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LAWS OF LARGE NUMBERS FOR TIGHT RANDOM ELEMENTS IN NORMED LINEAR SPACES

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A strong law of large numbers is proved for tight, independent random elements (in a separable normed linear space) which have uniformly bounded pth moments ($p > 1$). In addition, a weak law of large numbers is obtained for tight random elements with uniformly bounded pth moments ($p > 1$) where covergence in probability for the separable normed linear space holds if and only if convergence in probability for the weak linear topology holds.

1. Introduction. The consideration of stochastic processes as function-valued random variables motivated the study of random elements (random variables with values in normed linear space) by Doob (1947), Mourier (1953), Prohorov (1956), Beck (1963), Billingsley (1968) and others. The laws of large numbers for random elements have been obtained, and a summary of many of those results was presented by Padgett and Taylor (1973). These cancellation results arise from two sources (Beck (1976)). The first can be founded in the probabilistic structure of the sequence of random elements (independence, identical distributions, and boundedness of the moments). The second is the geometric properties of the spaces, usually called the cancellation conditions (such as B-convexity (Beck (1963)), super-reflexity, G_α-conditions (Woyczynski (1973)), and type p (Hoffmann-Jørgensen and Pisier (1976))). Only a few theorems do not require any cancellation condition on the space. For example, the pioneering theorem by Mourier (1953) states that the strong law of large numbers holds for independent, identically distributed random elements whose first moments exist.

In Section 2 some preliminaries are briefly listed. In Section 3 the strong law of large numbers is obtained for independent random elements (in a separable normed linear space) which are tight and have p-moments ($p > 1$) which are uniformly bounded. Since identically distributed random elements are tight but not conversely, this result relaxes the condition of identical distributions without requiring any cancellation condition on the space.

In Section 4 the weak law of large numbers is obtained for tight random elements in a separable normed linear space. These results generalize the results of Taylor (1972), Taylor and Padgett (1976). In particular, for tight random elements in separable normed linear spaces, the weak law of large numbers

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is shown to hold if and only if the weak law of large numbers holds for $\{f(V_n)\}$ for each continuous linear functional f. (Again, p-moment conditions are assumed.)

2. Preliminaries. Let X denote a real separable normed linear space with norm $\|\cdot\|$ and let $B(X)$ denote the Borel sigma-field generated by open subsets of X. Let (Ω, A, P) be a probability space and let V be a function from Ω into X. If $V^{-1}(B) \in A$ for every Borel set $B \in B(X)$, then V is said to be a *random element* in X (or an X-valued random variable). The *expected value* of V is defined to be the Pettis integral of V, denoted by EV. For $p > 0$, $E\|V\|^p = \int_\Omega \|V\|^p \, dP$ is called the pth *moment* of V. A random element is said to be *symmetric* if there exists a measure preserving function ϕ on Ω such that $P[V \circ \phi = -V] = 1$. Note that if V is symmetric, then $EV = 0$. The definitions of independence and identical distributions for random elements are similar to the (real-valued) random variable case.

A sequence $\{V_n\}$ of random elements is *tight* (Billingsley (1968)) if for each $\varepsilon > 0$, there exists a compact subset K_ε of X such that

$$P[V_n \in K_\varepsilon] > 1 - \varepsilon \qquad \text{for all} \quad n . \tag{2.1}$$

Since the continuous image of compact set is compact, $f(V_n)$ is tight if $\{V_n\}$ is tight and f is continuous (hence Borel-measurable) function from X into a normed linear space.

3. Strong laws of large numbers. The strong law of large numbers for tight, independent random elements in a separable normed linear space will be proved in two parts. First, the result will be obtained for random elements which take their values in a compact subset. In Theorem 2 the strong law of large numbers is then obtained by truncating the random elements to a compact subset and by applying Theorem 1.

Without loss of generality it can be assumed that $EV_n \equiv 0$ since $\{V_n\}$ being tight with uniformly bounded pth $(p > 1)$ moments implies that $\{V_n - EV_n\}$ is tight. Finally, the strong law of large numbers (SLLN) is said to hold if

$$\|n^{-1} \textstyle\sum_{k=1}^{n} (V_k - EV_k)\| \to 0 \qquad \text{with probability one.}$$

THEOREM 1. *Let K be a compact subset of a normed linear space X. Let $\{V_n\}$ be independent random elements taking values in K with $EV_n = 0$ for all n. Then the* SLLN *holds for $\{V_n\}$.*

PROOF. It can be assumed that X is a Banach space and that K is also convex (Rudin (1973), page 72). In the dual space X^* there is a countable set S which separates points of K. Let τ_S be the weakest topology on K making the elements of S continuous. Then for $\{x_n\} \subset K$ $x_n \to 0$ in τ_S if and only if $\|x_n\| \to 0$. For each $f \in S$,

$$n^{-1} \textstyle\sum_{k=1}^{n} f(V_k) \to 0 \tag{3.1}$$

with probability one since $\{f(V_k)\}$ is a sequence of independent, uniformly bounded random variables. Since S is countable,

