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# A FUNCTIONAL RELATIONSHIP BETWEEN THE DIFFERENT $r$-MEANS FOR INDICATOR FUNCTIONS

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Let $P$ be a probability measure defined on a $\sigma$-field $\mathfrak{F}$ over $\Omega$. Let $\mathfrak{L} \subset \mathfrak{F}$ be a $\sigma$-lattice and $r > 1$. For each $A \in \mathfrak{F}$ denote by $P_r(A/\mathfrak{L})$ the unique nearest point projection of $1_A$ onto the closed convex subspace of all "$\mathfrak{L}$-measurable" equivalence-classes of $L_r(\Omega, \mathfrak{F}, P)$.

It is shown that there exists a functional relationship between $P_r(A/\mathfrak{L})$ and $P_2(A/\mathfrak{L})$ of the form

$$P_r(A/\mathfrak{L}) = \varphi(P_2(A/\mathfrak{L}))$$

where the function $\varphi$ depends only on $r$ but not on $A$, $P$ or $\mathfrak{L}$. This relationship is applied to the theory of sufficiency.

The purpose of this note is to show that there exists a simple explicit functional relationship among the different $r$-means for indicator functions.

Let $P$ be a probability measure defined on a $\sigma$-field $\mathfrak{F}$ over $\Omega$. For each $r > 1$ denote by $\mathcal{L}_r(\Omega, \mathfrak{F}, P)$ the space of all real-valued random variables $X$ with $P[|X|^r] < \infty$, and denote by $L_r(\Omega, \mathfrak{F}, P)$ the system of corresponding equivalence classes. It is well known that $L_r(\Omega, \mathfrak{F}, P)$ is a uniformly convex Banach space; in such spaces there exist unique nearest point projections on closed convex sets.

Let $\mathfrak{L} \subset \mathfrak{F}$ be a $\sigma$-lattice. Denote by $P_r(X|\mathfrak{L})$ the unique projection of $X$ onto the closed convex set $L_r(\mathfrak{L})$, where $L_r(\mathfrak{L})$ is the set of all equivalence classes $\hat{Y} \in L_r(\Omega, \mathfrak{F}, P)$ containing a $\mathfrak{L}$-measurable function $Y$. $P_r(X|\mathfrak{L})$ is called the conditional $r$-mean of $X$ given the $\sigma$-lattice $\mathfrak{L}$ (for properties of $P_r(X|\mathfrak{L})$ see [3]). For $r = 2$ this is the usual concept of conditional expectation given a $\sigma$-lattice (see [2]). If $X$ is an indicator function $1_A$ we write $P_r(A|\mathfrak{L})$ instead of $P_r(1_A|\mathfrak{L})$.

We will show that there exists a functional relationship between $P_r(A|\mathfrak{L})$ and $P_2(A|\mathfrak{L})$ of the form

$$P_r(A|\mathfrak{L}) = \varphi(P_2(A|\mathfrak{L}))$$

where the function $\varphi$ depends only on $r$ but not on the set $A$ nor on the measure $P$ nor on the conditioning $\sigma$-lattice $\mathfrak{L}$.

If $a \in \mathbb{R}$, $r > 0$ let $a^r := |a|^r \operatorname{sign} a$.

The following remark gives a characterization of $r$-means by integration inequalities which is similar to the corresponding result for $r$-predictions (see, e.g., [1]).

REMARK. Let $r > 1$ and let $C \subset L_r(\Omega, \mathcal{Q}, P)$ be a closed convex cone. Let $X \in L_r(\Omega, \mathcal{Q}, P)$ and denote by $\pi^C X$ the nearest point projection of $X$ onto $C$.

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Then we have for $Y \in C$ that $Y = \pi^C X$ iff

(i) $P[(X - Y)^{r-1} Y] = 0$,
(ii) $P[(X - Y)^{r-1} Z] \leqslant 0$ for all $Z \in C$.

PROOF. That $\pi^C X$ fulfills (i) and (ii) can be seen by using the differential calculus in the same way as for $s$-predictors. This was also mentioned in [3].

Now let $Y \in C$ with property (i) and (ii). We have to prove that for each $Z \in C$

$$(*) \qquad \|X - Y\|_r \leqslant \|X - Z\|_r.$$

Let $Z \in C$ be given and $q$ with $1/r + 1/q = 1$. Using Hölder's inequality we obtain from (i) and (ii)

$$\begin{aligned} \|X - Y\|_r^r &\leqslant P\left[(X - Y)^{r-1}(X - Y)\right] \\ &\leqslant P\left[(X - Y)^{r-1}(X - Z)\right] \\ &\leqslant \|X - Y\|_r^{r/q} \cdot \|X - Z\|_r \end{aligned}$$

and hence ( $*$ ).

THEOREM. *Let $P$ be a probability measure on the $\sigma$-field $\mathfrak{F}$, $\mathfrak{L} \subset \mathfrak{F}$ a $\sigma$-lattice and $1 < r < \infty$. Then for each $A \in \mathfrak{F}$ we have*

$$P_r(A|\mathfrak{L}) = \frac{(P_2(A|\mathfrak{L}))^{1/(r-1)}}{(P_2(A|\mathfrak{L}))^{1/(r-1)} + (1 - P_2(A|\mathfrak{L}))^{1/(r-1)}}$$

*and hence*

$$P_2(A|\mathfrak{L}) = \frac{P_r(A|\mathfrak{L})^{r-1}}{P_r(A|\mathfrak{L})^{r-1} + (1 - P_r(A|\mathfrak{L}))^{r-1}}.$$

PROOF. Let $r \in (1, \infty)$ be fixed and put $s := 1/(r - 1) > 0$. The function

$$\varphi(x) := \frac{x^s}{x^s + (1 - x)^s}, \qquad x \in [0, 1]$$

is strictly increasing. Hence $\varphi(P_2(A|\mathfrak{L}))$ is an $\mathfrak{L}$-measurable function. Therefore it suffices to prove according to the preceding remark

(i) $P[\varphi(P_2(A|\mathfrak{L}))\{1_A - \varphi(P_2(A|\mathfrak{L}))\}^{r-1}] = 0$,
(ii) $P[(1_A - \varphi(P_2(A|\mathfrak{L}))^{r-1} Z] \leqslant 0$ for each $\mathfrak{L}$-measurable function $Z \in L_r(\Omega, \mathfrak{F}, P)$.

