(b) therefore deals with the case $\sigma^2 = 0$. The case where $\sigma^2 > 0$ will be treated in Corollary 4 for $\alpha < \frac{1}{2}$.

PROOF OF COROLLARY 3. As shown by Taqqu (1977, Corollary A2), $V(n) = \sum_{i=1}^n \sum_{j=1}^n r(i-j)$ is smoothly α-varying under the assumptions that $L(\cdot)$ is a positive slowly varying function and that (4.6) (resp. (4.8)) holds for $1 < \alpha < 2$ (resp. $0 < \alpha < 1$). Moreover, the asymptotic relations (4.7) and (4.9) hold. Since $L(\cdot)$ is also assumed to satisfy (4.3), Theorem 4 is applicable and the upper-lower class test (4.5) follows. □

COROLLARY 4. *Let* $r(n) = EX_1X_{n+1}$, $n = 1, 2, \cdots$. *Suppose that for some* $\varepsilon > \frac{3}{2}$,

$$|r(n)| \ll n^{-\varepsilon} . \tag{4.10}$$

Let $\sigma^2 = r(0) + 2\sum_1^\infty r(i)$. *If* $\sigma^2 > 0$, *then*

$$V(n) = \sigma^2 n + O(n^{(2-\varepsilon)^+}) , \tag{4.11}$$

and the upper-lower class test (1.5) *holds for* $0 < \phi(\cdot)\uparrow$.

PROOF. We note that

$$\begin{aligned} V(n) &= nr(0) + 2\sum_{i=1}^{n-1}(n-i)r(i) \\ &= n\sigma^2 - 2n\sum_{i=n}^\infty r(i) - 2\sum_{i=1}^{n-1} ir(i) = n\sigma^2 + O(n^{(2-\varepsilon)^+}) . \end{aligned}$$

Therefore (4.11) holds, and this implies that (4.2) holds with $\gamma = 1 - (2-\varepsilon)^+ > \frac{1}{2}$. Set $\alpha = 1$ and $L(n) \equiv 1$ in Theorem 4. The conditions of Theorem 4 are all satisfied and the upper-lower class test (4.5) reduces to (1.5) since $\sigma n^{\frac{1}{2}} = V^{\frac{1}{2}}(n)(1 + O(n^{-\lambda}))$ with $\lambda > \frac{1}{2}$. □

REMARK. Corollary 4 gives immediately the classical upper-lower class test (1.5) for sums of i.i.d. $N(0, \sigma^2)$ random variables. By an embedding argument, Stout and Philipp (1975, Corollary 5.1) have obtained the upper-lower class test (1.5) for partial sums of nonstationary Gaussian sequences satisfying certain conditions which in the particular case of stationarity reduce to (4.10) with $\varepsilon = 2$.

The following lemma, which will be useful in our proof of Theorem 4, shows that the conditions of Theorem 4 imply the conditions of Theorem 1 on the upper half of the law of the iterated logarithm. Since $\gamma > \alpha/2$ in (4.2), it is obvious that the conditions of Theorem 4 also imply the conditions of Theorem 3 on the lower half of the law of the iterated logarithm.

LEMMA 6. *Under the assumptions of Theorem* 4, (2.1) *and* (2.2) *both hold.*

Consequently by Theorem 1, *the upper half* (2.3) *of the law of the iterated logarithm holds. Therefore to prove the upper-lower class test* (4.5), *it suffices to consider the case*

$$\phi(t) \leqq (3 \log_2 t)^{\frac{1}{2}} \quad \textit{for all large} \quad t . \tag{4.12}$$

PROOF. By (4.1), there exist $c_1 > c_2 > 0$ such that

$$c_2 n^\alpha L(n) \leqq V(n) \leqq c_1 n^\alpha L(n) \quad \text{for all large} \quad n . \tag{4.13}$$

Let d denote the lim sup in (4.3). Then by (4.3) and (4.13), for every integer $K \geqq 2$,

$$\liminf_{n\to\infty} V(Kn)/V(n) \geqq (c_2/c_1)K^\alpha \liminf_{n\to\infty} L(Kn)/L(n) \geqq c_2 K^\alpha/(c_1 d) .$$

Therefore choosing K such that $K^\alpha > c_1 d/c_2$, we obtain (2.1). From (4.2), it is easy to see that (2.2) also holds. □

LEMMA 7. *If* (4.3) *holds for some* $\beta > 0$, *then it holds for all* $\beta > 0$.

