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MAXIMUM IN THE LÉVY-BAXTER THEOREM FOR GAUSSIAN RANDOM FIELDS

BY TAKAYUKI KAWADA

Kobe University of Commerce

The range of almost sure limits of F-variation for a class of Gaussian random fields is considered by adopting a class of sequences of partitions in the parameter space of the random field. The application to Lévy's Brownian motion explains, in the case of two-dimensional parameters, that the almost sure limit given by Berman is the maximum in a range.

1. Introduction. There is a series of results on the variation of stochastic processes, especially of the Brownian motion and of Gaussian processes, most of which give a single almost sure (a.s.) limit of the variation, even when a class of sequences of partitions is considered.

The a.s. limit of the variation for a stochastic process depends in general on the sequence of partitions. A main object of this paper is to define a class of the sequences of partitions which generates not only one value, but a range of a.s. limits of the variation for a class of Gaussian random fields. In Section 2, we define this class and F-variation as a generalization of the ordinary quadratic variation. In Section 3, the a.s. limit of the F-variation along each element of this class is indicated, and, moreover, a condition is given under which it attains the maximum in this range of the a.s. limits. An application of this model to Lévy's Brownian motion is given in Section 4. This shows in the case of two-dimensional parameters that the a.s. limit given by Berman [1] is the maximum in the range.

2. Preliminaries. Throughout this paper we shall consider a class of real, mean-zero Gaussian random fields $\{X(\mathbf{t});\ \mathbf{t} = (t_1, t_2, \cdots, t_d) \in [0, 1]^d\}$, and it is assumed that there exists a function S satisfying

$$E(X(\mathbf{t}) - X(\mathbf{s}))^2 = S(|\mathbf{t} - \mathbf{s}|),$$

where $|\mathbf{x}|^2 = x_1^2 + x_2^2 + \cdots + x_d^2$, for $\mathbf{x} = (x_1, x_2, \cdots, x_d)$. The function S is called the *structure function* of X. Then X has stationary increments.

The kth axis of the parameter space $[0, 1]^d$ is equally partitioned by a positive integer $a_k(n)$, $k = 1, 2, \cdots, d$, $n = 1, 2, \cdots$; i.e., the mesh in the kth axis is $a_k^{-1}(n)$. The number of cells in $[0, 1]^d$ obtained through this partition is $\prod_{k=1}^d a_k(n)$. Denote this by $N\{a_k(n)\}$. For the case $a_k(n) = a(n)$ $(k = 1, 2, \cdots, d)$ write $N(\mathbf{a}(n))$. Set $A_k(n) = N\{a_k(n)\}/N(\mathbf{a}_k(n))$, $k = 1, 2, \cdots, d$. The sequence $\{A_k(n)\}$ satisfies $\prod_{k=1}^d A_k(n) = 1$.

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Define the mixed-increment of $X(\mathbf{t})$ for $\mathbf{t} = \{t_k\}$ and $\mathbf{s} = \{s_k\}$, $(s_k < t_k : k = 1, 2, \cdots, d)$ by

$$\Delta_d^{s_d}\Delta_{d-1}^{s_{d-1}} \cdots \Delta_1^{s_1}X(\mathbf{t}),$$

where

$$\Delta_k^{s_k}X(\mathbf{t}) = X(t_1, \cdots, t_k, \cdots, t_d) - X(t_1, \cdots, s_k, \cdots, t_d).$$

Denote simply by $Y_{\mathbf{i},n}$ the mixed-increment for $\mathbf{t} = \{i_k a_k^{-1}(n)\}$ and $\mathbf{s} = \{(i_k - 1)a_k^{-1}(n)\}$, $(\mathbf{i} = \{i_k\}$; i_k is an integer such that $0 \leqslant i_k \leqslant a_k(n)$, $k = 1, 2, \cdots, d)$ and by $c(\mathbf{i}, n)$ the cell $\{\mathbf{u} = (u_1, u_2, \cdots, u_d) | (i_k - 1)a_k^{-1}(n) \leqslant u_k \leqslant i_k a_k^{-1}(n)\}$. In what follows, we omit the subscript n from several sequences dependent on n when it does not invite confusion. The increment-stationarity implies for each $\mathbf{i}$ that $E(Y_\mathbf{i}^2) = E(Y_\mathbf{1}^2)$, $\mathbf{1} = (1, 1, \cdots, 1)$. Moreover, $E(Y_\mathbf{1}^2)$ has an expression in terms of the structure-function S:

$$(1)\quad E(Y_\mathbf{1}^2)/2^{d-1} = \Sigma_{k=1}^d S\Big((a_k^{-2})^{\frac{1}{2}}\Big) - \Sigma_{i<j} S\Big((a_i^{-2} + a_j^{-2})^{\frac{1}{2}}\Big) + \Sigma_{i<j<k} S\Big((a_i^{-2} + a_j^{-2} + a_k^{-2})^{\frac{1}{2}}\Big) - \cdots + (-1)^{d-1} S\Big((a_1^{-2} + a_2^{-2} + \cdots + a_d^{-2})^{\frac{1}{2}}\Big).$$

Here, we introduce a class of sequences of partitions. Denote by $\mathbf{Q}$ the class of sequences of partitions $\{a_k^{-1}\}$ in the parameter space $[0, 1]^d$ satisfying the following conditions:

(D.1) $\Sigma_n G(n)/(N^3\{a_k\}E(Y_\mathbf{1}^2))^2 < \infty$;

(D.2) $\Sigma_{k=1}^d \Sigma_{n=1}^\infty N^{-1/d}(\mathbf{a}_k) < \infty$;

(D.3) Each $A_k = N\{a_k\}/N(\mathbf{a}_k)$ tends to a constant $L_k \in [0, \infty]$, as $n \to \infty$, $(k = 1, 2, \cdots, d)$,

where $G(n) = \Sigma_{(A)} g^2(n; i, j)$, $g(n; i, j) = \max^* D^{2d}(\mathbf{t}, \mathbf{s})S(|\mathbf{t} - \mathbf{s}|)$, $(D^{2d}(\mathbf{t}, \mathbf{s}) = \partial^{2d}/\partial t_1 \partial t_2 \cdots \partial t_d \partial s_1 \cdots \partial s_d)$, $\max^*$ denotes the maximum over all $(\mathbf{t}, \mathbf{s}) \in c(\mathbf{i}) \times c(\mathbf{j})$, $c(\mathbf{i})$ and $c(\mathbf{j})$ being mutually component-wise disjoint, and $\Sigma_{(A)}$ denotes the summation carried out over all pairs of such cells.

Next, we shall impose the condition (M) for X: there exist a positive exponent β and a continuous function M satisfying

(M.1) For a sequence $\{a_k^{-1}\}$ satisfying (D.3), $N^\beta\{a_k\}E(Y_\mathbf{1}^2) \to M(\{L_k\})$ as $n \to \infty$;

(M.2) If $|\mathbf{A}(n)| \to \infty$, $(\mathbf{A} = (A_1, A_2, \cdots, A_d))$, then $M(\{L_k\}) = 0$.

Further, we shall define F-variation. Take $F(u) = |u|^{2\delta}$, $(\delta \geqslant 1$; δ an integer). The F-variation for a sequence $\{a_k^{-1}\}$ is defined by

$$\Sigma_\mathbf{i} F(Y_\mathbf{i})/N_{\{a_k\}}^{1-\beta\delta},$$

where the summation extends over all cells constructed for the partition $\{a_k^{-1}\}$.

Finally, we set $\sigma_{\mathbf{1}}^2 = E(Y_{\mathbf{1}}^2)$ and we note an inequality concerning F and mixed-increments;

$$(2) \qquad E(F(Y_{\mathbf{i}})F(Y_{\mathbf{j}})) - E(F(Y_{\mathbf{i}}))E(F(Y_{\mathbf{j}})) \leqslant \text{const.}\ \rho^2(Y_{\mathbf{i}}, Y_{\mathbf{j}})F^2(\sigma_{\mathbf{1}}),$$

where ρ is the correlation coefficient between $Y_{\mathbf{i}}$ and $Y_{\mathbf{j}}$ (cf. [1]).

3. Model.

PROPOSITION 1. *For each sequence* $\{a_k^{-1}\} \in \mathbf{Q}$,

$$\lim_{n\to\infty}\Sigma_{\mathbf{i}}F(Y_{\mathbf{i}})/N_{\{a_k\}}{}^{1-\beta\delta} = F(M^{\frac{1}{2}}(\{L_k\}))\textstyle\int_{-\infty}^{\infty}x^{2\delta}\exp(-x^2/2)\,dx/(2\pi)^{\frac{1}{2}}, \quad \textit{if}\ \ L_k \neq 0,\ k = 1, 2, \cdots, d;$$
$$= 0 \qquad \textit{otherwise}$$

almost surely.

