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A STRONG LAW OF LARGE NUMBERS FOR SUBSEQUENCES OF RANDOM ELEMENTS IN SEPARABLE BANACH SPACES

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In 1967, Komlós proved that if $\{\xi_n\}$ is a sequence of real random variables for which $\sup_{n\geqslant 1}E|\xi_n| < \infty$, then there exists a subsequence $\{\eta_n\}$ of $\{\xi_n\}$ and an integrable random variable η such that for an arbitrary subsequence $\{\check{\eta}_n\}$ of $\{\eta_n\}$,

$$\lim_{n\to\infty}\frac{1}{n}(\check{\eta}_1 + \check{\eta}_2 + \cdots + \check{\eta}_n) = \eta \quad \text{a.s.}$$

In this paper, we attempt to extend this result to separable Banach space valued random elements. We impose a condition stronger than uniform integrability.

A. Introduction. In [1], page 218, Komlós proved that if ξ_n, $n \geqslant 1$ is a sequence of real random variables for which $\sup_{n\geqslant 1}E|\xi_n| < \infty$ then there exists a subsequence η_n, $n \geqslant 1$ of the sequence ξ_n, $n \geqslant 1$ and an integrable random variable η such that for an arbitrary subsequence $\tilde{\eta}_n$, $n \geqslant 1$ of the sequence η_n, $n \geqslant 1$

$$\lim_{n\to\infty}\frac{1}{n}(\tilde{\eta}_1 + \tilde{\eta}_2 + \cdots + \tilde{\eta}_n) = \eta \quad \text{a.s.}$$

Komlós' theorem is not generalizable as it stands to separable Banach spaces. Indeed, consider constant random vectors $V_n = e_n \in l_1$, where e_n, $n \geqslant 1$ is the canonical basis of l_1. Then evidently $\sup_{n\geqslant 1}E\|V_n\| < \infty$. ($V_n$'s are even uniformly bounded.) For any subsequence V_n' of V_n, the sequence $n^{-1}(V_1' + V_2' + \cdots + V_n')$ does not satisfy Cauchy condition. Clearly, a similar example can be constructed in any space containing uniformly $l_1^{(n)}$, i.e., in any non-Beck convex space.

The following theorem is an attempt to generalize Komlós' theorem to general separable Banach spaces. The L_1-boundedness is replaced by the condition (C) below which is stronger than uniform integrability of V_n, $n \geqslant 1$.

B. Theorem. *Let V_n, $n \geqslant 1$ be a sequence of random elements defined on a probability space $(\Omega, \mathcal{Q}, P)$ and taking values in a separable Banach space B. Suppose the following condition is satisfied.*

(C) *For any sequence A_k, $k \geqslant 1$ of Borel subsets of B with $A_k \downarrow \varnothing$ as $k \to \infty$*

$$\lim_{k\to\infty}\sup_{n\geqslant 1}\int_{V_n^{-1}(A_k)}\|V_n\|\,dP = 0.$$

Then there exists a subsequence V_n^*, $n \geqslant 1$ of V_n, $n \geqslant 1$ and a random element $V_0 \in L_1(B)$ satisfying

$$\lim_{s\to\infty}s^{-1}\Sigma_{i=1}^{s}V_i^* = V_0 \quad \text{a.s.},$$

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and the same holds for any further subsequence of V_n^*, $n \geqslant 1$. *Moreover, the* a.s. *convergence above may be replaced by the convergence in* $L^1(B)$.

C. Proof. The proof is carried out in the following steps.

(a) In any separable Banach space B there exists a sequence x_n, $n \geqslant 1$ in B and a sequence $f_n : B \to \{x_k, k \geqslant 1\}$, $n \geqslant 1$ of functions satisfying

$$\|f_n(x) - x\| \leqslant \frac{1}{n} \qquad \text{for every} \quad x \text{ in } B.$$

Define

$$\begin{aligned} f_n^m(x) &= f_n(x) \qquad \text{if} \quad f_n(x) \in \{x_1, x_2, \cdots, x_m\} \\ &= 0 \qquad\qquad \text{otherwise} \end{aligned}$$

for $n, m = 1, 2, 3, \cdots$. Clearly, for each $n, m = 1, 2, 3, \cdots$, the sequence of random variables

$$\alpha_{n,k}^m = \|f_n(V_k) - f_n^m(V_k)\|, \ k = 1, 2, 3, \cdots,$$

is bounded in $L_1(\mathbb{R})$ so that by Komlós' theorem and Cantor diagonal selection procedure there exists a subsequence $V_{k_p}, p \geqslant 1$ of V_n, $n \geqslant 1$ such that $\alpha_{n,k_p}^m, p \geqslant 1$ converges in Cesàro mean to an α_n^m a.s. together with every further subsequence and for every $m, n \geqslant 1$.