PROOF. Suppose there exist $\beta > 0$, $0 < a < 1 < b$ and n_0 such that $aL(n) \leqq L(j) \leqq bL(n)$ for $n \geqq n_0$ and $n(\log_2 n)^{-\beta} \leqq j \leqq n$. Then for $n(\log_2 n)^{-2\beta} \leqq j \leqq n(\log_2 n)^{-\beta}$ and n large,

$$a^2 L(n) \leqq aL(n(\log_2 n)^{-\beta}) \leqq L(j) \leqq bL(n(\log_2 n)^{-\beta}) \leqq b^2 L(n) .$$

Proceeding inductively, we thus obtain that for each $k \geqq 1$,

$$a^k L(n) \leqq L(j) \leqq b^k L(n) ,$$

if n is large and $n(\log_2 n)^{-k\beta} \leqq j \leqq n$. □

LEMMA 8. *Assume the hypotheses of Theorem* 4. *Given* $0 < \varepsilon < \min\{\alpha/2, \gamma - \alpha/2\}$, *there exist* m_0 *and* λ_0 *such that if* $\lambda_0 n \geqq m \geqq m_0$, *then*

$$|ES_m S_n|/\{V(m)V(n)\}^{\frac{1}{2}} \leqq (m/n)^\varepsilon .$$

PROOF. As indicated in the proof of (3.8), for $n \geqq m$,

$$\begin{aligned} &2ES_m S_n/\{V(m)V(n)\}^{\frac{1}{2}} \\ &\qquad = (V(m)/V(n))^{\frac{1}{2}} + (V(n) - V(n-m))/\{V(m)V(n)\}^{\frac{1}{2}} . \end{aligned} \tag{4.14}$$

By (4.2), there exist m_0 and $0 < \lambda_0 < 1$ such that if $\lambda_0 n \geqq m \geqq m_0$, then

$$|V(n) - V(n-m)| \leqq M(m/(n-m))^\gamma V(n-m) . \tag{4.15}$$

By (4.13), (4.14) and (4.15), choosing m_0 and λ_0 suitably, we have for $\lambda_0 n \geqq m \geqq m_0$,

$$\begin{aligned} &|ES_m S_n|/\{V(m)V(n)\}^{\frac{1}{2}} \\ &\qquad \leqq C(m/n)^{\alpha/2}(L(m)/L(n))^{\frac{1}{2}} \\ &\qquad\qquad + C\{m/(n-m)\}^\gamma\{(n-m)/(mn)^{\frac{1}{2}}\}^\alpha\{L(n-m)/(L(m)L(n))^{\frac{1}{2}}\} , \end{aligned} \tag{4.16}$$

where C is a positive constant. For $\lambda_0 n \geqq m$, we have $n \geqq n - m \geqq (1 - \lambda_0)n$, and so by (4.3), $L(n-m) \leqq C'L(n)$ if $\lambda_0 n \geqq m \geqq m_0$, where $C' > 0$ and m_0 is chosen large enough. Take any $\rho > 0$. By (4.4), choosing $m_0 \geqq m_\rho$ and $\lambda_0 \leqq \lambda_\rho$, we obtain that $L(m)/L(n) \leqq (n/m)^\rho$ and $L(n)/L(m) \leqq (n/m)^\rho$ if $\lambda_0 n \geqq m \geqq m_0$. Applying these estimates to (4.16), we obtain that for $\lambda_0 n \geqq m \geqq m_0$,

$$|ES_m S_n|/\{V(m)V(n)\}^{\frac{1}{2}} \leqq C''\{(m/n)^{(\alpha-\rho)/2} + (m/n)^{\gamma-(\alpha+\rho)/2}\}$$

for some $C'' > 0$. □

We now make use of the above lemmas to estimate the first exit probability (1.10). Our estimates, which are crucial to the proof of Theorem 4, are given in Lemmas 9 and 10 below.

LEMMA 9. *Assume the hypotheses of Theorem* 4. *Let* $\phi(t)$ *be a positive function such that* $\lim_{t\to\infty} \phi(t) = \infty$ *and* (4.12) *holds. Then*

$$P[\max_{2^k \leqq n \leqq 2^{k+1}} (V(n))^{-\frac{1}{2}} S_n \geqq \phi(2^k)] \ll (\phi(2^k))^{(2/\alpha)-1} \exp(-\tfrac{1}{2}\phi^2(2^k)) . \tag{4.17}$$

PROOF. By Lemma 7, we can choose $\beta > 3/\alpha$ and let $l(k) = [2^k/(\log k)^\beta]$, $m(k) = [2(\log k)^\beta]$. For each fixed k, set $n_i = 2^k + il(k)$, $i = 0, 1, \cdots, m(k)$. By (4.2), for k large and $i = 0, \cdots, m(k)$, if $n_i + M \leqq n \leqq n_{i+1}$, then

$$|V(n)/V(n_i) - 1| \leqq M\{(n_{i+1} - n_i)/n_i\}^\gamma \leqq M(\log k)^{-\beta\gamma} . \tag{4.18}$$

