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AN INVERSE BALAYAGE PROBLEM FOR BROWNIAN MOTION

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Let B be a standard n-dimensional Brownian motion, let A be compact and let ν be a probability measure on ∂A. We treat the following inverse exit problem: describe the set $M(\nu)$ of all probability measures μ on A such that $P^{\mu}\{B(T) \in \cdot\} = \nu(\cdot)$, where T is the time of first exit from A. Elements of $M(\nu)$ are characterized in terms of integrals of harmonic functions with respect to them. For $n = 1$, extreme points of $M(\nu)$ are computed in closed form; for $n \geqslant 2$, extreme points of $M(\nu)$ are characterized. Geophysical and potential-theoretic aspects of the problem are discussed.

1. Introduction. Motivated by a model for a distillation process used in petroleum refining, Ray and Margo [14] posed the following problems for a Markov chain X with finite state space $S = S_1 \cup S_2$. Let states in S_1 be transient and states in S_2 be absorbing. A probability ν on S_2 is the balayage of a probability μ on S_1 provided $\nu(\cdot) = P^{\mu}\{X(T) \in \cdot\}$, where T is the first hitting time of S_2. Then, find all ν that are the balayage of at least one μ and for each such ν find all μ of which ν is the balayage. Subsequent papers by Ray [13] and Pittenger [11] treated various aspects of the problem but failed to obtain either complete or explicit results. Karr and Pittenger [10] obtained characterizations, when S_1 is finite, of the compact, convex set

$$M(\nu) = \{\mu : \nu \text{ is the balayage of } \mu\},$$

of the extreme points of $M(\nu)$, and of those measures in $M(\nu)$ that are maximal with respect to a partial ordering defined in terms of the excessive functions of X. In particular, [10] contains an explicit identification of the extreme points of $M(\nu)$ when X is a one-dimensional random walk and S_1 is an interval whose endpoints comprise S_2. The approaches in [11], [13] and [14] were algebraic, whereas that in [10] relies heavily on probability and potential theory.

It is natural to consider the same questions for processes with continuous time and state space and the purpose of this note is to describe some work in this direction. Our context is Brownian motion in n-dimensional Euclidean space and our problem, specifically, is the following.

Let (B_t) be n-dimensional Brownian motion, let $A \subset \mathbb{R}^n$ be compact, and let ν be a probability measure on ∂A.

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(1) Describe the set $M(\nu)$ of probability measures μ on A such that $\nu(\cdot) = P^{\mu}\{B(T) \in \cdot\}$, where $T = \inf\{t > 0 : B_t \notin A\}$.

(2) Characterize the extreme points of $M(\nu)$.

For simplicity we assume in this note that A is the closure of a simply connected open set.

In Section 2 we solve the one-dimensional problem explicitly; the result is qualitatively the same as that obtained in [10] for one-dimensional random walks. Section 3 contains characterizations of the elements and extreme points of $M(\nu)$ in higher dimensions, but the latter characterization can hardly be called explicit. We also discuss a geophysical interpretation of the problem and an example which demonstrates that the nice structure of the one-dimensional problem fails to carry over to higher dimensions.

2. The one-dimensional problem. Here $A = [a, b]$, $\partial A = \{a, b\}$ and ν is a prescribed probability measure on $\{a, b\}$. For a given probability λ on A, the *direct* problem of computing the P^{λ}-distribution of B_T, where $T = \inf\{t : B_t \notin A\}$ is easily solved: it is well known that for $x \in A$,

$$P^{x}\{B_T = a\} = (b - x)/(b - a), \tag{1}$$

and then one need only integrate with respect to λ. Our problem is the *inverse* of this: find all μ such that $P^{\mu}\{B(T) \in \cdot\} = \nu(\cdot)$. The following result constitutes a complete solution to the problem.