(3.2) $$\|n^{-1} \sum_{k=1}^{n} V_k\| \to 0$$

with probability one. □

THEOREM 2. *Let X be a separable normed linear space and let $\{V_n\}$ be tight, independent random elements in X such that $E\|V_n\|^p \leqq M$ for all n where $p > 1$. Then, the* SLLN *holds for $\{V_n\}$.*

PROOF. First, assume (w.l.o.g.) that $EV_n = 0$ for all n. Given $\varepsilon > 0$, let

(3.3) $$\delta = (M^{-1}(\varepsilon/4)^p)^{1/(p-1)} .$$

Let K be a compact (also assume that it is again convex and symmetric) such that

(3.4) $$P[V_n \in K] > 1 - \delta .$$

Define

(3.5) $$Y_n = V_n I_{[V_n \in K]} \qquad \text{and} \qquad Z_n = V_n - Y_n .$$

Then $\{Y_n - EY_n\}$ takes their values in $2K$, and hence

(3.6) $$\|n^{-1} \sum_{k=1}^{n} (Y_k - EY_k)\| \to 0$$

with probability one by Theorem 1. By Hölder's inequality, for each n

(3.7) $$\begin{aligned} E\|Z_n\| &= E\|V_n\| I_{[V_n \notin K]} \\ &\leqq (E\|V_n\|^p)^{1/p} (\delta^{(p-1)/p}) \\ &\leqq (M\delta^{p-1})^{1/p} \leqq \varepsilon/4 . \end{aligned}$$

Also, for each n

(3.8) $$E|\|Z_n\| - E\|Z_n\||^p \leqq 2^p M .$$

By Chung's (1947) SLLN for random variables,

(3.9) $$n^{-1} \sum_{k=1}^{n} (\|Z_k\| - E\|Z_k\|) \to 0$$

with probability one. Since a sequence of $\varepsilon_n \to 0$, $\delta_n \to 0$, and corresponding compact sets K_n could be chosen, a countable number of null sets can be excluded in (3.6) and (3.9). Thus, for almost all $\omega \in \Omega$, there is an $n_0(\omega)$ such that for $n \geqq n_0(\omega)$,

(3.10) $$n^{-1}\|\sum_{k=1}^{n} (Y_k - EY_k)\| < \varepsilon/4$$

and

(3.11) $$n^{-1}|\sum_{k=1}^{n} (\|Z_k\| - E\|Z_k\|)| < \varepsilon/4 .$$

Since from (3.7)

$$n^{-1}\|\sum_{k=1}^{n} EZ_k\| \leqq n^{-1} \sum_{k=1}^{n} E\|Z_k\| \leqq \varepsilon/4 ,$$

then (3.11) implies that

(3.12) $$\|n^{-1} \sum_{k=1}^{n} Z_k\| \leqq n^{-1} \sum_{k=1}^{n} \|Z_k\| < \varepsilon/2 .$$

Next, $EY_k = -EZ_k$ and (3.10) provides that

(3.13) $$\|n^{-1} \sum_{k=1}^{n} Y_k\| < \varepsilon/2 .$$

Finally, from (3.5), (3.12) and (3.13),

$$\|n^{-1} \sum_{k=1}^{n} V_k\| < \varepsilon .$$ □

4. Weak laws of large numbers. Weak laws of large numbers (WLLN's) are proved in this section. Theorem 3 is a WLLN for random elements which take their values in a compact subset. Theorem 4 is a WLLN for tight random elements which have pth ($p > 1$) bounded moments.

THEOREM 3. *Let K be a compact subset of a normed linear space X. Let $\{V_n\}$ be random elements taking their values in K with $EV_n = 0$ for all n. Then for each $f \in X^*$*

$$|n^{-1} \sum_{k=1}^{n} f(V_k)| \to 0 \qquad \text{in probability}$$

if and only if

$$\|n^{-1} \sum_{k=1}^{n} V_k\| \to 0 \qquad \text{in probability.}$$

The proof of Theorem 3 is similar to the proof of Theorem 1. To obtain convergence in probability in (3.2) from (3.1) in the proof of Theorem 1, recall that a sequence converges in probability if and only if every subsequence has a further subsequence which converges with probability one.

THEOREM 4. *Let X be a separable normed linear space and let $\{V_n\}$ be tight random elements in X such that $E\|V_n\|^p \leqq M$ for all n ($p > 1$). For each $f \in X^*$,*

$$|n^{-1} \sum_{k=1}^{n} f(V_k)| \to 0 \qquad \text{in probability}$$

if and only if

$$\|n^{-1} \sum_{k=1}^{n} V_k\| \to 0 \qquad \text{in probability.}$$

PROOF. Given $\varepsilon > 0$, let

(4.1) $$\delta = (M^{-1}(\varepsilon^2/6)^p)^{1/(p-1)} .$$

Let K be a convex, symmetric, compact set such that $P[V_n \in K] > 1 - \delta$ for all n. Again (w.l.o.g.), assume that $EV_n = 0$ for each n. Define

