The Annals of Probability
1979, Vol. 7, No. 1, 62–74

ENVELOPES OF VECTOR RANDOM PROCESSES AND THEIR CROSSING RATES

BY DANIELE VENEZIANO

Massachusetts Institute of Technology

Vector-valued random processes, $\mathbf{X}(t)$, can be "enveloped" by set-valued random processes, $\mathbb{S}(t)$, to which they belong with probability 1 during any finite length of time. When applied to scalar processes, the set-definition of envelope includes and is richer than the familiar point-definitions. Several random set-envelope processes in n-dimensional space, R_n, are defined and the mean rates at which they "cross" given regions of R_n are calculated. Comparison is made with the mean crossing rates of associated enveloped Gaussian processes, $\mathbf{X}(t)$.

1. Introduction. Enveloping scalar random processes has proved to be an effective means of studying their extremal properties (Cramér and Leadbetter, 1967; Crandall and Mark, 1961; Rice, 1944, 1945). In particular, the mean rate at which the envelope of a random process, $X(t)$, upcrosses a high level, r, has been used to approximate the probability distribution of the time T, $T \geqslant 0$, at which $X(t)$ first exceeds the same level (Crandall, 1970; Lyon, 1961; Vanmarcke, 1969, 1975). In reliability applications, $F_T(\tau)$ is the probability that a system with state $X(t)$ and survival condition $X \leqslant r$ has failed at time τ. Similarly, for a system with vector-state $\mathbf{X}(t) = [X_1(t), \cdots, X_n(t)]^T$, and survival condition $\mathbf{X} \in D$ (D = a region of R_n), $F_T(\tau)$ is the probability that $\mathbf{X}(t)$ is not in D, sometimes in $[0, \tau]$. In the multivariate problem, $F_T(\tau)$ can be estimated from the mean rate at which $\mathbf{X}(t)$ leaves D (Veneziano et al., 1977), but by analogy with the scalar case one may expect that mean crossing rates of appropriately defined envelopes will generate better approximations. The notion of envelope is extended here from scalar to vector random functions by defining stochastic set processes in n-dimensional space to which $\mathbf{X}(t)$ belongs with probability 1. The mean rate at which some set processes in R_n experience "crossings" is derived and compared with the associated mean outcrossing rate of $\mathbf{X}(t)$.

2. Set processes in R_n. In one-dimension, a scalar process $S(t)$ such that $S(t) \geqslant |X(t)|$ for all t is called an envelope of $X(t)$. For easy physical interpretation it is desirable that the difference, $S(t) - |X(t)|$, be not very large, particularly at the points of stationarity of $X(t)$, and that $S(t)$ be a smoother process than $X(t)$. The definition that follows, due to Cramér and Leadbetter (1967), satisfies these requirements for narrow-band processes and is mathematically convenient for mean-crossing-rate-calculations. Let $U(\omega)$ and $V(\omega)$ be the random cosine and sine

Received July 6, 1977; revised December 5, 1977.
AMS 1970 *subject classifications.* Primary 60G10.
Key words and phrases. Stochastic vector processes, envelopes, first crossing, reliability.

spectral components of $X(t)$, such that

$$X(t) = \int_0^\infty [U(\omega)\cos \omega t + V(\omega)\sin \omega t]\, d\omega. \tag{2.1}$$

Then $S(t)$ is defined:

$$S(t) = [X^2(t) + \hat{X}^2(t)]^{\frac{1}{2}}, \tag{2.2}$$

in which

$$\hat{X}(t) = \int_0^\infty [U(\omega)\sin \omega t - V(\omega)\cos \omega t]\, d\omega. \tag{2.3}$$

A multivariate generalization of $S(t)$ becomes conceptually simple if one replaces the crossing level r with the interval $D = (-\infty, r]$ and the envelope $S(t)$ with the random set process

$$\mathbb{S}(t) = \{X:|X| \leqslant S(t)\}. \tag{2.4}$$

At any time t, $\mathbb{S}(t)$ is a vertical line segment with random, time-dependent extremes. $\mathbb{S}$ will be said to be in the *crossed state* at time t iff $\mathbb{S}(t)$ is not entirely in D. Because $\mathbb{S}(t)$ is in the crossed state if and only if $S(t) > r$, the two definitions of envelope, (2.2) and (2.4), are equivalent with respect to level crossing. However, the concept of set envelope is richer than the concept of point envelope and can be extended more naturally to the multivariate case.

DEFINITION. $\mathbb{S}(t)$ is a set-envelope process (or simply a set process) of $\mathbf{X}(t)$ if $\mathbf{X}(t) \in \mathbb{S}(t)$ with probability 1 during any finite length of time.

In application—e.g., to system reliability—it is desirable that at any given time t, $\mathbb{S}(t)$ be contained in a small region of R_n and that in some sense $\mathbb{S}(t)$ be smoother than $\mathbf{X}(t)$. Three set processes that satisfy these requirements and for which mean crossing rates are rather simple to calculate are defined in the next section.

3. Three set-envelope processes in R_n. Without loss of generality, one can define set envelopes, $\mathbb{S}(t)$, only for zero-mean processes, $\mathbf{X}(t)$. Processes with nonzero and possibly time-dependent mean are enveloped by translations of $\mathbb{S}(t)$.

Let $d^2(t) = \Sigma_{i=1}^n S_i^2(t)$ with $S_i(t)$ = envelope of $X_i(t)$ in the sense of (2.2). Then $\mathbf{X}(t)$ is enveloped by the spherical set (*disk process* of $\mathbf{X}(t)$),

$$\mathbb{S}_D(t) = \{\mathbf{X}:|\mathbf{X}| \leqslant d(t)\}. \tag{3.1}$$

Envelope sets smaller then $\mathbb{S}_D$ can be defined in terms of the scalar processes, $S_i(t)$, the smallest set of this type being

$$\mathbb{S}_R(t) = \{\mathbf{X}:|X_i| \leqslant S_i(t), i = 1, \cdots, n\}. \tag{3.2}$$

$\mathbb{S}_R(t)$, the *rectangle process* of $\mathbf{X}(t)$, is a random parallelepiped with center at the origin and vertices on the boundary of $\mathbb{S}_D(t)$.

