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SOME REMARKS ON A MIXING CONDITION[1]

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It is shown that the only stationary countable state Markov chains satisfying a mixing condition referred to as "of Markov type" are sequences of independent random variables.

1. Introduction. In a number of papers (see for example [2] and [3]) a mixing condition referred to as a Markov type regularity condition has been used to obtain asymptotic properties of normalized partial sums for random processes, especially for results on large deviations. It has already been noted that this condition has in some ways a limited range of applicability. The object of this discussion is to show that of all stationary countable state Markov chains, only sequences of independent random variables satisfy this condition.

The condition is usually specified in the following form. Let

$$\mathfrak{B}_s^t = \mathfrak{B}(X_s, X_{s+1}, \cdots, X_t), \qquad s \leqslant t,$$

be the Borel field generated by the random variables X_k, $s \leqslant k \leqslant t$. The condition is then

$$\gamma(s, t) = \sup_{A \in \mathfrak{B}_t^t;\, B \in \mathfrak{B}_{s+1}^{t-1};\, C \in \mathfrak{B}_s^s} |P\{A|B, C\} - P\{A|B\}| \to 0, \tag{1}$$

as $t - s \to \infty$.

PROPOSITION 1. *The only stationary countable state Markov chains that satisfy condition* (1) *are sequences of independent random variables.*

2. The proof. It is obvious that a sequence of independent random variables satisfies (1). Let us assume that a stationary countable state Markov chain satisfies (1) and prove that it must then be a sequence of independent random variables. By taking $B = \Omega$ we see that (1) implies that

$$\gamma(s, t) = \sup_{A \in \mathfrak{B}_t^t;\, C \in \mathfrak{B}_s^s} |P\{A|C\} - P\{A\}| \to 0,$$

as $t - s \to \infty$. This is a version of a Doeblin condition called D_0 in [1] and for any two states i, j it implies that there is an $m = m(i, j) > 0$ such that

$$p_{i,j}^{(m)}, p_{j,i}^{(m)} > 0. \tag{2}$$

Consider the event

$$B = \left\{ \begin{matrix} X_0 = i, X_m = j, X_{2m} = i, \cdots, X_{2rm} = i \\ or \\ X_0 = j, X_m = i, X_{2m} = j, \cdots, X_{2rm} = j \end{matrix} \right\}.$$

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Now

(3) $$P(B) = (p_i + p_j)\left(p_{i,j}^{(m)} p_{j,i}^{(m)}\right)^r.$$

Here p_i is the stationary probability

$$P(X_s = i) = p_i.$$

Let C and A be the events

$$C = \{X_{-1} = \gamma\}$$
$$A = \{X_{2rm+1} = \alpha\}.$$

Then

$$P(AB) = p_i\left(p_{i,j}^{(m)} p_{j,i}^{(m)}\right)^r p_{i\alpha} + p_j\left(p_{i,j}^{(m)} p_{j,i}^{(m)}\right)^r p_{j\alpha}$$
$$P(BC) = p_\gamma p_{\gamma i}\left(p_{i,j}^{(m)} p_{j,i}^{(m)}\right)^r + p_\gamma p_{\gamma j}\left(p_{i,j}^{(m)} p_{j,i}^{(m)}\right)^r$$

(4) $$P(ABC) = p_\gamma p_{\gamma i}\left(p_{i,j}^{(m)} p_{j,i}^{(m)}\right)^r p_{i\alpha} + p_\gamma p_{\gamma j}\left(p_{i,j}^{(m)} p_{j,i}^{(m)}\right)^r p_{j\alpha}.$$

It follows that

$$P(A|B) = (p_i p_{i\alpha} + p_j p_{j\alpha})/(p_i + p_j),$$

and

$$P(A|B, C) = (p_{\gamma i} p_{i\alpha} + p_{\gamma j} p_{j\alpha})/(p_{\gamma i} + p_{\gamma j}),$$

if at least one of $p_{\gamma i}$, $p_{\gamma j}$ are positive. Since (2), (3) hold for all positive integral r the condition (1) implies that

(5) $$(p_i p_{i\alpha} + p_j p_{j\alpha})/(p_i + p_j) = (p_{\gamma i} p_{i\alpha} + p_{\gamma j} p_{j\alpha})/(p_{\gamma i} + p_{\gamma j}),$$

for all α and all γ for which $p_{\gamma i} + p_{\gamma_j} > 0$. If $p_{i\alpha} \neq p_{j\alpha}$ for some α, then from (5) it follows that

(6) $$\frac{p_i}{p_i + p_j} = \frac{p_{\gamma i}}{p_{\gamma i} + p_{\gamma j}},$$

for all γ for which $p_{\gamma i} + p_{\gamma_j} > 0$. This implies that

$$p_{\gamma j} = (p_j/p_i)p_{\gamma i},$$

for all such γ. In fact this is obviously still valid even if $p_{\gamma i} + p_{\gamma_j} = 0$ (even if this holds for all γ). Relation (6) was obtained for a particular pair i, j. However, it holds for any pair i, j since we could find an integer $m(i, j) > 0$ for which (2) holds. Now if $p_{i\alpha} = p_{j\alpha}$ for all α and all pairs i, j, we are done. If this is not so, there is an α and a pair i, j such that $p_{i\alpha} \neq p_{j\alpha}$. Consider the maximal sets of equality of the $p_{i\alpha}$ (for this α as functions of i). There are at least two such nonvacuous sets. By comparing i, i' in any two such distinct maximal sets of equality we find

$$p_{\gamma i'} = \left(\frac{p_{i'}}{p_i}\right)p_{\gamma i},$$

for all γ. But this must hold for any i', i. This leads back to

$$p_{\alpha i} = p_i,$$

(by summing on i'). This is constancy of column vectors, which we assumed was not the case initially. Hence it must be so.

3. Concluding remarks. Suppose one considers a stationary Markov sequence $\{X_k, k = \cdots, -1, 0, 1, \cdots\}$ with invariant measure μ and transition probability function $P(x, \cdot)$ absolutely continuous with respect to the invariant measure μ for almost all x (with respect to μ). Let us call the set of all such stationary Markov sequences A. A small modification of the argument given for Proposition 1 would yield the following corresponding result.

PROPOSITION 2. *The only stationary Markov sequences belonging to set A that satisfy* (1) *are sequences of independent random variables.*

A condition somewhat like (1) that has been suggested [2] is

$$\rho(s, t) = \sup_{B \in \mathfrak{B}_{s+1}^{t-1};\, C \in \mathfrak{B}_s^s} \int |x| |P\{dx|B, C\} - P\{dx|B\}| \to 0, \tag{7}$$

as $t - s \to \infty$. One can directly show that Proposition 1 holds under condition (7) instead of condition (1) by the same argument.

Apparently the reason that results of this type hold is due to the fact that one takes the supremum of expressions of the type used in (1) and (7) over all sets B in $\mathfrak{B}_{s+1}^{t-1}$. If one takes the supremum over an appropriately chosen subcollection of sets B in $\mathfrak{B}_{s+1}^{t-1}$ such results would not follow, and one might still be able to get conclusions of the type desired on large deviations using the correspondingly modified mixing conditions.

The investigation leading to this paper was motivated in part by an interesting observation made by I. Zhurbenko. This was to the effect that the class of Gaussian processes satisfying the mixing condition in question was very small. I have seen a preprint of Statulevicius in which the definition of Markov type regularity is modified so that a nominal collection of processes satisfies the condition.

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