(2) THEOREM. *Let $m = \nu(a)a + \nu(b)b$. Then $\mu \in M(\nu)$ if and only if $\mu([a, b]) = 1$ and*

$$\int_{[a,b]} x\mu(dx) = m. \tag{3}$$

The extreme points of $M(\nu)$ are precisely the following:

(a) *ε_m, the point mass at m;*

and

(b) *the family $\{\eta_{x,y} : a \leqslant x < m < y \leqslant b\}$, where*

$$\eta_{x,y} = \frac{y - m}{y - x}\varepsilon_x + \frac{m - x}{y - x}\varepsilon_y. \tag{4}$$

PROOF. Since (1) implies that

$$P^{\mu}\{B_T = a\} = (b - a)^{-1}\left(b - \int_{[a,b]} x\mu(dx)\right),$$

it follows at once that (3) is necessary and sufficient in order that a probability measure μ on $[a, b]$ belong to $M(\nu)$.

Now it is immediate that ε_m is in $M(\nu)$ and is an extreme point. That each $\eta_{x,y}$ belongs to $M(\nu)$ follows by direct computation, while the argument that $\eta_{x,y}$ is an extreme point of $M(\nu)$ is straightforward: if $\eta_{x,y}$ equals $\alpha\mu_1 + (1 - \alpha)\mu_2$, with μ_i in $M(\nu)$, then $\mu_i(\{x, y\}) = 1$, which forces $\mu_1 = \mu_2 = \eta_{x,y}$.

It remains to show there are no other extreme points. To do this recall a result used in the proof of the Skorohod representation theorem (cf. [5], page 277 or [7], page 68), which asserts that if μ satisfies (3), then there exists a probability measure Q on $(-\infty, m) \times (m, \infty) \cup \{(m, m)\}$ such that for each Borel set B

(5) $$\mu(B) = \int Q(dx, dy)\eta_{x,y}(B),$$

with $\eta_{m,m} = \varepsilon_m$ and $-\infty < x \leqslant m \leqslant y < +\infty$. But if $\mu([a, b]) = 1$ then almost surely with respect to Q, $\eta_{x,y}[a, b] = 1$, which forces $a \leqslant x$ and $y \leqslant b$. Hence the integral in (5) is over $[a, m) \times (m, b] \cup \{(m, m)\}$, and the proof is complete. ▯

We note that if mass outside $A = [a, b]$ is permitted, the results above permit the identification of that larger set of probabilities. The calculations are straightforward and are omitted.

To place Theorem (2) in the context of the potential theoretic discussion of the next section, we note the following equivalences; cf. also Proposition (8) and Theorem (10) below and Proposition 1 and Theorem 4 of [10].

(6) PROPOSITION. *For μ a probability measure on $A = [a, b]$, the following are equivalent*:

(a) $\mu \in M(\nu)$;

(b) $\int x\mu(dx) = m$;

(c) $\int h\, d\mu = \int h\, d\nu$ *for all harmonic functions h on A*;

(d) $U\mu = U\nu$ *at one point of A^c, where $U\mu$ is the Newtonian potential of μ, defined by* $U\mu(y) = -\int |x - y|\, \mu(dx)$;

(e) $E^{\mu}[T] = \int x^2\nu(dx) - \int x^2\mu(dx)$;

(f) *There is a (possibly randomized) stopping time S such that $P^m\{S \leqslant T\} = 1$ and $\mu(\cdot) = P^m\{B(S) \in \cdot\}$.*

PROOF. Equivalence of (a) through (d) is computational. Since $(B_t^2 - t)$ is a martingale, (e) holds when (a) does by virtue of the optional sampling theorem. Conversely, if (e) is satisfied then

$$\begin{aligned} \nu(a)a^2 + \nu(b)b^2 &= \int_{[a,b]}\big(x^2 + E^x[T]\big)\mu(dx) \\ &= \int_{[a,b]}\big(x^2 + (b - x)(x - a)\big)\mu(dx) \end{aligned}$$

and $\int x\mu(dx) = m$ follows.

Sufficiency of (f) is a consequence of the strong Markov property and the fact that T is a terminal time, and necessity is implied by a theorem of Baxter and Chacon [4]. ▯

In the same way we obtain the following result.