At first we prove (i): Relation (i) is equivalent to

$$(1) \qquad P\left[\varphi(P_2(A|\mathfrak{L}))\{1 - \varphi(P_2(A|\mathfrak{L}))\}^{r-1} 1_A\right] = P\left[\varphi^r(P_2(A|\mathfrak{L})) 1_{\bar{A}}\right];$$

(1) is equivalent to

$$(2) \qquad P\left[\varphi(P_2(A|\mathfrak{L}))\left\{\{1 - \varphi(P_2(A|\mathfrak{L}))\}^{r-1} + \varphi^{r-1}(P_2(A|\mathfrak{L}))\right\} 1_A\right] = P\left[\varphi^r(P_2(A|\mathfrak{L}))\right];$$

and (2) is equivalent to

$$(3)\qquad P\left[\frac{1_A\cdot\varphi(P_2(A|\mathfrak{L}))}{\{P_2(A|\mathfrak{L})^s+(1-P_2(A|\mathfrak{L}))^s\}^{r-1}}\right] = P\left[\frac{P_2(A|\mathfrak{L})\cdot\varphi(P_2(A|\mathfrak{L}))}{\{P_2(A|\mathfrak{L})^s+(1-P_2(A|\mathfrak{L}))^s\}^{r-1}}\right].$$

Now let

$$\psi(x)=\frac{\varphi(x)}{(x^s+(1-x)^s)^{r-1}},\qquad x\in[0,1].$$

Then (3) is equivalent to

$$(4)\qquad P[\psi(P_2(A|\mathfrak{L}))1_A]=P[\psi(P_2(A|\mathfrak{L}))P_2(A|\mathfrak{L})].$$

Since $\psi$ is a Borel function, (4) immediately follows from Corollary $A$, page 343 of [1]. This proves (i).

Relation (ii) is equivalent to

$$(5)\qquad P[1_A\{1-\varphi(P_2(A|\mathfrak{L}))\}^{r-1}Z]\leqslant P[1_{\bar{A}}Z\varphi^{r-1}(P_2(A|\mathfrak{L}))];$$

and (5) is equivalent to

$$(6)\qquad P[1_AZ\{(1-\varphi(P_2(A|\mathfrak{L})))^{r-1}+\varphi^{r-1}(P_2(A|\mathfrak{L}))\}]\leqslant P[Z\varphi^{r-1}(P_2(A|\mathfrak{L}))]$$

and (6) is equivalent to

$$(7)\qquad P\left[\frac{1}{\{P_2(A|\mathfrak{L})^s+(1-P_2(A|\mathfrak{L}))^s\}^{r-1}}\right]\leqslant P\left[\frac{P_2(A|\mathfrak{L})Z}{\{P_2(A|\mathfrak{L})^s+(1-P_2(A|\mathfrak{L}))^s\}^{r-1}}\right].$$

Put

$$\rho(x)=\frac{1}{\{x^s+(1-x)^s\}^{r-1}},\qquad x\in[0,1].$$

Then $\rho$ is a nonnegative Borel function. Now apply Theorem 7.19, page 342 of [2], to $z := Z$ and $y := 1_A$. This yields (7) and hence (ii).

Definition. Let $\mathfrak{P}|\mathfrak{F}$ be a family of probability measures, $\mathfrak{L}\subset\mathfrak{F}$ a $\sigma$-lattice and $1<r<\infty$. Then $\mathfrak{L}$ is $r$-sufficient for $\mathfrak{P}|\mathfrak{F}$ iff for every $A\in\mathfrak{F}$ there exists an $\mathfrak{L}$-measurable function $g\in P_r(A|\mathfrak{L})$ for all $P\in\mathfrak{P}$.

We remark that if $\mathfrak{L}$ is a sub-$\sigma$-field of $\mathfrak{F}$ and $r=2$, this is the usual definition of sufficiency.

COROLLARY. *Let $\mathcal{P}|\mathcal{F}$ be a family of probability measures, $\mathfrak{L} \subset \mathcal{F}$ a $\sigma$-lattice and $1 < r < \infty$. Then $\mathfrak{L}$ is r-sufficient for $\mathcal{P}|\mathcal{F}$* iff *$\mathfrak{L}$ is 2-sufficient for $\mathcal{P}|\mathcal{F}$.*

The question is still open whether there exists a functional relationship for step functions between $r$-means and 2-means given $\sigma$-lattices. For $\sigma$-fields instead of $\sigma$-lattices such a relationship was proved in [4].

If a similar relationship would be true for $\sigma$-lattices one would directly obtain a very general martingale theorem of the following form:

$$P_r(X|\mathfrak{L}_n) \to P_r(X|\mathfrak{L}_\infty) \qquad P - \text{a.e.}$$

if $\mathfrak{L}_n$ decreases or increases to $\mathfrak{L}_\infty$ and $X \in \mathfrak{L}_r$. With our functional relationship we obtain such a martingale theorem only for indicator functions $X = 1_A$.

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