PROOF. In the first place we show that

$$\Sigma_n \operatorname{Var}(\Sigma_{\mathbf{i}}F(Y_{\mathbf{i}})/N_{\{a_k\}}{}^{1-\beta\delta}) < \infty.$$

In the relation

$$\operatorname{Var}(\Sigma_{\mathbf{i}}F(Y_{\mathbf{i}})) = \Sigma_{(\mathbf{i},\mathbf{j})}(E(F(Y_{\mathbf{i}})F(Y_{\mathbf{j}})) - E(F(Y_{\mathbf{i}}))E(F(Y_{\mathbf{j}}))),$$

we separate the right hand side into $\Sigma_{(A)}$ and $\Sigma_{(B)}$; the former is for the pairs $(\mathbf{i}, \mathbf{j})$ for which $c(\mathbf{i})$ and $c(\mathbf{j})$ are mutually component-wise disjoint, and the latter is for the others. In $\Sigma_{(A)}$, the relation

$$E(Y_{\mathbf{i}}Y_{\mathbf{j}}) = \left(-\tfrac{1}{2}\right)\textstyle\iint_{c(\mathbf{i})\times c(\mathbf{j})}D_{(\mathbf{t},\mathbf{s})}{}^{2d}S(|\mathbf{t}-\mathbf{s}|)\,d\mathbf{t}\,d\mathbf{s}$$

and the inequality (2) work. Then we have, applying the same letter C for various constants,

$$\begin{aligned}\Sigma_{(A)}/N_{\{a_k\}}{}^{2(1-\beta\delta)} &\leqslant CN_{\{a_k\}}{}^{2(\beta\delta-1)}F^2(\sigma_{\mathbf{1}})\Sigma_{(A)}|\rho(Y_{\mathbf{i}}, Y_{\mathbf{j}})|^2 \\ &\leqslant CN_{\{a_k\}}{}^{2(\beta\delta-1)}F^2(\sigma_{\mathbf{1}})\left(N_{\{a_k\}}{}^{-2}\right)^2\Sigma_{(A)}g^2(\mathbf{i},\mathbf{j})/E^2(Y_{\mathbf{1}}^2) \\ &\leqslant C\left(N_{\{a_k\}}{}^{\beta}F_{(\sigma_{\mathbf{1}})}{}^{1/\delta}\right)^{2\delta}G(n)/\left(N_{\{a_k\}}{}^{3}E(Y_{\mathbf{1}}^2)\right)^2 \\ &\leqslant CF\left(N_{\{a_k\}}{}^{\beta}\sigma_{\mathbf{1}}^2\right)G(n)/\left(N_{\{a_k\}}{}^{3}E(Y_{\mathbf{1}}^2)\right)^2.\end{aligned}$$

The convergence of $\Sigma_{(A)}$ stems from (D.1) and from (M).

Next, we have for the summand in $\Sigma_{(B)}$ a majorization by $E^{\frac{1}{2}}(F^2(Y_{\mathbf{i}}))\cdot E^{\frac{1}{2}}(F^2(Y_{\mathbf{j}}))$, which is equal to $E(F^2(Y_{\mathbf{1}}))$, and note that the number of the pair $(\mathbf{i}, \mathbf{j})$ appearing in $\Sigma_{(B)}$ is bounded by a constant multiple of $N_{\{a_k\}}{}^2(\Sigma_{k=1}^d a_k^{-1})$. Then we obtain

$$\begin{aligned}\Sigma_{(B)}/N_{\{a_k\}}{}^{2(1-\beta\delta)} &\leqslant C(\Sigma_{k=1}^d a_k^{-1})N_{\{a_k\}}{}^2E(F^2(Y_{\mathbf{1}}))/N_{\{a_k\}}{}^{2(1-\beta\delta)} \\ &\leqslant C(\Sigma_{k=1}^d a_k^{-1})F(N_{\{a_k\}}{}^{\beta}\sigma_{\mathbf{1}}^2).\end{aligned}$$

This, (D.2) and (M) imply the assertion for the part $\Sigma_{(B)}$.