Equation (3.2) uses the scalar envelope processes to constrain $\mathbf{X}(t)$ along the coordinate directions. With some additional information one can constrain $\mathbf{X}(t)$ in all directions. Consider the random process, $X_\alpha(t) = \boldsymbol{\alpha}^T\mathbf{X}(t)$, being the component of $\mathbf{X}(t)$ in the direction of the unit vector, $\boldsymbol{\alpha}$. The associated envelope process in the sense of Cramér and Leadbetter (1967) is

$$(3.3)\qquad S_{\alpha}(t) = \left[X_{\alpha}^2(t) + \hat{X}_{\alpha}^2(t)\right]^{\frac{1}{2}} = \left\{\boldsymbol{\alpha}^T\left[\mathbf{X}(t)\mathbf{X}^T(t) + \hat{\mathbf{X}}(t)\hat{\mathbf{X}}^T(t)\right]\boldsymbol{\alpha}\right\}^{\frac{1}{2}}$$
$$= \left[\boldsymbol{\alpha}^T\mathbf{G}(t)\boldsymbol{\alpha}\right]^{\frac{1}{2}}$$

in which $\mathbf{G}(t)$ is the random, positive semidefinite matrix with (i, j)th element $[X_i(t)X_j(t) + \hat{X}_i(t)\hat{X}_j(t)]$. For any given $\boldsymbol{\alpha}$, $\mathbf{X}(t)$ belongs to the set $\{\mathbf{X}:|\boldsymbol{\alpha}^T\mathbf{X}| \leqslant [\boldsymbol{\alpha}^T\mathbf{G}(t)\boldsymbol{\alpha}]^{\frac{1}{2}}\}$; hence $\mathbf{X}(t)$ belongs to the ellipsoid (called here the *ellipse set* of $\mathbf{X}(t)$)

$$(3.4)\qquad \mathbb{S}_E(t) = \left\{\mathbf{X}:|\boldsymbol{\alpha}^T\mathbf{X}| \leqslant \left[\boldsymbol{\alpha}^T\mathbf{G}(t)\boldsymbol{\alpha}\right]^{\frac{1}{2}} \quad \text{for all} \quad \boldsymbol{\alpha}\right\}.$$

This definition of $\mathbb{S}_E(t)$ holds whether $\mathbf{G}(t)$ is singular or not; if $\mathbf{G}(t)$ is nonsingular, then an equivalent definition is $\mathbb{S}_E(t) = \{\mathbf{X} : \mathbf{X}^T\mathbf{G}^{-1}(t)\mathbf{X} \leqslant 1\}$. For $n > 2$, $\mathbf{G}(t)$ is always singular and $\mathbb{S}_E(t)$ is a random ellipse in the plane that contains the origin, the point $\mathbf{X}(t)$, and the direction of $\hat{\mathbf{X}}(t)$.

Properties of $\mathbb{S}_D$, $\mathbb{S}_R$, *and* $\mathbb{S}_E$. For any given t and with $\partial_{\mathbb{S}}$ = boundary of $\mathbb{S}$,

1. $\mathbb{S}_D(t) \supset \mathbb{S}_R(t) \supset \mathbb{S}_E(t)$.
2. $\mathbf{X}(t) \in \partial_{\mathbb{S}_D}(t)$ iff $\hat{X}_i(t) = 0$ for all i,
   $\mathbf{X}(t) \in \partial_{\mathbb{S}_R}(t)$ iff $\hat{X}_i(t) = 0$ for at least one i,
   $\mathbf{X}(t) \in \partial_{\mathbb{S}_E}(t)$ if $n > 1$ (for $\mathbf{X} = \mathbf{X}(t)$ and for $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha}^T\hat{\mathbf{X}}(t) = 0$ the inequality of (3.4) holds as an equality).
3. $\mathbb{S}_D(t)$ is invariant under isotropic scaling and rotation,
   $\mathbb{S}_R(t)$ is invariant under scaling but not under rotation,
   $\mathbb{S}_E(t)$ is invariant under all homogeneous linear transformations.

Let $\mathbf{Y}(t) = \mathbf{A}\mathbf{X}(t)$, $\mathbf{A}$ a given square matrix, and denote by $\mathbb{S}_{D_{\mathbf{X}}}(t)$ and $\mathbb{S}_{D_{\mathbf{Y}}}(t)$ the disk processes of $\mathbf{X}(t)$ and $\mathbf{Y}(t)$, respectively. The set processes, $\mathbb{S}_{R_{\mathbf{X}}}(t)$, $\mathbb{S}_{R_{\mathbf{Y}}}(t)$, $\mathbb{S}_{E_{\mathbf{X}}}(t)$, and $\mathbb{S}_{E_{\mathbf{Y}}}(t)$ are defined in a similar way. For any $\mathbf{X}$ in $\mathbb{S}_{D_{\mathbf{X}}}(t)$, $\mathbf{Y} = \mathbf{A}\mathbf{X} \in \mathbb{S}_{D_{\mathbf{Y}}}(t)$ if $\mathbf{A}$ is proportional to an orthogonal matrix, but not necessarily otherwise. Therefore, the disk process is invariant under rotation and under isotropic scaling. For all $\mathbf{X} \in \mathbb{S}_{R_{\mathbf{X}}}(t)$, $\mathbf{Y} = \mathbf{A}\mathbf{X}$ belongs to $\mathbb{S}_{R_{\mathbf{Y}}}(t)$ if $\mathbf{A}$ is diagonal, but not necessarily otherwise. Hence, the rectangle process is invariant under scaling of the components of $\mathbf{X}$ but not under rotation. Finally, if $\mathbf{X} \in \mathbb{S}_{E_{\mathbf{X}}}(t)$, then $\mathbf{Y} = \mathbf{A}\mathbf{X} \in \mathbb{S}_{E_{\mathbf{Y}}}(t)$ for any given $\mathbf{A}$, indicating that the ellipse process is invariant under all homogeneous linear transformations. One can use these invariance properties to obtain the following alternative definitions of $\mathbb{S}_{R_{\mathbf{X}}}$ and $\mathbb{S}_{E_{\mathbf{X}}}$ in terms of $\mathbb{S}_{D_{\mathbf{Y}}}$:

4. $\mathbb{S}_{R_{\mathbf{X}}}(t) = \{\mathbf{X} : \mathbf{Y} = \mathbf{A}\mathbf{X} \in \mathbb{S}_{D_{\mathbf{Y}}}(t)$ for all diagonal $\mathbf{A}\}$,
   $\mathbb{S}_{E_{\mathbf{X}}}(t) = \{\mathbf{X} : \mathbf{Y} = \mathbf{A}\mathbf{X} \in \mathbb{S}_{D_{\mathbf{Y}}}(t)$ for all $\mathbf{A}\}$.
5. For $n = 1$, $\mathbb{S}_D(t) = \mathbb{S}_R(t) = \mathbb{S}_E(t) = \{X:|X| \leqslant S(t)\}$.