(7) PROPOSITION. *For $\mu \in M(\nu)$ the following are equivalent*:

(a) μ *is an extreme point*;

(b) $\mu = \varepsilon_m$ *or* $\mu = \eta_{x,y}$ *as given by* (4) *for some x, y*;

(c) $\mu(\cdot) = P^m\{B(T_{\operatorname{supp}\mu}) \in \cdot\}$, *where* supp μ *denotes the support of* μ.

Of Propositions (6) and (7) very little carries over to the higher-dimensional case.

As an application of the foregoing, note that $E^{\mu}[T]$ can be viewed as a linear functional on the compact, convex set $M(\nu)$ and therefore attains its maximum and minimum at extreme points, the latter at $\mu = \eta_{a,b} = \nu$ and the former at $\mu = \varepsilon_m$, since by (6e)

$$E^{\eta_{x,y}}[T] = (m - a)(b - m) - (m - x)(y - m).$$

Extreme values and points at which they are attained may be similarly evaluated for any linear functional.

Finally, note that the analysis of Theorem (2) applies also to a regular diffusion $(X_t; P^x)$ with scale function S. If we put $m = \nu(a)S(a) + \nu(b)S(b)$ then the extreme points of $M(\nu)$ are precisely

(a) $\varepsilon_{S^{-1}(m)}$,

and

(b) the family $\{\eta_{x,y} : a \leqslant x < S^{-1}(m) < y \leqslant b\}$, where

$$\eta_{x,y} = \frac{S(y) - m}{S(y) - S(x)}\varepsilon_x + \frac{m - S(x)}{S(y) - S(x)}\varepsilon_y.$$

3. The inverse problem for $n \geqslant 2$. To illustrate some aspects of the inverse problem in higher dimensions, we consider the following analogue of the problem of Section 2. Let $A = G \cup \partial G$, where G is a bounded, simply connected open set, and let ν be a probability measure on $\partial A = \partial G$. Denote by H the set of functions harmonic in G and continuous on A and by S the set of functions superharmonic in G and continuous on A. We then have the following analogue of Proposition (6).

(8) PROPOSITION. *For each probability measure μ on A the following are equivalent*:

(a) $\mu \in M(\nu)$;

(b) $\int h\, d\mu = \int h\, d\nu$ *for all* $h \in H$;

(c) $\int f\, d\mu \geqslant \int f\, d\nu$ *for all* $f \in S$;

(d) $U\mu \geqslant U\nu$ *and* $U\mu = U\nu$ *on* A^c, *where* $U\mu$ *is the logarithmic* $(n = 2)$ *or Newtonian* $(n \geqslant 3)$ *potential of* μ.

PROOF. If $\{\rho_x : x \in A\}$ is the set of harmonic measures for A, as defined in [8], for example, then $P^x\{B(T) \in dy\} = \rho_x(dy)$ for each x (cf. Itô-McKean [9]). For μ a probability measure on A and $f \in C(\partial A)$ it follows that

$$E^{\mu}[f(B_T)] = \int_{\partial A} f(y) \int_A \mu(dx)\rho_x(dy),$$

so that $\mu \in M(\nu)$ if and only if

$$\int_{\partial A} f(y)\nu(dy) = \int_{\partial A} f(y) \int_A \mu(dx)\rho_x(dy) \tag{9}$$

for all $f \in C(\partial A)$. Since each $h \in H$ admits the representation

$$h(x) = \int_{\partial A} \rho_x(dy)h(y),$$

the equivalence of (a) and (b) is established. Equivalence of (b), (c) and (d) is demonstrated in [1]. ▯

Proposition (8) shows that our inverse problem admits an interesting geophysical interpretation: to what extent can the mass distribution of the earth be inferred from its potential outside the earth? This interpretation is discussed in more detail in [2]. The problem of finding conditions on a mass distribution under which it is uniquely determined by its exterior potential has a lengthy history, cf. [12]. The general inverse problem of potential theory (in which (8d) is taken to define $M(\nu)$) has also been investigated, cf. [1, 2, 3]. Choquet [6] employs an expression similar to (9) in an analysis of the Dirichlet problem. Our probabilistic approach and interpretation appear to be new.