In the second place we observe that the almost sure limit exists for each $\{a_k^{-1}\}$ in $\mathbf{Q}$. The stationarity of increments implies that

$$E(\Sigma_\mathbf{i} F(Y_\mathbf{i}))/N_{\{a_k\}}^{1-\beta\delta} = N_{\{a_k\}}^{\beta\delta}\sigma_\mathbf{1}^{2\delta}\int_{-\infty}^{\infty} x^{2\delta}\exp(-x^2/2)\,dx/(2\pi)^{\frac{1}{2}}$$
$$= F\Big(\big(N_{\{a_k\}}^{\beta}\sigma_\mathbf{1}^2\big)^{\frac{1}{2}}\Big)\int_{-\infty}^{\infty} x^{2\delta}\exp(-x^2/2)\,dx/(2\pi)^{\frac{1}{2}}.$$

If some A_k diverges here (or, equivalently, converges to zero under the condition $\Pi_{k=1}^d A_k = 1$), $F((N_{\{a_k\}}^{\beta}\sigma_\mathbf{1}^2)^{\frac{1}{2}})$ tends to zero by the condition (M.2). If every A_k converges to a positive constant L_k, $(k = 1, 2, \cdots, d)$, the almost sure limit is $F(M_{(\{L_k\})}^{\frac{1}{2}})\int_{-\infty}^{\infty} x^{2\delta}\exp(-x^2/2)\,dx/(2\pi)^{\frac{1}{2}}$. This completes the proof.

REMARK 1. $\mathbf{Q}$ contains a sequence $\{a_k^{-1}\}$ along which the a.s. limit of F-variation of X vanishes.

REMARK 2. If $L_j \neq 0$, $j = 1, 2, \cdots, d$ holds in Proposition 1, then we have $a_i(n) = 0(a_1(n))$ as $n \to \infty$, $i = 2, 3, \cdots, d$. (The converse statement is trivial). In fact, assume $a_2/a_1 \to 0$ in A_1. Since $L_1 \neq 0$, there exists a number $i(1)$, $(1 \leqslant i(1) \leqslant d)$ for which $a_{i(1)}/a_1 \to \infty$. Next, by $L_{i(1)} \neq 0$, we have $i(2)$ such that $a_{i(2)}/a_{i(1)} \to \infty$. Take $A_{i(2)}$ and repeat the above procedure. Continue it until we find a number $i(p)$, $(p \leqslant q)$, equal to some $i(h) \in \{1, i(1), i(2), \cdots, i(p-2)\}$ (we can find such $i(p)$ after, at most, $d!$ repetitions of the procedure) and take the product; $(a_{i(h)}/a_{i(h+1)})(a_{i(h+1)}/a_{i(h+2)}) \cdots (a_{i(p-1)}/a_{i(p)}) = 1$. The left-hand side tends to zero as $n \to \infty$. This is a contradiction. The other cases are proved similarly.

PROPOSITION 2. *If M is differentiable, then a sufficient condition for the existence of a maximum of the F-variation limits is the existence of solutions of*

$$(3) \qquad x_j D_j M(\{x_k\}) = x_1 D_1 M(\{x_k\}), \qquad D_j = \partial/\partial x_j,\ j = 1, 2, \cdots, d,$$

subject to the conditions $\Pi_{i=1}^d x_i = 1$, $x_i > 0$, $i = 1, 2, \cdots, d$.

PROOF. The value of the limit depends only on $F(M_{(\{x_k\})}^{\frac{1}{2}})$, subject to the condition $\Pi_{i=1}^d x_i = 1$, $x_i > 0$, $i = 1, \cdots, d$. The rest follows immediately using Lagrange's multiplier and condition (M.2).