4. Outcrossing rates of Gaussian vector processes, $\mathbf{X}(t)$. Let $\mathbf{X}(t)$ be an n-variate stationary Gaussian process. In calculating the mean crossing rate of $\mathbf{X}(t)$ out of a given region D, it is often convenient to formulate the problem in a *standard reference* such that for any given t, $\begin{bmatrix} \mathbf{X}(t) \\ \dot{\mathbf{X}}(t) \end{bmatrix}$ is a zero-mean vector with independent components and covariance matrix

(4.1) $$\Sigma\begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} = \left[\begin{array}{c|c} \operatorname{diag}(\lambda_{0_i}) & \mathbf{0} \\ \hline \mathbf{0} & \operatorname{diag}(\lambda_{2_i}) \end{array}\right].$$

Similarly, for calculation of the mean crossing rates of $\mathbb{S}_D(t)$, $\mathbb{S}_R(t)$, or $\mathbb{S}_E(t)$, a convenient reference is one in which $\begin{bmatrix} \mathbf{X}(t) \\ \hat{\mathbf{X}}(t) \end{bmatrix}$ is a zero-mean vector with independent components and covariance matrix

(4.2) $$\Sigma\begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} = \left[\begin{array}{c|c} \operatorname{diag}(\lambda_{0_i}) & \mathbf{0} \\ \hline \mathbf{0} & \operatorname{diag}(\lambda_{0_i}) \end{array}\right],$$

and in which the scalar envelopes $S_i(t)$ are independent of the envelope derivatives $\dot{S}_j(t)$. In (4.1) and (4.2), λ_{i_j} is the ith absolute spectral moment of $X_j(t)$,

(4.3) $$\lambda_{i_j} = \int_0^\infty \omega^i G_j(\omega)\, d\omega$$

in which $G_j(\omega)$ = one-sided mean power spectral density function of $X_j(t)$. It is not always possible to reduce a stationary Gaussian process to standard form by means of linear transformations. However, the class of processes for which this is possible is large enough (e.g., it includes processes with independent components) for results in this and in later sections to be of practical interest (Veneziano et al., 1977). There is no additional loss of generality in setting $\lambda_{0_i} = \lambda_0 = 1$ for all i, in which case $\mathbf{X}(t)$ will be said to be in (standard) *reference 1*, or in setting $\lambda_{2_i} = \lambda_2 = 1$ ($\mathbf{X}(t)$ in *reference 2*), or in setting $B_i = \lambda_{2_i} - \lambda_{1_i}^2/\lambda_{0_i} = 1$ ($\mathbf{X}(t)$ in *reference 3*). B_i is the variance of $\dot{S}_i(t)$, the derivative process of $S_i(t)$. In all cases, (4.1) is assumed to hold if the mean crossing rate of $\mathbf{X}(t)$ is being calculated, whereas (4.2) is assumed to hold if an envelope mean crossing rate is being calculated.

Reference 2 is particularly convenient to find $\nu_\mathbf{X}$, the mean rate with which $\mathbf{X}(t)$ outcrosses the boundary ∂_D of any given region, D. In fact in this reference (Bolotin, 1971; Veneziano et al., 1977),

(4.4) $$\nu_\mathbf{X} = (2\pi)^{-\frac{1}{2}} p_\mathbf{X}(\partial_D)$$

in which

(4.5) $$p_\mathbf{X}(\partial_D) = \int_{\partial_D} f_\mathbf{X}(\mathbf{x})\, da(\mathbf{x})$$

is the integral of the probability density of $\mathbf{X}(t)$ over ∂_D ($da(\mathbf{x})$ = differential area of ∂_D at $\mathbf{x}$). More explicit results for particular configurations of D are found in Belyaev (1968), Belyaev and Nosko (1969), Hasofer (1974), Veneziano et al. (1977). These results will be used in later sections, to compare $\nu_\mathbf{X}$ with ν_D, ν_R, and ν_E, the mean crossing rates of set processes $\mathbb{S}_D(t)$, $\mathbb{S}_R(t)$, and $\mathbb{S}_E(t)$, respectively.

5. Mean crossing rates of $\mathbb{S}_D(t)$. In any given reference, $\mathbb{S}_D(t)$ is in the crossed state if and only if the radius of $\mathbb{S}_D(t)$, $d(t) = [\Sigma_{i=1}^n S_i^2(t)]^{\frac{1}{2}}$, is larger then $\beta =$ distance of ∂_D from the mean of $\mathbf{X}(t)$. Hence, ν_D equals the mean rate at which $d(t)$ upcrosses β.

Calculation of ν_D is particularly simple *if* λ_{0_i} *and* B_i *do not depend on i*. Then, in standard reference 1 ($\lambda_{0_i} = 1$ and $B_i = B = \lambda_{2_i} - \lambda_{1_i}^2$), $d(t)$ and $\dot{d}(t)$ are independent, $d(t)$ with χ_{2n} distribution and $\dot{d}(t)B^{-\frac{1}{2}}$ with standard normal distribution. Their joint probability density function is

$$(5.1) \qquad f_{d,\dot{d}}(r, \dot{r}) = \frac{r(r^2/2)^{n-1}\exp(-r^2/2)}{(n-1)!}\ \frac{\exp(-\dot{r}^2/2B)}{(2\pi)^{\frac{1}{2}}B^{\frac{1}{2}}}.$$

Use of (5.1) in Rice's formula for the mean upcrossing rate of scalar processes (Rice, 1944, 1945) gives

$$(5.2) \qquad \nu_D = (2\pi)^{-\frac{1}{2}}B^{\frac{1}{2}}\chi_{2n}(\beta)$$

in which $\chi_\nu(\cdot) =$ probability density function of the chi-variate with ν degrees of freedom. ν_D depends on the shape of D only through β. The expected duration of $\mathbb{S}_D(t)$ crossings, $E[T_D]$, equals the ratio between the probability that $\mathbb{S}_D(t)$ is in the crossed state at the generic time t, $P[d(t) > \beta]$, and the mean crossing rate, ν_D, i.e.,

$$(5.3) \qquad E[T_D] = \frac{(2\pi)^{\frac{1}{2}}}{B^{\frac{1}{2}}}\ \frac{[1 - \bar{\chi}_{2n}(\beta)]}{\chi_{2n}(\beta)}$$

in which $\bar{\chi}_\nu(\cdot) =$ cumulative distribution function of the chi-variate with ν degrees of freedom.