On the other hand, if $n_i \leqq n \leqq n_i + M$, then since (2.2) holds by Lemma 6 and

$$|V(n) - V(n_i)| \leqq |V(n) - V(n_i + 2M)| + |V(n_i + 2M) - V(n_i)| ,$$

(4.18) also holds with M replaced by $M' = 2M$, provided that k is large enough. Therefore for all large k and $i = 0, \cdots, m(k)$,

$$\begin{aligned}(V(n_i)/V(n))^{\frac{1}{2}}\{\phi(2^k) - (\phi(2^k))^{-1}\} \\ \leqq \{1 + 2M'(\log k)^{-\beta\gamma}\}^{\frac{1}{2}}\{\phi(2^k) - (\phi(2^k))^{-1}\} < \phi(2^k) .\end{aligned} \tag{4.19}$$

To see the last inequality above, we note that $\beta\gamma > 1$ by our choice of β and so $(\log k)^{-\beta\gamma}\phi(2^k) = o(1/\phi(2^k))$ by (4.12).

From (4.19), it follows that for all large k,

$$\begin{aligned}&\bigcap_{i=0}^{m(k)} [\max_{n_i \leqq n \leqq n_{i+1}} (S_n - S_{n_i}) \leqq (V(n_i))^{\frac{1}{2}}/\phi(2^k)] \\ &\quad \cap [\max_{0 \leqq i \leqq m(k)} (V(n_i))^{-\frac{1}{2}} S_{n_i} \leqq \phi(2^k) - 2(\phi(2^k))^{-1}] \\ &\qquad \subset \bigcap_{i=0}^{m(k)} [\max_{2^k \leqq n \leqq 2^{k+1}} S_n \leqq (V(n_i))^{\frac{1}{2}}\{\phi(2^k) - (\phi(2^k))^{-1}\}] \\ &\qquad \subset [\max_{2^k \leqq n \leqq 2^{k+1}} (V(n))^{-\frac{1}{2}} S_n < \phi(2^k)] .\end{aligned}$$

Therefore, for all large k,

$$\begin{aligned}&P[\max_{2^k \leqq n \leqq 2^{k+1}} (V(n))^{-\frac{1}{2}} S_n \geqq \phi(2^k)] \\ &\quad \leqq P[\max_{0 \leqq i \leqq m(k)} (V(n_i))^{-\frac{1}{2}} S_{n_i} \geqq \phi(2^k) - 2(\phi(2^k))^{-1}] \\ &\quad + \textstyle\sum_{i=0}^{m(k)} P[\max_{n_i \leqq n \leqq n_{i+1}} (S_n - S_{n_i}) \geqq (V(n_i))^{\frac{1}{2}}/\phi(2^k)] .\end{aligned} \tag{4.20}$$

But by stationarity,

$$\begin{aligned}&\textstyle\sum_{i=0}^{m(k)} P[\max_{n_i \leqq n \leqq n_{i+1}} (S_n - S_{n_i}) \geqq (V(n_i))^{\frac{1}{2}}/\phi(2^k)] \\ &\quad \leqq 2(\log k)^\beta P[\max_{j \leqq 2^k(\log k)^{-\beta}} S_j \geqq \min_{0 \leqq i \leqq m(k)} (V(n_i))^{\frac{1}{2}}\phi/(2^k)] .\end{aligned} \tag{4.21}$$

By our convention, $\log_2 n \geqq 1$ for $n \geqq 1$. Also (2.3) holds by Lemma 6. Hence letting

$$a_k = \min_{2^k \leqq n \leqq 2^{k+2}} (V(n))^{\frac{1}{2}} / \{\phi(2^k) \max_{j \leqq 2^k (\log k)^{-\beta}} (2V(j) \log_2 2^k)^{\frac{1}{2}}\}, \tag{4.22}$$

we can apply Lemma 3 to obtain

$$\begin{aligned} P[\max_{j \leqq 2^k (\log k)^{-\beta}} S_j \geqq \min_{0 \leqq i \leqq m(k)} (V(n_i))^{\frac{1}{2}} / \phi(2^k)] \\ \leqq P[\sup_{j \geqq 1} (2V(j) \log_2 j)^{-\frac{1}{2}} S_j \geqq a_k] \leqq \exp(-\tfrac{1}{2} a_k^2). \end{aligned} \tag{4.23}$$

By (4.1) and (4.3) together with Lemma 7, as $k \to \infty$,

$$\min_{2^k \leqq n \leqq 2^{k+2}} V(n) \gg 2^{k\alpha} \min_{2^k \leqq n \leqq 2^{k+2}} L(n) \gg 2^{k\alpha} L(2^k). \tag{4.24}$$