(4.2) $$Y_n = V_n I_{[V_n \in K]} \qquad \text{and} \qquad Z_n = V_n - Y_n .$$

Thus, for each n

(4.3) $$P[\|n^{-1} \sum_{k=1}^{n} V_k\| > \varepsilon] \leqq P[\|n^{-1} \sum_{k=1}^{n} Y_k\| > \varepsilon/2] + P[\|n^{-1} \sum_{k=1}^{n} Z_k\| > \varepsilon/2] .$$

Similar to (3.7) $E\|Z_n\| \leqq \varepsilon^2/6$ for each n. Hence,

(4.4) $$P\|n^{-1} \sum_{k=1}^{n} Z_k\| > \varepsilon/2] \leqq (2/\varepsilon)n^{-1} \sum_{k=1}^{n} E\|Z_k\| \leqq \varepsilon/3 .$$

From Theorem 3 and (4.4) there exists $N(\varepsilon)$ such that

(4.5) $$P[\|n^{-1} \sum_{k=1}^{n} Y_k\| > \varepsilon/2] \leqq P[\|n^{-1} \sum_{k=1}^{n} (Y_k - EY_k)\| > \varepsilon/6] < \varepsilon/2$$

since $\{Y_n - EY_n\}$ take their values in $2K$ and $EY_k = -EZ_k$. Thus, the proof is complete by (4.3), (4.4) and (4.5). □

Let $\{b_i\} \subset X$ denote a Schauder basis and let $\{f_i\}$ denote the coordinate functionals. For $t = 1, 2, \cdots$, define

$$U_t(x) = \sum_{i=1}^{t} f_i(x)b_i$$

to be the finite-dimensional, partial sum operators. Then Lemma 5 characterizes compact subsets, and the converse is sufficient for relative compactness when $\sup_{x \in K} \|x\| < \infty$.

LEMMA 5. *Let K be a compact set in a Banach space which has a Schauder basis, then for $\eta > 0$, there exists a positive integer N such that $\|x - U_n(x)\| < \eta$ for all $x \in K$ and $n \geqq N$.*

PROOF. Let $g_n(x) = \sup_{k \geq n} \|x - U_k(x)\|$. Then

$$\begin{aligned} |g_n(x) - g_n(y)| &\leqq \sup_{k \geq n} |\|x - U_k(x)\| - \|y - U_k(y)\|| \\ &\leqq \sup_{k \geq n} \|(x - y) - U_k(x - y)\| \leqq (m + 1)\|x - y\| \end{aligned}$$

where m is the basis constant. Hence g_n is uniformly continuous for each n. Moreover, $\{g_n\}$ decreases monotonically (pointwise) to zero, and hence converges uniformly to zero by Dine's theorem (Royden (1972), page 162). □

Since the coordinate functionals separate points, the following WLLN can be obtained from Theorems 3 and 4.

COROLLARY. *Let X be a Banach space which has a Schauder basis and let $\{V_n\}$ be tight random elements in X such that $EV_n = 0$ and $E\|V_n\|^p \leqq M$ for all n $(p > 1)$. For each coordinate fuctional f_i,*

$$|n^{-1} \textstyle\sum_{k=1}^{n} f_i(V_k)| \to 0 \qquad \textit{in probability}$$

if and only if

$$\|n^{-1} \textstyle\sum_{k=1}^{n} V_k\| \to 0 \qquad \textit{in probability.}$$

It is interesting to compare the weak law of large numbers of Theorem 2 with the results in Taylor (1972). The condition of identical distributions is eliminated by assuming tightness while a pth $(p > 1)$ moment condition is needed instead of the first moment. Also, Theorem 2, its corollary, and other obvious corollaries, provide weak laws of large numbers for tight weakly uncorrelated (or coordinate uncorrelated) random elements. Finally, using the characterizations of Billingsley (1968) for tightness on the space $C[0, 1]$, these results can be applied to stochastic processes which have continuous sample paths.

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random variables. Let $\{V_n\}$ be independent random variables such that

$$
\begin{aligned}
V_n &= n && \text{with probability } \tfrac{1}{2}n\log(n+2)\\
&= -n && \text{with probability } \tfrac{1}{2}n\log(n+2)\\
&= 0 && \text{otherwise.}
\end{aligned}
$$

Note that $EV_n = 0$, $E|V_n| = 1/\log(n+2)$ (which actually converges to 0), and $P[|V_n| \geqq n] = 1/n\log(n+2)$. Thus, the random variables $\{V_n\}$ are tight, and the Borel lemmas imply that $|V_n| \geqq n$ infinitely often with probability one. Hence, the strong law of large numbers does not hold.

The authors are also indebted to the referee for suggesting a much shorter proof for Theorem 2. The original proof consisted of the four parts (1) uniformly boundedness and use of a Schauder basis, (2) deletion of the uniformly boundedness condition, (3) deletion of the symmetry condition, and (4) proof for arbitrary normed linear spaces.

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