Since the mean rate of $\mathbf{X}(t)$ crossings depends on the shape of D, the ratio $\nu_{\mathbf{X}}/\nu_D =$ expected number of $\mathbf{X}$ outcrossings per $\mathbb{S}_D$ crossing depends on the shape of D and not just on β. For example, let D be the *centered sphere* (center at 0) with radius β. For $\lambda_{2_i} = \lambda_2$ and $\lambda_{0_i} = 1$, (4.4) gives $\nu_{\mathbf{X}} = (2\pi)^{-\frac{1}{2}}\lambda_2^{\frac{1}{2}}\ \chi_n(\beta)$. Hence

$$(5.4) \qquad \nu_{\mathbf{X}}/\nu_D = \left(\frac{\lambda_2}{\lambda_2 - \lambda_1^2}\right)^{\frac{1}{2}} \frac{\chi_n(\beta)}{\chi_{2n}(\beta)}.$$

If λ_{0_i} *or* B_i *are functions of i*, then (5.1) and (5.2) cannot be used. One can work in standard reference 3, in which case $\dot{d}(t)$ has the normal distribution implied by (5.1) and $d(t)$ has probability density function

$$(5.5) \qquad f_d(r) = \int_{\Omega_r} f_{\mathbf{S}}(\mathbf{s})\, da(\mathbf{s})$$

in which $f_{\mathbf{S}}(\mathbf{s}) = \Pi_{i=1}^n f_{S_i}(s_i)$, $f_{S_i}(s) = \lambda_{0_i}^{-\frac{1}{2}}\chi_2(s\lambda_{0_i}^{-\frac{1}{2}})$, $\Omega_r = \{\mathbf{s} : |\mathbf{s}| = r,\ s_i \geq 0, i = 1, \cdots, n\}$ = portion of centered spherical surface with radius r in the positive orthant of R_n, and $da(\mathbf{s})$ = differential area of Ω_r at $\mathbf{s}$. With $f_d(\cdot)$ in (5.5), the mean crossing rate of $\mathbb{S}_D(t)$ is

$$(5.6) \qquad \nu_D = f_d(\beta)(2\pi)^{-\frac{1}{2}}B^{\frac{1}{2}}.$$

Alternatively, one can work in *standard reference 1*. Then the variables $S_i(t)$ are independent with identical Rayleigh distribution, $d(t)$ has the χ_{2n} distribution implied by (5.1), and, given $d(t) = r$, $\dot{d}(t)$ has conditional probability density

$$f_{\dot{d}|d}(\dot{r}, r) = \chi_{2n}^{-1}(r)\int_{\Omega_r} f_{\mathbf{S}}(\mathbf{s}) f_{\dot{d}|\mathbf{S}}(\dot{r}, \mathbf{s})\, da(\mathbf{s}) \tag{5.7}$$

in which $(\dot{d}|\mathbf{S} = \mathbf{s})$ is a normal variate with zero mean and variance $|\mathbf{s}|^{-2}\Sigma_{i=1}^{n} s_i^2 B_i$. In this case ν_D is found from

$$\nu_D = \chi_{2n}(\beta)\int_0^\infty \dot{r} f_{\dot{d}|d}(\dot{r}, \beta)\, d\dot{r}. \tag{5.8}$$

For $n = 2$ and with $\gamma_S^2 = \mathrm{Var}[\dot{S}_2(t)]/\mathrm{Var}[\dot{S}_1(t)] = B_2/B_1$ ($\gamma_S^2 \geqslant 1$, say), integration of (5.8) gives

$$\frac{\nu_D}{\nu_{D|\gamma_S^2=1}} = \frac{2}{3}\frac{\gamma_S^3 - 1}{\gamma_S^2 - 1} \tag{5.9}$$

in which $\nu_{D|\gamma_S^2=1}$ is the mean crossing rate in (5.2) for $n = 2$.

6. Mean crossing rates of $\mathcal{S}_R(t)$. Since point envelope vectors $\mathbf{S} = [S_1, \cdots, S_n]$ and rectangles $\mathcal{S}_R$ are in one-to-one correspondence, $\mathcal{S}_R(t)$ crossings with respect to a region D of R_n are in one-to-one correspondence with $\mathbf{S}(t)$ exits out of the region D_R, defined

(6.1)

$$D_R = \{\mathbf{S} : \mathbf{X} \in D \quad \text{for all} \quad \mathbf{X} \quad \text{such that} \quad |X_i| \leqslant S_i, i = 1, \cdots, n\}.$$

A simple case is when D is a *centered sphere* in R_n; then $D_R = D$, $\mathcal{S}_R$ crossings coincide with $\mathcal{S}_D$ crossings, and the results of the previous section apply without modification. Some problems with nonspherical regions in *standard reference 3* are considered next. In all cases,

$$\nu_R = (2\pi)^{-\frac{1}{2}} B^{\frac{1}{2}} p_{\mathbf{S}}(\partial_{D_R}) \tag{6.2}$$

in which $p_{\mathbf{S}}(\partial_{D_R})$ = integral of the $\mathbf{S}(t)$ density over the boundary of D_R. If D is the *rectangle* $\{\mathbf{X} : -c_i \leqslant X_i \leqslant d_i, i = 1, \cdots, n\}$ with c_i, d_i nonnegative constants, then D_R is the rectangle $\{\mathbf{S} : 0 \leqslant S_i \leqslant b_i = \min(c_i, d_i), i = 1, \cdots, n\}$ and

$$\nu_R = (2\pi)^{-\frac{1}{2}} B^{\frac{1}{2}} \Sigma_{i=1}^{n} \frac{b_i}{\lambda_{0_i}^{1/2}} \exp\left(-b_i^2/2\lambda_{0_i}\right)\bar{\chi}/\bar{\chi}_2\left(b_i/\lambda_{0_i}^{1/2}\right) \tag{6.3}$$

with $\bar{\chi} = \Pi_{j=1}^{n}\bar{\chi}_2(b_j/\lambda_{0_i}^{1/2})$. The expected duration of $\mathcal{S}_R(t)$ crossings is $(1 - \bar{\chi})/\nu_R$ in which the numerator is the probability that $\mathcal{S}_R(t)$ is in the crossed state at any given time t.