Since (2.6) holds by Lemmas 1 and 6,

$$\begin{aligned} \max_{j \leqq 2^k (\log k)^{-\beta}} V(j) &\ll 2^{k\alpha} (\log k)^{-\alpha\beta} L([2^k (\log k)^{-\beta}]) \\ &\ll 2^{k\alpha} (\log k)^{-\alpha\beta} L(2^k) \quad \text{by (4.3)}. \end{aligned} \tag{4.25}$$

Putting (4.12), (4.24) and (2.25) into (4.22), we obtain that for some $C > 0$ and all large k,

$$\begin{aligned} (\log k)^\beta \exp(-\tfrac{1}{2} a_k^2) &\leqq (\log k)^\beta \exp(-C(\log k)^{\alpha\beta - 2}) \\ &= o(\{(\phi(2^k))^{(2/\alpha)-1} \exp(-\tfrac{1}{2}\phi^2(2^k))\}), \end{aligned} \tag{4.26}$$

since $\alpha\beta > 3$ and $\phi^2(2^k) = O(\log k)$.

From (4.20), (4.21), (4.23) and (4.26), it suffices to prove that

$$\begin{aligned} P[\max_{0 \leqq i \leqq m(k)} (V(n_i))^{-\frac{1}{2}} S_{n_i} \geqq \phi(2^k) - 2(\phi(2^k))^{-1}] \\ \ll (\phi(2^k))^{(2/\alpha)-1} \exp(-\tfrac{1}{2}\phi^2(2^k)). \end{aligned} \tag{4.27}$$

To prove (4.27), let $\rho_k(i, j)$ be the covariance between $(V(n_i))^{-\frac{1}{2}} S_{n_i}$ and $(V(n_j))^{-\frac{1}{2}} S_{n_j}$ for $0 \leqq i, j \leqq m(k)$. We note that for all large k and $m(k) \geqq i > j \geqq 0$,

$$\begin{aligned} 1 - \rho_k(i, j) &= 1 - \tfrac{1}{2}(V(n_i)V(n_j))^{-\frac{1}{2}} \{V(n_i) + V(n_j) - V(n_i - n_j)\} \\ &\leqq \tfrac{1}{2}(V(n_i)V(n_j))^{-\frac{1}{2}} V(n_i - n_j) \\ &\leqq C''(n_i - n_j)^\alpha L(n_i - n_j) / \{n_i^{\alpha/2} n_j^{\alpha/2} (L(n_i)L(n_j))^{\frac{1}{2}}\} \\ &\leqq C'\{(i - j)/(\log k)^\beta\}^\alpha \leqq C\{(i - j)/m(k)\}^\alpha \end{aligned} \tag{4.28}$$

for some positive constants C, C', and C'' by (4.1), (4.3) and the fact that $n_i - n_j = (i - j)[2^k/(\log k)^\beta]$ and $2^k \leqq n_i, n_j \leqq 2^{k+2}$.

Let d be a positive integer such that $d^\alpha > C$. Then

$$\begin{aligned} P[\max_{0 \leqq i \leqq m(k)} (V(n_i))^{-\frac{1}{2}} S_{n_i} \geqq \phi(2^k) - 2(\phi(2^k))^{-1}] \\ \leqq \sum_{\nu=1}^d P[\max_{\nu m(k)/d \geqq i \geqq (\nu-1)m(k)/d} (V(n_i))^{-\frac{1}{2}} S_{n_i} \geqq \phi(2^k) - 2(\phi(2^k))^{-1}]. \end{aligned} \tag{4.29}$$

Let $r(t)$, $|t| \leqq d^{-1}$, be an even positive definite function such that $r(0) = 1$ and $r(t) \leqq 1 - C|t|^\alpha$ for $|t| \leqq d^{-1}$. By Pólya's criterion (cf. Lukacs (1970), page 83), such functions exist. Now let $\{Z(t), 0 \leqq t \leqq d^{-1}\}$ be a separable stationary Gaussian process with covariance function $r(t)$. For $\nu = 1, \cdots, d$ and

$\nu m(k)/d \geqq i > j \geqq (\nu - 1)m(k)/d$, we obtain from (4.28) that for all large k,

$$\rho_k(i, j) \geqq 1 - C\{(i - j)/m(k)\}^\alpha \geqq \operatorname{Cov}\{Z(i/m(k)), Z(j/m(k))\}. \tag{4.30}$$