4. Lévy's d-parameter Brownian motion. Let $\{X(\mathbf{t}) : \mathbf{t} \in [0, 1]^d\}$, $(d \geqslant 2)$ be Gaussian random fields with $E(X(\mathbf{t})) = 0$ and $E(X(\mathbf{t})X(\mathbf{s})) = (\frac{1}{2})(|\mathbf{t}|^\alpha + |\mathbf{s}|^\alpha - |\mathbf{t} - \mathbf{s}|^\alpha)$, $\alpha \in (0, 2)$. The structure function $S(u)$ of X is $S(u) = |u|^\alpha$, $\alpha \in (0, 2)$. Lévy's d-parameter Brownian motion corresponds to the case $\alpha = 1$. Since we have (i) $D_{(\mathbf{t},\mathbf{s})}{}^{2d}E(X(\mathbf{t})X(\mathbf{s})) = (-\frac{1}{2})D_{(\mathbf{t},\mathbf{s})}{}^{2d}|\mathbf{t} - \mathbf{s}|^\alpha$ and (ii) $D_{(\mathbf{x},\mathbf{x})}{}^{2d}r^\alpha \leqslant \text{const.}/r^{2d-\alpha}$, $(r = (\Sigma_{k=1}^d x_k^2)^{\frac{1}{2}})$, we can set $G(n) = \text{const.}/(\min_k\{a_k^{-1}\})^{2(2d-\alpha)}$ in (D.1). Define a function M_α for $\{x_k\} = (x_1, x_2, \cdots, x_d)$ $(x_i > 0, i = 1, 2, \cdots, d)$ by

$$M_\alpha(\{x_k\})/2^{d-1} = \Sigma_{k=1}^d\big(x_k^2\big)^{\alpha/2} - \Sigma_{i<j}\big(x_i^2 + x_j^2\big)^{\alpha/2} + \Sigma_{i<j<k}\big(x_i^2 + x_j^2 + x_k^2\big)^{\alpha/2}$$
$$- \cdots + (-1)^{d-1}\big(x_1^2 + x_2^2 + \cdots + x_d^2\big)^{\alpha/2}.$$

This $M_\alpha(\{x_k\})$ satisfies the condition (M.2) (cf. Appendix). For a partition $\{a_k^{-1}\}$ in $[0, 1]^d$ (see (1)) $E(Y_1^2) = M_\alpha(\{a_k^{-1}\})$, and $N^\beta\{a_k\}E(Y_1^2) = M_\alpha(\{A_k^{1/d}\})$, ($\beta = \alpha/d$). Moreover, for the $M_\alpha(\{x_k\})$ in equation (3) $\{x_k = 1, k = 1, 2, \cdots, d\}$ is a trivial solution. But we obtain in the case $d = 2$ the following theorem:

THEOREM. *For $d = 2$ let $\{X(\mathbf{t}) : \mathbf{t}[0, 1]^d\}$ be the present Gaussian random fields. Set $F(u) = |u|^{2\delta}$, ($\delta \geqslant 1$: δ an integer). Define $\mathbf{Q}_0 = \{\{a_k^{-1}\} \in \mathbf{Q}; \lim_{n\to\infty} a_2(n)/a_1(n) = 1\}$. Then the almost sure limit of the F-variation of X, $\Sigma_\mathbf{i} F(Y_\mathbf{i})/N_{\{a_k\}}^{1-\beta\delta}$, ($\beta = \alpha/d$), along any $\{a_k^{-1}\} \in \mathbf{Q}_0$, attains the maximum*

$$F\left(M_\alpha^{\frac{1}{2}}(\{1\})\right)\int_{-\infty}^{\infty} x^{2\delta} \exp(-x^2/2)\, dx/(2\pi)^{\frac{1}{2}}, \tag{4}$$

in the range of almost sure limits for all sequences in $\mathbf{Q}$.

PROOF. $M_\alpha(\{x_k\})$ satisfies the condition in Proposition 2; more strictly equation (3) has a unique solution. For $\{x_k\}$, set $x_2 = x_1(1 + x)$, $(x > -1)$ and eliminate x_1 from the equation. Then we get $(y - 1)^{\alpha/2} - (y - 2)/y^{1-\alpha/2} = 1$, $(y = 1 + (1 + x)^2)$. Denote the left-hand side by $g(y)$ and solve for y in $g(y) = 1$. A calculation shows that $g(1) = g(2) = 1$, $g(y)$ is the unique maximum in [1, 2], and $g'(y) < 0$ for $y \geqslant 2$. Thus we have the unique solution $y = 2$; i.e., $x = 0$. Accordingly, the original equation (3) has the unique solution $x_1 = x_2 = 1$. Thus the a.s. limit of the F-variation attains the maximum when $\lim_{n\to\infty} A_2(n) = \lim_{n\to\infty} A_1(n)$. The relation $(a_2/a_1)^2 = A_1/A_2$ completes the proof.