For *cubic regions* with $c_i = d_i = \beta$ and for processes $X_i(t)$ with the same first three spectral moments (say $\lambda_{0_i} = \lambda_0 = 1, \lambda_{1_i} = \lambda_1, \lambda_{2_i} = \lambda_2$, and $B_i = B = \lambda_2 - \lambda_1^2$), the mean rate at which $\mathbf{X}(t)$ leaves D is (Veneziano et al. (1977))

$$\nu_{\mathbf{X}} = \frac{n}{\pi}\lambda_2^{\frac{1}{2}}\ \bar{\chi}_1^{n-1}(\beta)\exp(-\beta^2/2) \tag{6.4}$$

where $\bar{\chi}_1(\beta) = 1 - 2\Phi(-\beta)$ and $\Phi(\cdot)$ = standard normal cdf. From (6.3) and (6.4) one finds that the expected number of $\mathbf{X}(t)$ crossings per $\mathcal{S}_R(t)$ crossing is

$$\frac{\nu_{\mathbf{X}}}{\nu_R} = \frac{2}{(2\pi)^{\frac{1}{2}}} \frac{1}{\beta} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1^2} \right)^{\frac{1}{2}} [\bar{\chi}_1(\beta)/\bar{\chi}_2(\beta)]^{n-1}. \tag{6.5}$$

For $\beta \geqslant 3$, a very good approximation to $\nu_{\mathbf{X}}/\nu_R$ is obtained by setting $\bar{\chi}_1(\beta)/\bar{\chi}_2(\beta) = 1$. This approximation coincides with a well-known result for scalar processes (Cramér and Leadbetter, 1967; Vanmarcke, 1969), which in turn is a special case of (6.5).

A case of interest for reliability applications is when D is a *generic polygon in the plane* (then also D_R is a polygon). A simple expression for $\nu_{\mathbf{X}}$ was obtained by Veneziano et al. (1977) in terms of scalar mean-crossing-rate results. A convenient expression can be found also for ν_R, and is given below without derivation. Since ν_R is contributed additively by the mean rates at which $\mathbf{S}(t)$ outcrosses each side of ∂_{D_R}, one need only find the mean outcrossing rate, ν_{R_i}, for the generic side i. With the notation of Figure 1a it can be shown that:

$$\nu_{R_i} = (2\pi)^{-\frac{1}{2}} \sigma_i \left[g(Y_{i_2}, d_i, \alpha_i) - g(Y_{i_1}, d_i, \alpha_i) \right] \tag{6.6}$$

in which $\sigma_i^2 = B_1 \cos^2\alpha_i + B_2 \sin^2\alpha_i$ and

$$\begin{aligned} g(Y, d, \alpha) = &\Big\{ -\frac{2^{\frac{1}{2}}}{2} \sin 2\alpha \left[\Gamma\left(\tfrac{3}{2}, d_1^2\right) - \Gamma\left(\tfrac{3}{2}, Y^2\right) \right] \\ &+ d \cos 2\alpha \left[\exp(-Y^2/2) - \exp(-d_1^2/2) \right] \\ &+ \tfrac{1}{2} d^2 \sin 2\alpha \left[\exp(-Y^2) - \exp(-d_1^2) \right] \Big\} \exp(-d^2/2), \\ &\qquad \text{if} \quad -d_1 \leqslant Y \leqslant 0 \\ = &\Big\{ -\frac{2^{\frac{1}{2}}}{2} \sin 2\alpha \left[\Gamma\left(\tfrac{3}{2}, d_1^2\right) + \Gamma\left(\tfrac{3}{2}, Y^2\right) \right] \\ &+ d \cos 2\alpha \left[\exp(-Y^2/2) - \exp(-d_1^2/2) \right] \\ &+ \tfrac{1}{2} d^2 \sin 2\alpha \left[2 - \exp(-Y^2) - \exp(-d_1^2) \right] \Big\} \exp(-d^2/2), \\ &\qquad \text{if} \quad 0 \leqslant Y \leqslant d \tan\alpha \end{aligned} \tag{6.7}$$

with $d_1 = d \tan^{-1}\alpha$ and $\Gamma(\cdot, \cdot\cdot)$ = incomplete gamma function (Abramowitz and Stegun, 1965). Function $g(Y, d, \alpha)$ in (6.7) is the integral of the joint density of $S_1(t)$ and $S_2(t)$ over the segment AB in Figure 1b.

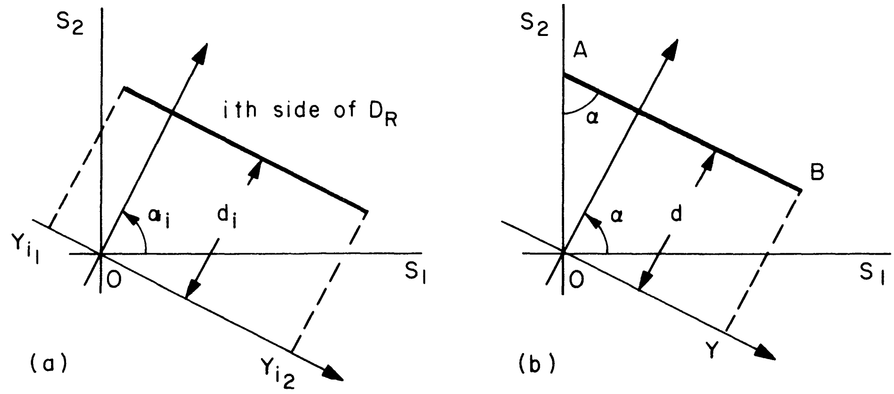

FIG. 1. Notation for the mean crossing rate of rectangle set processes. D is a polygon in the plane.

7. Mean crossing rates of $\mathcal{S}_E(t)$. Consideration will be limited to bivariate processes $\mathbf{X}(t)$ in standard reference 1, or 2, or 3, with D either a centered disk or a convex polygon. In all these cases $\mathcal{S}_E(t)$ is a random ellipse with fixed center at 0; therefore $\mathcal{S}_E(t)$ can be described by three parameters, themselves components of a stochastic vector process of time. A set of parameters that is convenient for mean-crossing-rate calculations is $S_1(t)$, $S_2(t)$, and $\phi(t) = \tan^{-1}[\hat{X}_2(t)/X_2(t)] - \tan^{-1}[\hat{X}_1(t)/X_1(t)]$. In terms of $\mathbf{C} = [S_1, S_2, \phi]$, the ellipse $\mathcal{S}_E$ is

$$\mathcal{S}_E(\mathbf{C}) = \left\{\mathbf{X} : \boldsymbol{\alpha}^T\mathbf{X} \leqslant \left[\boldsymbol{\alpha}^T\mathbf{Q}(\mathbf{C})\boldsymbol{\alpha}\right]^{\frac{1}{2}} \text{ for all } \boldsymbol{\alpha} = \left[\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right]\right\} \tag{7.1}$$

in which

$$\mathbf{Q}(\mathbf{C}) = \left[\begin{array}{cc} S_1^2 & S_1S_2\cos\phi \\ S_1S_2\cos\phi & S_2^2 \end{array}\right].$$