Therefore by Lemma 5 (i.e., Slepian's lemma) and the stationarity of $Z(\cdot)$, for all large k,

$$\begin{aligned} &\textstyle\sum_{\nu=1}^d P[\max_{\nu m(k)/d \geqq i \geqq (\nu-1)m(k)/d} (V(n_i))^{-\frac{1}{2}} S_{n_i} \geqq \phi(2^k) - 2(\phi(2^k))^{-1}] \\ &\quad \leqq d\, P[\max_{0 \leqq i \leqq m(k)/d} Z(i/m(k)) \geqq \phi(2^k) - 2(\phi(2^k))^{-1}] \\ &\quad \leqq d\, P[\sup_{0 \leqq t \leqq d^{-1}} Z(t) \geqq \phi(2^k) - 2(\phi(2^k))^{-1}] \\ &\quad \ll (\phi(2^k))^{(2/\alpha)-1} \exp(-\tfrac{1}{2}\{\phi(2^k) - 2(\phi(2^k))^{-1}\}^2) \quad \text{as} \quad k \to \infty. \end{aligned} \tag{4.31}$$

The last relation above follows from Slepian's lemma and a theorem of Qualls and Watanabe (1972, Theorem 2.1). From (4.29) and (4.31), (4.27) follows as desired. □

LEMMA 10. *Assume the hypotheses of Theorem* 4. *Let c be an integer* > 1 *and let $\phi(t)$ be a positive function such that*

$$\log_2 t \leqq \phi^2(t) \leqq 3 \log_2 t \qquad \textit{for all large} \quad t. \tag{4.32}$$

For each k define the set

$$G_k = \{c^k + i[c^k/\phi^{2/\alpha}(c^{k+1})] : i = 0, 1, \cdots, [\tfrac{1}{2}\phi^{2/\alpha}(c^{k+1})]\}. \tag{4.33}$$

Then as $k \to \infty$,

$$P[\max_{n \in G_k} (V(n))^{-\frac{1}{2}} S_n \geqq \phi(c^{k+1})] \gg (\phi(c^{k+1}))^{(2/\alpha)-1} \exp(-\tfrac{1}{2}\phi^2(c^{k+1})). \tag{4.34}$$

PROOF. For each fixed k, let $n_i = c^k + i[c^k/\phi^{2/\alpha}(c^{k+1})]$ and let $\rho_k(i, j)$ be the covariance between $(V(n_i))^{-\frac{1}{2}} S_{n_i}$ and $(V(n_j))^{-\frac{1}{2}} S_{n_j}$. As in (4.28), for $0 \leqq j < i \leqq [\frac{1}{2}\phi^{2/\alpha}(c^{k+1})]$,

$$\begin{aligned} 1 - \rho_k(i, j) &= \tfrac{1}{2}\{2 - (V(n_i)/V(n_j))^{\frac{1}{2}} - (V(n_j)/V(n_i))^{\frac{1}{2}}\} \\ &\quad + \tfrac{1}{2}(V(n_i)V(n_j))^{-\frac{1}{2}} V(n_i - n_j). \end{aligned} \tag{4.35}$$

By (4.1), (4.3) and (4.32), for all large k and $0 \leqq j < i \leqq [\frac{1}{2}\phi^{2/\alpha}(c^{k+1})]$, since $c^k \leqq n_i, n_j \leqq 2c^k$ and $n_i - n_j = (i - j)[c^k/\phi^{2/\alpha}(c^{k+1})]$, we obtain

$$\begin{aligned} &\tfrac{1}{2}(V(n_i)V(n_j))^{-\frac{1}{2}} V(n_i - n_j) \\ &\quad \geqq D'(n_i - n_j)^\alpha L(n_i - n_j)/\{n_i^{\alpha/2} n_j^{\alpha/2} (L(n_i)L(n_j))^{\frac{1}{2}}\} \\ &\quad \geqq D\{(i - j)/\phi^{2/\alpha}(c^{k+1})\}^\alpha \end{aligned} \tag{4.36}$$

for some positive constants D, D'. By (4.2), there exist $\gamma > \alpha/2$, $M \geqq 1$, and $0 < \theta < \frac{1}{2}$ such that for all large k and $0 \leqq j < i \leqq \theta[\phi^{2/\alpha}(c^{k+1})]$,

$$\begin{aligned} &|2 - (V(n_i)/V(n_j))^{\frac{1}{2}} - (V(n_j)/V(n_i))^{\frac{1}{2}}| \\ &\quad = |\{1 - (V(n_i)/V(n_j))^{\frac{1}{2}}\}\{1 - (V(n_j)/V(n_i))^{\frac{1}{2}}\}| \\ &\quad \leqq \tfrac{1}{3}M^2\{(n_i - n_j)/n_j\}^{2\gamma} \leqq \tfrac{1}{3}M^2\{(i - j)/\phi^{2/\alpha}(c^{k+1})\}^{2\gamma}. \end{aligned} \tag{4.37}$$