(b) Because of (C) and because for each $n = 1, 2, \cdots$, the sets $\{f_n \neq f_n^m\} \searrow \varnothing$ as $m \to \infty$

$$\begin{aligned} \int_\Omega \|f_n(V_k) - f_n^m(V_k)\| dP &= \int_{V_k^{-1}\{f_n \neq f_n^m\}} \|f_n(V_k) - f_n^m(V_k)\| dP \\ &\quad + \int_{V_k^{-1}\{f_n = f_n^m\}} \|f_n(V_k) - f_n^m(V_k)\| dP \\ &\leqslant \frac{1}{n} + \int_{V_k^{-1}\{f_n \neq f_n^m\}} \|V_k\| dP \\ &\leqslant \frac{2}{n} \end{aligned}$$

for m greater than or equal to a certain m_n. By Fatou's lemma

$$E\alpha_n^m \leqslant \liminf_{s\to\infty} E\frac{1}{s}\Sigma_{r=1}^s \|f_n(V_{k_r}) - f_n^m(V_{k_r})\|$$

so that $E\alpha_n^{m_n} \leqslant (2/n)$ for every $n \geqslant 1$. This implies that $\lim_{n\to\infty} E\alpha_n^{m_n} = 0$, and we can choose a subsequence n_r, $r \geqslant 1$ of $\{1, 2, 3, \cdots\}$ such that $\alpha_{n_r}^{m_{n_r}}$, $r \geqslant 1$ converges to 0 a.s.

(c) For any fixed n and m, f_n^m takes values in a finite dimensional space, and because Komlós' theorem trivially holds in finite dimensional spaces, by Cantor diagonalization procedure, we can find a subsequence V_n^*, $n \geqslant 1$ of V_{k_r}, $r \geqslant 1$ and a β_n^m such that almost surely $f_n^m(V_k^*)$, $k \geqslant 1$ converges in Cesàro mean to β_n^m.

(d) We conclude by proving that Cesàro means of V_n^*, $n \geqslant 1$ are Cauchy a.s. Take an $\omega \in \Omega$ from the set of probability 1 for which conclusions of (a), (b) and (c) simultaneously hold.

Let $\varepsilon > 0$. Find $n_r \geqslant 1$ such that $n_r \geqslant (4/\varepsilon)$ and $|\alpha_{n_r}^{m_{n_r}}(\omega)| < (\varepsilon/4)$. Then by (a) and (c) we can find $s_0 \geqslant 1$ such that for $s \geqslant s_0$

$$\left|s^{-1}\Sigma_{p=1}^{s}\|f_{n_r}(V_p^*(\omega)) - f_{n_r}^{m_{n_r}}(V_p^*(\omega))\| - \alpha_{n_r}^{mn_r(\omega)}\right| < \frac{\varepsilon}{4}$$

and

$$\|s^{-1}\sum_{p=1}^{s} f_{n_r}{}^{m_{n_r}}(V_p^*(\omega)) - \beta_{n_r}{}^{m_{n_r}}(\omega)\| < \frac{\varepsilon}{4}\cdot$$

Thus, for $s \geqslant s_0$

$$\begin{aligned}\|s^{-1}(V_1^*(\omega) + V_2^*(\omega) + \cdots + V_s^*(\omega)) - \beta_{n_r}^{m_{n_r}}(\omega)\| \\ \leqslant \|s^{-1}\Sigma_{p=1}^{s}V_p^*(\omega) - s^{-1}\sum_{p=1}^{s} f_{n_r}(V_p^*(\omega))\| \\ + \|s^{-1}\Sigma_{p=1}^{s}\left[f_{n_r}(V^*{}_p(\omega)) - f_{n_r}^{m_{n_r}}(V_p^*(\omega))\right]\| - \alpha_{n_r}^{m_{n_r}}(\omega) \\ + \alpha_{n_r}^{m_{n_r}}(\omega) + \|s^{-1}\Sigma_{p=1}^{s} f_{n_r}^{m_{n_r}}(V_p^*(\omega)) - \beta_{n_r}^{m_{n_r}}(\omega)\| \leqslant 4.\frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

This shows that the Cesáro means of V_n^*, $n \geqslant 1$ satisfy a.s. Cauchy condition. The same proof shows that the same is true for any further subsequence of V_n^*, $n \geqslant 1$ and the L_1-convergence follows from the uniform integrability implied by (C).

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