If $\cos\phi \neq 1$, $\mathbf{Q}(\mathbf{C})$ is nonsingular and a simpler characterization of $\mathcal{S}_E$ is $\mathcal{S}_E(\mathbf{C}) = \{\mathbf{X} : \mathbf{X}^T\mathbf{Q}^{-1}(\mathbf{C})\mathbf{X} \leqslant 1\}$. Crossings of $\mathcal{S}_E(t)$ can be studied as crossings of $\mathbf{C}(t)$ out of the region D_E of R_3, defined

$$D_E = \{\mathbf{S}_E : \mathcal{S}_E(\mathbf{C}) \subset D\}. \tag{7.2}$$

For example, if D is the *centered disk* with radius β, then D_E is the set of points $\mathbf{C}$ such that the principal semiaxes of $\mathcal{S}_E$ are not larger than β. A sketch of D_E for this case is shown in Figure 2a, for $0 \leqslant \phi \leqslant \pi$. In the interval, $\pi \leqslant \phi \leqslant 2\pi$, one can use symmetry of D_E with respect to the plane $\phi = \pi$. If D is a *convex polygon*, then $D_E = \cap_j D_{E_j}$ where D_{E_j} is the region that corresponds to D_j = the half plane bounded by the jth side that contains the polygon. It is therefore sufficient to find D_E for D = the half plane bounded by a generic line, $aX_1 + bX_2 = 1$, and containing the origin. This is

$$D_E = \left\{\mathbf{C} : \left[\boldsymbol{\eta}^T\mathbf{Q}(\mathbf{C})\boldsymbol{\eta}\right]^{\frac{1}{2}} \leqslant d\right\}, \tag{7.3}$$

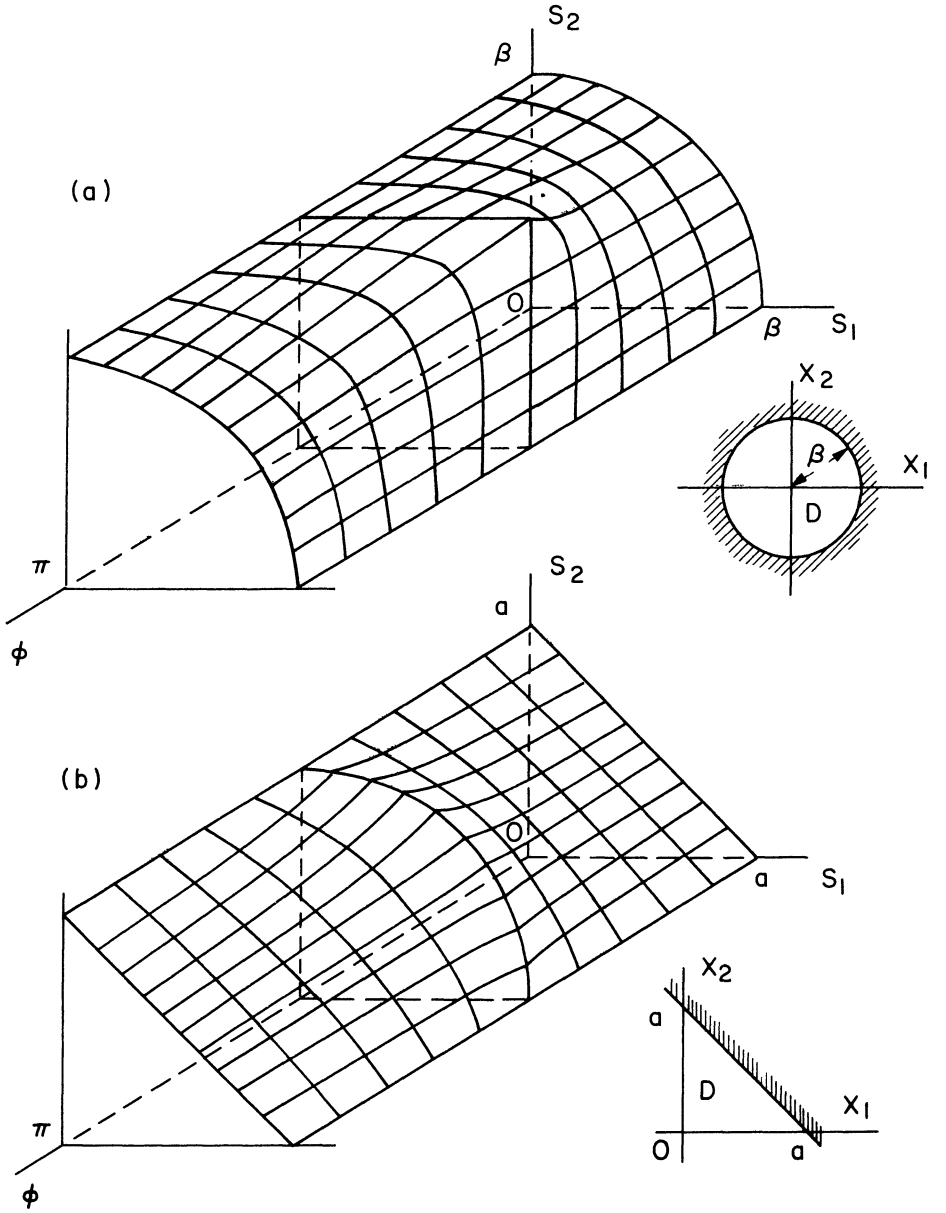

FIG. 2. Region D_E in $S_1S_2\phi$ space that corresponds: (a) to a centered disk, (b) to a half plane in X_1X_2 space.

in which $\boldsymbol{\eta}$ is the unit vector normal to the line $aX_1 + bX_2 = 1$ and $d = (a^2 + b^2)^{-\frac{1}{2}}$ is the distance between the same line and the origin. D_E is sketched in Figure 2b for the case $a = b$. As in the previous case, D_E is symmetrical with respect to the plane $\phi = \pi$.