Noting that $\theta \geqq (i - j)/\phi^{2/\alpha}(c^{k+1})$ in (4.37) and applying (4.36), (4.37) to (4.35), we can choose $\theta > 0$ sufficiently small (say $M^2\theta^{2\gamma-\alpha} < D$) so that for all large k and $0 \leqq j < i \leqq \theta[\phi^{2/\alpha}(c^{k+1})]$,

$$1 - \rho_k(i, j) \geqq \tfrac{1}{2}D\{(i-j)/\phi^{2/\alpha}(c^{k+1})\}^\alpha . \tag{4.38}$$

Let $0 < d < D$. Let $r(t)$, $|t| \leqq \theta$, be an even positive definite function such that $r(0) = 1$ and

$$r(t) = 1 - \tfrac{1}{2}d|t|^\alpha + o(|t|^\alpha) \quad \text{as} \quad t \to 0 ; \tag{4.39a}$$

$$r(t) \geqq 1 - \tfrac{1}{2}D|t|^\alpha \quad \text{for all} \quad |t| \leqq \theta . \tag{4.39b}$$

Such functions exist by Pólya's criterion. Let $\{Y(t),\ 0 \leqq t \leqq \theta\}$ be a separable stationary Gaussian process with covariance function $r(t)$. Set $u(k) = \phi^{2/\alpha}(c^{k+1})$. For all large k and $\theta u(k) \geqq i > j \geqq 0$, we obtain from (4.38) and (4.39b) that

$$\rho_k(i, j) \leqq 1 - \tfrac{1}{2}D\{(i-j)/u(k)\}^\alpha \leqq \operatorname{Cov}\{Y(i/u(k)),\ Y(j/u(k))\} .$$

Hence by Slepian's lemma, for all large k,

$$\begin{aligned} &P[\max_{n \in G_k} (V(n))^{-\frac{1}{2}} S_n \geqq \phi(c^{k+1})] \\ &\qquad \geqq P[\max_{0 \leqq i \leqq \theta u(k)} (V(n_i))^{-\frac{1}{2}} S_{n_i} \geqq \phi(c^{k+1})] \\ &\qquad \geqq P[\max_{0 \leqq i \leqq \theta u(k)} Y(i/u(k)) \geqq \phi(c^{k+1})] . \end{aligned} \tag{4.40}$$

Let $\Delta(x) = (dx^2)^{-1/\alpha}$ and note that $\Delta(\phi(c^{k+1})) = d^{-1/\alpha}/u(k)$. In view of (4.39a), we can apply a result of Qualls and Watanabe (1972, Lemma 2.3) to obtain

$$\begin{aligned} &P[\max_{0 \leqq i \leqq \theta u(k)} Y(i/u(k)) \geqq \phi(c^{k+1})] \\ &\qquad = P[\max_{0 \leqq i \leqq \theta/\{d^{1/\alpha}\Delta(\phi(c^{k+1}))\}} Y(id^{1/\alpha}\Delta(\phi(c^{k+1}))) \geqq \phi(c^{k+1})] \\ &\qquad \sim \rho(\phi(c^{k+1}))^{(2/\alpha)-1} \exp\{-\tfrac{1}{2}\phi^2(c^{k+1})\} \quad \text{as} \quad k \to \infty , \end{aligned} \tag{4.41}$$

where ρ is a positive constant. From (4.40) and (4.41), (4.34) follows as desired. □

REMARK. With the same notation and assumptions as in Lemma 10, we have obtained that

$$\begin{aligned} (\phi(c^{k+1}))^{(2/\alpha)-1} \exp(-\tfrac{1}{2}\phi^2(c^{k+1})) &\ll P[\max_{n \in G_k} (V(n))^{-\frac{1}{2}} S_n \geqq \phi(c^{k+1})] \\ &\leqq P[\max_{c^k \leqq n \leqq c^{k+1}} (V(n))^{-\frac{1}{2}} S_n \geqq \phi(c^{k+1})] \\ &\ll (\phi(c^{k+1}))^{(2/\alpha)-1} \exp(-\tfrac{1}{2}\phi^2(c^{k+1})) . \end{aligned}$$

The last relation above follows from a straightforward modification of Lemma 9. Now the consecutive points of G_k are at a common distance $[c^k/\phi^{2/\alpha}(c^{k+1})]$ apart. By (4.32), when $\alpha = 1$, this distance is $\gg$ and $\ll$ $c^k/\log k$. If instead of the geometric subsequence $\{c^k\}$, we consider the subsequence $\{\nu_j\}$, where $\nu_j = [c^{(j/\log j)}]$, then $\nu_{j+1} - \nu_j \sim \nu_j/\log j$. Thus, partitioning the "large" block $[c^k, c^{k+1}]$ into "small" blocks of size $[c^k/\log k]$ is "asymptotically equivalent" to partitioning $[c^k, c^{k+1}]$ with those points of $\{\nu_j\}$ which lie inside the interval $[c^k, c^{k+1}]$. The subsequence $\{\nu_j\}$ is used in most standard proofs of the classical upper-lower class test (1.5) for the i.i.d. case. Partly because $\{\nu_j\}$ is a little too fine, these proofs of the lower half of (1.5) are rather difficult.