COROLLARY. *For $d = 2$ let $\{B(\mathbf{t}) : \mathbf{t} \in [0, 1]^d\}$ be Lévy's two-dimensional Brownian motion. Set $F(u) = |u|^{2d}$. Then the sequence $\{a_k^{-1}(n) = 2^{-n}; k = 1, 2\}$ is in* $\mathbf{Q}_0$; i.e., *for the* a.s. *limit of F-variation of B it gives the maximum, which is* (4) *for $\alpha = 1$ and $\delta = 2$* (cf. [1], [2]).

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APPENDIX

When $|\mathbf{A}| \to \infty$, one can assume without loss of generality that $A_i \to \infty$, ($i = 1, 2, \cdots, p : p \leqslant d - 1$), A_{kj} $(= A_j/A_k, 1 \leqslant k < j \leqslant p) \to c_{kj} \in [0, \infty)$ and $A_h \to c_h \in [0, \infty)$, $(h = p + 1, p + 2, \cdots, d)$ where $c_{p+1} = c_{p+2} = \cdots = c_{p+q-1} = 0 (1 \leqslant q - 1 \leqslant d - p)$. Then we rewrite:

$$\begin{aligned} M_\alpha(\{A_k\}) = \Sigma_{l=1}^{p} A_l^\alpha \Big\{ 1 - \Sigma_{l<i_1}\left(1 + A_{li_1}^2\right)^{\alpha/2} + \Sigma_{l<i_1<i_2}\left(1 + A_{li_1}^2 + A_{li_2}^2\right)^{\alpha/2} - \cdots \\ + (-1)^{d-l}\Sigma_{l<i_1<i_2<\cdots<i_{d-l}}\left(1 + A_{li_1}^2 + A_{li_2}^2 + \cdots + A_{li_{d-l}}^2\right)^{\alpha/2}\Big\} \\ + \Sigma_{h=p+1}^{d}\Big\{A_h^\alpha - \Sigma_{h<j_1}\left(A_h^2 + A_{j_1}^2\right)^{\alpha/2} + \Sigma_{h<j_1<j_2}\left(A_h^2 + A_{j_1}^2 + A_{j_2}^2\right)^{\alpha/2} \\ - \cdots + (-1)^{d-h}\Sigma_{h<j_1<j_2<\cdots<j_{d-h}}\left(A_h^2 + A_{j_1}^2 + \cdots + A_{j_{d-h}}^2\right)^{\alpha/2}\Big\}. \end{aligned}$$

In the first summation take $l = 1$ and estimate the order in the bracket as $|\mathbf{A}| \to \infty$. For $l > 1$, the proof is quite similar, as follows. Count the number of ± 1 and, in general, of the terms like $(1 + A_{1i_1}^2 + \cdots + A_{1i_q}^2)^{\alpha/2}$, $2 \leqslant i_1 < i_2 < \cdots < i_q \leqslant p$. For the former, count the terms including only A_{1i_k}, $(p + 1 \leqslant i_k \leqslant d)$, applying the Taylor expansion. For the latter, exclude $(1 + A_{1i_1}^2 + \cdots + A_{1i_q}^2)^{\alpha/2}$, if necessary, as $(T + \Sigma' A_{1k}^2)^{\alpha/2} = T^{\alpha/2}(1 + (\Sigma' A_{1k}^2/T))^{\alpha/2}$, where Σ' denotes a sum over some k's, $(k > p)$, and apply the Taylor expansion. They both accumulate to $\Sigma_{k=0}^{d-p}(-1)^k {}_{d-p}C_k$, which vanishes. The order of A_1, as $|\mathbf{A}| \to \infty$, is $0(A_1^\alpha \Sigma_{k>p} A_{1k}^2)$. For the second summation, count the number of the terms which tend to $\pm(c_{j_1}^2 + c_{j_2}^2 + \cdots + c_{j_m}^2)^{\alpha/2}$, $(p + q \leqslant j_1 < j_2 < \cdots < j_m \leqslant d)$, taking into account that $A_h \to 0$, $(h = p + 1, p + 2, \cdots, p + q - 1)$, which accumulates to $\Sigma_{k=0}^{q-1}(-1)^k {}_{q-1}C_k$, which also vanishes.

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Kobe University of Commerce
Tarumi, Kobe, Japan (655)