For calculation of ν_E, the mean rate at which $\mathbf{C}(t)$ leaves D_E, one needs to know the joint probability density function of $S_1(t)$, $S_2(t)$, $\phi(t)$, $\dot{S}_1(t)$, $\dot{S}_2(t)$, and $\dot{\phi}(t)$. Using results in the Appendix and independence properties in the standard references, this density function is found to be

(7.4) $$f_{S_1, S_2, \phi, \dot{S}_1, \dot{S}_2, \dot{\phi}} = f_{S_1} f_{S_2} f_{\phi} f_{\dot{S}_1} f_{\dot{S}_2} f_{\dot{\phi}|S_1, S_2}$$

in which

$$f_{S_i}(s) = \frac{s}{\lambda_{0_i}} \exp\left(-s^2/2\lambda_{0_i}\right), \quad s \geq 0$$
$$= 0, \quad s < 0, \qquad i = 1, 2;$$
$$f_{\dot{S}_i}(\dot{s}) = (2\pi B_i)^{-\frac{1}{2}} \exp\left(-\dot{s}^2/2B_i\right) \quad i = 1, 2;$$
$$f_{\phi}(\phi) = 1/2\pi, \quad 0 \leq \phi < 2\pi,$$
$$= 0, \quad \text{otherwise};$$
$$f_{\dot{\phi}|S_1, S_2}(\dot{\phi}, s_1, s_2) = \frac{1}{(2\pi)^{\frac{1}{2}} \sigma_{\dot{\phi}|S_1, S_2}(s_1, s_2)} \exp\left[-\frac{\left(\dot{\phi} - m_{\dot{\phi}}\right)^2}{2\sigma^2_{\dot{\phi}|S_1, S_2}(s_1, s_2)}\right];$$

and

$$m_{\dot{\phi}} = \frac{\lambda_{1_2}}{\lambda_{0_2}} - \frac{\lambda_{1_1}}{\lambda_{0_1}}$$
$$\sigma^2_{\dot{\phi}|S_1, S_2}(s_1, s_2) = \frac{B_1}{s_1^2} + \frac{B_2}{s_2^2}.$$

If $\mathbf{n}(\mathbf{C}) = [n_1(\mathbf{C}), n_2(\mathbf{C}), n_3(\mathbf{C})]^T$ is the vector of director cosines of the external normal to ∂_{D_E} at $\mathbf{C}$, then

(7.5) $$\nu_E = \int_{\partial_{D_E}} f_{S_1}(s_1) f_{S_2}(s_2) f_{\phi}(\phi) E_0^{\infty}\left[\dot{Y}(\mathbf{c})\right] da(\mathbf{c})$$

in which $\mathbf{c} = [s_1, s_2, \phi]^T$, $da(\mathbf{c})$ = differential area of ∂_{D_E} at $\mathbf{c}$, $\dot{Y}(\mathbf{c}) = n_1(\mathbf{c})\dot{S}_1 + n_2(\mathbf{c})\dot{S}_2 + n_3(\mathbf{c})\dot{\phi}$ = component of $\dot{\mathbf{C}}$ at $\mathbf{c}$ in the direction of $\mathbf{n}(\mathbf{c})$, and $E_0^{\infty}[\dot{Y}] = \int_0^{\infty} u f_{\dot{Y}}(u)\, du$. $\dot{Y}(\mathbf{c})$ has normal distribution with parameters

(7.6) $$E\left[\dot{Y}(\mathbf{c})\right] = n_3(\mathbf{c}) m_{\dot{\phi}}$$
$$\operatorname{Var}\left[\dot{Y}(\mathbf{c})\right] = n_1^2(\mathbf{c}) B_1 + n_2^2(\mathbf{c}) B_2 + n_3^2(\mathbf{c}) \sigma^2_{\dot{\phi}|S_1, S_2}(s_1, s_2).$$

8. Appendix: Joint distribution of S, Θ, $\dot{S}$, and $\dot{\Theta}$. Let $X(t)$ be a stationary Gaussian process with continuous derivative. We obtain here the joint distribution of $S(t)$, $\Theta(t)$, $\dot{S}(t)$ and $\dot{\Theta}(t)$, with $S(t)$ in (2.2), $\Theta(t) = \tan^{-1}[\hat{X}(t)/X(t)]$, and $\hat{X}(t)$ in (2.3).

From Cramér and Leadbetter (1967), pages 249 and 251, vector $\xi(t, \tau) = [X(t), \hat{X}(t), X(t + \tau), \hat{X}(t + \tau)]^T$ has normal distribution with zero mean and covariance matrix

$$\Sigma_\xi = \begin{bmatrix} \lambda_0 & 0 & a & b \\ 0 & \lambda_0 & -b & a \\ a & -b & \lambda_0 & 0 \\ b & a & 0 & \lambda_0 \end{bmatrix} \tag{8.1}$$

in which

$$\begin{aligned} a &= a(\tau) = \lambda_0 - \frac{\lambda_2 \tau^2}{2} + 0(\tau^2) \\ b &= b(\tau) = \lambda_1 \tau + 0(\tau) \end{aligned} \tag{8.2}$$

and λ_i = ith absolute spectral moment of $X(t)$. Let

$$\tilde{\xi}(t, \tau) = \begin{bmatrix} \tilde{\xi}_1(t) \\ \tilde{\xi}_2(t, \tau) \end{bmatrix} \tag{8.3}$$

with

$$\tilde{\xi}_1(t) = \begin{bmatrix} X(t) \\ \hat{X}(t) \end{bmatrix} \quad \text{and} \quad \tilde{\xi}_2(t, \tau) = \begin{bmatrix} \Delta X(t, \tau) \\ \Delta \hat{X}(t, \tau) \end{bmatrix} = \begin{bmatrix} X(t + \tau) - X(t) \\ \hat{X}(t + \tau) - \hat{X}(t) \end{bmatrix}. \tag{8.4}$$

Then $\tilde{\xi}(t, \tau)$ has normal distribution with zero mean and covariance matrix

$$\Sigma_{\tilde{\xi}} = \begin{bmatrix} \lambda_0 \mathbf{I}_2 & \begin{bmatrix} (a - \lambda_0) & b \\ -b & (a - \lambda_0) \end{bmatrix} \\ \begin{bmatrix} (a - \lambda_0) & -b \\ b & (a - \lambda_0) \end{bmatrix} & 2(\lambda_0 - a)\mathbf{I}_2 \end{bmatrix}. \tag{8.5}$$

Since $\tilde{\xi}_1(t)$ has circular Gaussian distribution with zero mean, $S(t)\lambda_0^{-\frac{1}{2}}$ and $\Theta(t)$ are independent variables, the former with Rayleigh distribution, the latter with uniform distribution in $[0, 2\pi]$. The conditional distribution of $[\dot{S}(t), \dot{\Theta}(t)]$ given $[S(t), \Theta(t)]$ is found as follows.