With our results in Lemmas 9 and 10, we now proceed to complete the proof of Theorem 4 by a simple adaptation of the Borel–Cantelli lemma. Such an easy adaptation (of the "independent half" of the Borel–Cantelli lemma) is possible because the geometric subsequence $\{c^k\}$ yields a nice asymptotic independence property. The following easy lemma gives an equivalent statement (involving the geometric subsequence $\{c^k\}$) of the integral criterion in the upper-lower class test (4.5).

LEMMA 11. *Let $\phi(t)$ be a positive nondecreasing function on* $[1, \infty)$. *Let $c > 1$ and $\theta > 0$. Then*

$$\int_1^\infty t^{-1}(\phi(t))^{(2/\alpha)-1} \exp(-\tfrac{1}{2}\phi^2(t))\, dt < \infty \tag{4.42}$$

if and only if

$$\textstyle\sum_1^\infty (\phi(\theta c^k))^{(2/\alpha)-1} \exp(-\tfrac{1}{2}\phi^2(\theta c^k)) < \infty\,. \tag{4.43}$$

PROOF. Since $\phi(t)$ is nondecreasing and the function $g(x) = x^{(2/\alpha)-1}\exp(-\frac{1}{2}x^2)$ is eventually monotone, it follows that (4.43) holds if and only if

$$\int_1^\infty (\phi(\theta c^u))^{(2/\alpha)-1} \exp(-\tfrac{1}{2}\phi^2(\theta c^u))\, du < \infty\,. \tag{4.44}$$

Applying a change of variable $t = \theta c^u$ to (4.42), it is clear that (4.42) and (4.44) are equivalent. □

PROOF OF THEOREM 4. By Lemma 6, we may assume without loss of generality that $\phi^2(t) \leqq 3 \log_2 t$ for all large t. First we assume the integral in (4.5) converges. Then $\phi(t) \uparrow \infty$ and hence by Lemmas 9 and 11 we obtain that

$$\textstyle\sum_{k=1}^\infty P[\max_{2^k \leqq n \leqq 2^{k+1}} (V(n))^{-\frac{1}{2}} S_n \geqq \phi(2^k)] < \infty\,.$$

Since $\phi(\cdot)$ is nondecreasing, it follows by the Borel–Cantelli lemma that

$$P[S_n \geqq (V(n))^{\frac{1}{2}}\phi(n) \text{ i.o.}] = 0\,,$$

proving the upper half of the theorem.

Now assume that the integral in (4.5) diverges. As is well known (cf. Jain, Stout, and Jogdeo (1975), Lemma 2.3 and page 131), we may assume without loss of generality that for all large t,

$$2 \log_2 t \leqq \phi^2(t) \leqq 3 \log_2 t\,. \tag{4.45}$$

Let $c > 3$ be a positive integer to be specified later. Let $t_{ki} = c^k + i[c^k/\phi^{2/\alpha}(c^{k+1})]$, $i = 0, \cdots, [\frac{1}{2}\phi^{2/\alpha}(c^{k+1})]$, and $G_k = \{t_{ki} : i = 0, \cdots, [\frac{1}{2}\phi^{2/\alpha}(c^{k+1})]\}$. Define

$$E_k = [\max_{n \in G_k} (V(n))^{-\frac{1}{2}} S_n < \phi(c^{k+1})]\,.$$

By Lemmas 10 and 11, $\sum_{k=1}^\infty P(E_k') = \infty$ ($E_k' = \Omega \backslash E_k$). As in the Borel–Cantelli lemma, we write

$$1 - P[E_k' \text{ i.o.}] = \lim_{m\to\infty} \textstyle\prod_m^\infty P(E_k) + \lim_{m\to\infty} \{P(\bigcap_m^\infty E_k) - \prod_m^\infty P(E_k)\}\,. \tag{4.46}$$

In order to complete the proof of the theorem, noting the monotonicity of $\phi(\cdot)$, it suffices to prove $P[E_k' \text{ i.o.}] = 1$. The first limit in (4.46) is zero because $\sum_{k=1}^\infty P(E_k') = \infty$. Therefore, we need only prove the asymptotic independence of the E_k's (for c chosen sufficiently large) in the sense that

$$\lim_{m\to\infty} \lim_{n\to\infty} |P(\bigcap_m^n E_k) - \prod_m^n PE_k| = 0 . \tag{4.47}$$