Our next result characterizes the extreme points of $M(\nu)$ in the multidimensional case but, unfortunately, not very explicitly. Note, however, the analogy to Theorem 2 of [10].

(10) THEOREM. *For each* $\mu \in M(\nu)$ *the following are equivalent*:

(a) μ *is an extreme point*;
(b) *there exists* no $\eta \in M(\nu)$ *such that* $\eta \neq \mu$, $\eta \ll \mu$ *and* $d\eta/d\mu$ *is bounded.*

PROOF. Both implications are shown by contraposition. If (b) fails, there is $0 \leqslant g \in L^\infty(\mu)$ such that $d\eta = g\, d\mu$ defines an element η of $M(\nu)$. If $\alpha > 0$ is sufficiently small that $1 - \alpha g \geqslant 0$ a.e. (μ) then the definition

$$\lambda(A) = (1-\alpha)^{-1}\textstyle\int_A (1 - \alpha g)\, d\mu \tag{11}$$

gives a probability measure λ such that for $h \in H$,

$$\begin{aligned}\textstyle\int h\, d\lambda &= (1-\alpha)^{-1}\left[\textstyle\int h\, d\mu - \alpha \textstyle\int h\, d\eta\right]\\ &= \textstyle\int h\, d\nu.\end{aligned}$$

Hence $\lambda \in M(\nu)$ and (11) then evidently implies that $\mu = \alpha\eta + (1-\alpha)\lambda$; therefore (a) also fails.

Conversely, if (a) fails there exist $\alpha \in (0, 1)$ and $\mu_1, \mu_2 \in M(\nu)$, distinct from μ, such that

$$\mu = \alpha\mu_1 + (1-\alpha)\mu_2. \tag{12}$$

The relation (12) implies that $\mu_1 \ll \mu$ and also that if $f_1 = d\mu_1/d\mu$ then $f_1 \leqslant \alpha^{-1}$ a.e. (μ), and consequently (b) also fails. ▯

Theorem (10) is evidently valid for any process (X_t) with continuous sample paths provided one replaces H in (8) by the set of functions harmonic for (X_t).

The statements (6f) and (7c) show that in one dimension the Brownian motion (B_t) begun at m "sweeps out" all other elements of $M(\nu)$ on its way to the boundary ∂A. We conclude this note with an example demonstrating that a similarly pleasant situation does not obtain, in general, in higher dimensions. It therefore appears, as is also true for the discrete time and state space processes treated in [10], that the one-dimensional case is (in yet another respect) very special.

EXAMPLE. Let G be the open unit disk in $\mathbb{R}^2$ and ν the uniform distribution on ∂G. Then $\varepsilon_{(0,0)} \in M(\nu)$ and it is easily seen that $\varepsilon_x \notin M(\nu)$ for all other $x \in A$. For $\|y\|$ sufficiently small, the $P^{(0,0)}$-hitting distribution η of the circle C of radius $\frac{1}{2}$ about y belongs to $M(\nu)$ by the strong Markov property. For $\|y\|$ possibly still smaller, but still positive, η is absolutely continuous with respect to (one-dimensional) Lebesgue measure m on C with, furthermore, $d\eta/dm$ bounded away from zero, say $d\eta/dm \geqslant \beta > 0$. The measure λ given by

$$\lambda(B) = \beta\varepsilon_y(B) + \int_{B\cap C}(d\eta/dm - \beta)\, dm$$

then agrees with η on harmonic functions by the very definition of harmonic functions and is hence in $M(\nu)$. However, since a singleton is a polar set for two-dimensional Brownian motion neither $\varepsilon_{(0,0)}$ nor λ can lead to the other in the manner of (6f).

Further results concerning the inverse problem will appear in a future paper.

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