From well-known results on conditional Gaussian vectors, $[\tilde{\xi}_2(t, \tau)/\tilde{\xi}_1(t)]$ has bivariate normal distribution with mean

$$\mathbf{m}_{2|1} = \frac{1}{\lambda_0} \begin{bmatrix} (a - \lambda_0)X(t) - b\hat{X}(t) \\ bX(t) + (a - \lambda_0)\hat{X}(t) \end{bmatrix} \tag{8.6}$$

and covariance matrix

$$\Sigma_{2|1} = \left\{ 2(\lambda_0 - a) - \frac{1}{\lambda_0} \left[(\lambda_0 - a)^2 + b^2 \right] \right\} \mathbf{I}_2. \tag{8.7}$$

$\dot{\tilde{\xi}}_1(t)$ is the limit of $(1/\tau)\tilde{\xi}_2(t, \tau)$ as $\tau \to 0$. Substituting for a and b in (8.2), using the fact that if $\tilde{\xi}_2$ has mean $\mathbf{m}_2$ and covariance matrix Σ_2, then $(1/\tau)\tilde{\xi}_2$ has mean $(1/\tau)\mathbf{m}_2$ and covariance matrix $(1/\tau^2)\Sigma_2$, and taking the limit as $\tau \to 0$ one finds that $[\dot{\tilde{\xi}}_1(t)|\tilde{\xi}_1(t)]$ has normal distribution with parameters

(8.8) $$E\left[\dot{\tilde{\xi}}_1(t)|\tilde{\xi}_1(t)\right] = \frac{\lambda_1}{\lambda_0}\begin{bmatrix} -\hat{X}(t) \\ X(t) \end{bmatrix}$$

$$\Sigma_{\dot{\tilde{\xi}}_1(t)|\tilde{\xi}_1(t)} = (\lambda_2 - \lambda_1^2/\lambda_0)\mathbf{I}_2.$$

Next denote by $\dot{S}(t)$ and $\dot{S}_\Theta(t)$ the radial and tangential components of $\dot{\tilde{\xi}}_1(t)$, respectively. In terms of previous quantities, $\dot{S}(t)$ and $\dot{S}_\Theta(t)$ are

(8.9) $$\begin{bmatrix} \dot{S}(t) \\ \dot{S}_\Theta(t) \end{bmatrix} = \begin{bmatrix} c & d \\ d & -c \end{bmatrix}\dot{\tilde{\xi}}_1(t)$$

in which $c = X(t)/S(t)$ and $d = \hat{X}(t)/S(t)$. Given $\tilde{\xi}_1$ (or, equivalently, given $S(t)$ and $\Theta(t)$), c and d are known and the conditional joint distribution of $\dot{S}(t)$ and $\dot{S}_\Theta(t)$ is normal with mean $[0, \lambda_1 S(t)/\lambda_0]^T$ and covariance matrix $(\lambda_2 - \lambda_1^2/\lambda_0)\mathbf{I}_2$. Finally, given $S(t)$ and $\Theta(t)$, $\dot{S}(t)$ and $\dot{\Theta}(t) = \dot{S}_\Theta(t)/S(t)$ have conditional bivariate normal distribution with parameters

(8.10) $$\mathbf{m} = \begin{bmatrix} 0 \\ \lambda_1/\lambda_0 \end{bmatrix}, \qquad \Sigma = (\lambda_2 - \lambda_1^2/\lambda_0)\begin{bmatrix} 1 & 0 \\ 0 & 1/S^2(t) \end{bmatrix}.$$

This completes the derivation of the joint distribution of $S(t)$, $\Theta(t)$, $\dot{S}(t)$ and $\dot{\Theta}(t)$. Note that the distribution of $\dot{S}(t)$ implied by (8.10)—normal with zero mean and variance $(\lambda_2 - \lambda_1^2/\lambda_0)$—coincides with the well-known result by Cramér and Leadbetter (1967).

REFERENCES

[1] ABRAMOWITZ, M. and STEGUN, I. A., eds. (1965). *Handbook of Mathematical Functions*. Dover, New York.

[2] BELYAEV, YU. K. (1968). On the number of exits across the boundary of a region by a vector stochastic process. *Theor. Probability Appl.* **13** 320–324.

[3] BELYAEV, YU. K. and NOSKO, V. P. (1969). Characteristics of excursions above a high level for a Gaussian process and its envelope. *Theor. Probability Appl.* **14** 296–309.

[4] BOLOTIN, V. V. (1971). *Primenenie Metodov Teorii Verovatnostei i Teorii Nadejnosti v Raschetah Soorujenni*. Izdatelstvo Literaturi po Stroitelstvi, Moskov.

[5] CRAMÉR, H. and LEADBETTER, M. R. (1967). *Stationary and Related Stochastic Processes*. Wiley, New York.

[6] CRANDALL, S. H. (1970). First-crossing probabilities of the linear oscillator. *J. Sound Vib.* **12**(3) 285–299.

[7] CRANDALL, S. H. and MARK, W. D. (1963). *Random Vibration in Mechanical Systems*. Academic Press, New York.

[8] HASOFER, A. M. (1974).The upcrossing rate of a class of stochastic processes. In *Studies in Probability and Statistics* (papers in honor of E. J. G. Pitman; E. J. Williams, ed.), 153–160. Jerusalem Academic Press.

[9] LYON, R. H. (1961). On the vibration statistics of a randomly excited hard-spring oscillator II. *J. Acoust. Soc. Am.* **33**(10) 1395–1403.

[10] RICE, S. O. (1944). Mathematical analysis of random noise. *Bell System Tech. J.* **23** 282; (1945) **24** 46.

[11] VANMARCKE, E. H. (1969). First-passage and other failure criteria in narrow-band random vibration: a discrete state approach. Res. Report No. R69-68, Dept. of Civil Engrg., M.I.T.

[12] VANMARCKE, E. H. (1975). On the distribution of the first-passage time for normal stationary random processes. *J. Appl. Mech.*, Transactions, ASME **42** 215–220.

[13] VENEZIANO, D., GRIGORIU, M. and CORNELL, C. A. (1977). Vector-process models for system reliability. *J. Engrg. Mech. Div.*, ASCE **103** (EM3) 441–460.

DEPARTMENT OF CIVIL ENGINEERING
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS 02139