Let $l(k) = [\frac{1}{2}\phi^{2/\alpha}(c^{k+1})]$. As is well known (cf. Qualls and Watanabe (1971), Lemma 1.5),

$$|P(\bigcap_m^n E_k) - \prod_m^n PE_k| \leqq \sum_{m\leqq k<h\leqq n} \sum_{i=0}^{l(k)} \sum_{j=0}^{l(h)} |r(t_{ki}, t_{hj})| \int_0^1 f(\phi(c^{k+1}), \phi(c^{h+1}); \lambda r(t_{ki}, t_{hj}))\, d\lambda \tag{4.48}$$

where $r(t_{ki}, t_{hj})$ is the correlation coefficient between $(V(t_{ki}))^{-\frac{1}{2}}S_{t_{ki}}$ and $(V(t_{hj}))^{-\frac{1}{2}}S_{t_{hj}}$, and $f(x, y; \rho)$ is the standard bivariate normal density with correlation coefficient ρ. Note that for $h > k$,

$$t_{hj} \geqq c^h \geqq (\tfrac{2}{3})c^{h-k}t_{ki} \qquad \text{for} \quad i \leqq l(k)\,, \quad j \leqq l(h)\,.$$

Take $0 < \varepsilon < \min\{\alpha/2, \gamma - \alpha/2\}$. Choose $c > 3$ such that $(\frac{2}{3})c > \lambda_0^{-1}$, where $\lambda_0 = \lambda_0(\varepsilon)$ is as given by Lemma 8. Then, by Lemma 8, for $h > k \geqq m_0$ and $i \leqq l(k)$, $j \leqq l(h)$, we have

$$|r(t_{hj}, t_{ki})| \leqq (t_{ki}/t_{hj})^\varepsilon \leqq (\tfrac{3}{2})^\varepsilon c^{-\varepsilon(h-k)} < \tfrac{1}{8}\,, \tag{4.49}$$

by choosing c sufficiently large. Therefore taking $k_0 (\geqq m_0)$ large enough, we obtain by (4.45) and (4.49) that for $h > k \geqq k_0$ and $i \leqq l(k), j \leqq l(h)$, $0 \leqq \lambda \leqq 1$,

$$\begin{aligned} &f(\phi(c^{k+1}), \phi(c^{h+1}); \lambda r(t_{ki}, t_{hj})) \\ &\quad \leqq (2\pi)^{-1}(1 - (\tfrac{1}{8})^2)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}[\phi^2(c^{k+1}) + \phi^2(c^{h+1}) \\ &\qquad - 2|r(t_{ki}, t_{hj})|\phi(c^{k+1})\phi(c^{h+1})]\} \\ &\quad \leqq \pi^{-1} \exp\{-\log k - \log h + \tfrac{1}{2}(\log h)^{\frac{1}{2}}(\log k)^{\frac{1}{2}}\} \leqq \pi^{-1}k^{-1}h^{-\frac{1}{2}}\,. \end{aligned} \tag{4.50}$$

Therefore using (4.45), (4.49) and (4.50), there exists $A > 0$ such that for $m \geqq k_0$,

$$\begin{aligned} &\sum_{m\leqq k\leqq h} \sum_{i=0}^{l(k)} \sum_{j=0}^{l(h)} |r(t_{ki}, t_{hj})| \int_0^1 f(\phi(c^{k+1}), \phi(c^{h+1}); \lambda r(t_{ki}, t_{hj}))\, d\lambda \\ &\quad \leqq \sum_{m\leqq k<h} (l(k) + 1)(l(h) + 1)(\tfrac{3}{2})^\varepsilon c^{-\varepsilon(h-k)}\pi^{-1}k^{-1}h^{-\frac{1}{2}} \\ &\quad \leqq A \sum_{m\leqq k<h} (\log h)^{2/\alpha}c^{-\varepsilon(h-k)}k^{-1}h^{-\frac{1}{2}} \\ &\quad = A \sum_{k=m}^\infty k^{-1} \sum_{j=1}^\infty (k + j)^{-\frac{1}{2}}(\log (k + j))^{2/\alpha}c^{-\varepsilon j} \\ &\quad = O(\sum_{k=m}^\infty k^{-\frac{5}{4}} \sum_{j=1}^\infty c^{-\varepsilon j}) \to 0 \qquad \text{as} \quad m \to \infty\,. \end{aligned}$$

Hence by (4.48), (4.47) holds, proving the theorem. □

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