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STOCHASTIC COMPACTNESS OF SAMPLE EXTREMES

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Let $Y_1, Y_2, \cdots$ be independent and identically distributed random variables with common distribution function F and let $X_n = \max\{Y_1, \cdots, Y_n\}$ for $n = 1, 2, \cdots$. Necessary and sufficient conditions (in terms of F) are derived for the existence of a sequence of positive constants $\{a_n\}$ such that the sequence $\{X_n/a_n\}$ is stochastically compact. Moreover, the relation between the stochastic compactness of partial maxima and partial sums of the Y_n's is investigated.

1. Introduction. In the following compactness properties of sequences of sample maxima are investigated. This parallels results of Feller (1965) with respect to compactness properties of sequences of partial sums. It will become clear, that, in the case of sample maxima, it is not evident how the concept of stochastic compactness should be defined. Therefore this concept will be defined in two different ways (Sections 2, 4). For both definitions necessary and sufficient conditions are obtained (Sections 3, 4). These conditions are in terms of the distribution function, in contrast to those for the compactness of sample sums. They resemble a generalisation of the concept of regular variation called R-O-variation (Seneta (1976)). The relation between the present results and the theory of R-O-variation will be considered in Section 4. The relation between the compactness and the weak convergence of sample maxima and the relation between the compactness of partial maxima and the compactness of partial sums will be considered in Sections 5 and 6 respectively. In Section 7 some examples and counterexamples are presented.

2. Stochastic boundedness and compactness.

2.1. Definitions. Let $\{X_n\}$ be a sequence of real-valued random variables with distribution functions (df) $\{F_n\}$.

DEFINITION 2.1.1. The sequence of random variables $\{X_n\}$ is *stochastically bounded* if for all $\varepsilon > 0$ there is an $x_0 > 0$ such that $P(|X_n| > x_0) < \varepsilon$ for all n.

REMARK 2.1.1. This definition is equivalent to each of the following assertions.

1. Every subsequence $\{X_{n_k}\}$ of $\{X_n\}$ contains a further subsequence $\{X_{n_r}\}$ such that $\{X_{n_r}\} \to_w X$ with X a nondefective random variable.
2. $\lim_{x\to\infty} P(|X_n| > x) = 0$ uniformly in n.

We now define the concept of stochastic compactness of sample maxima. For that purpose we consider a sequence of identically and independently distributed

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(i.i.d.) random variables $\{Y_n\}$ with common df F. We suppose that:

(1) F is continuous (see Remark 2.2.2);

(2) $F(0) = 0$ (see Remark 2.2.1);

(3) $F(x) < 1$ for all x (this is to avoid trivialities).

We now define for $n = 1, 2, \cdots$

$$X_n = \max\{Y_1, \cdots, Y_n\}.$$

DEFINITION 2.1.2. The sequence of sample maxima $\{X_n\}$ is *stochastically compact* if there is a sequence of positive constants $\{a_n\}$ such that the sequences $\{X_n/a_n\}$ and $\{a_n/X_n\}$ are stochastically bounded.

REMARK 2.1.2.

1. An alternative formulation of this definition is that the sequence of sample maxima $\{X_n\}$ is stochastically compact if there is a sequence of positive constants $\{a_n\}$ such that every convergent subsequence of $\{X_n/a_n\}$ has a limit distribution concentrated on $(0, \infty)$.
2. Instead of $\{X_n/a_n\}$ we can consider the sequence $\{\ln(X_n/a_n)\}$. This is possible because $\{X_n\}$ is a sequence of positive random variables. The stochastic compactness of the sequence of sample maxima $\{X_n\}$ then is equivalent to the assertion that there is a sequence of real constants $\{b_n\}$ such that the sequence $\{\ln X_n - b_n\}$ is stochastically bounded. This is an alternative way to present the results because $\ln X_n = \max\{\ln Y_1, \cdots, \ln Y_n\}$.
3. Helly's theorem implies that every subsequence of $\{X_n/a_n\}$ contains a further subsequence that is weakly convergent. In the definition of stochastic compactness, however, certain limit distributions are excluded. In the definition above limit distributions which assign positive probability to the points 0 or ∞ are not admitted (note that one can always choose the constants $\{a_n\}$ such that, e.g., $X_n/a_n \to_w 0$). In his definition of stochastic compactness of partial sums Feller excludes degenerate limit distributions. In the sequel we will consider an alternative definition of stochastic compactness of maxima where also a larger class of limit distributions is excluded.

2.2. Necessary and sufficient conditions for stochastic compactness. From Remark 2.1.1 it follows that the stochastic compactness of the sequence of sample maxima $\{X_n\}$ is equivalent to the following two assertions:

$$\lim_{x\to\infty} \liminf_{n\to\infty} F^n(a_n x) = 1. \tag{2.2.1}$$

$$\lim_{x\downarrow 0} \limsup_{n\to\infty} F^n(a_n x) = 0. \tag{2.2.2}$$

REMARK 2.2.1. From (2.2.1) and (2.2.2) we see that the requirement $F(0) = 0$ does not result in a loss of generality. For if $0 < F(0) < 1$, then for $x > 0$

$$P(|X_n| > a_n x) = 1 - F^n(a_n x) + F^n(-a_n x -). \tag{2.2.3}$$

However, the last term on the right-hand side of (2.2.3) clearly converges to 0 as $n \to \infty$.

LEMMA 2.2.1.

(1) $\lim_{x\to\infty} \liminf_{n\to\infty} F^n(a_n x) = 1 \Leftrightarrow \lim_{x\to\infty} \limsup_{n\to\infty} n(1 - F(a_n x)) = 0.$

(2) $\lim_{x\downarrow 0} \limsup_{n\to\infty} F^n(a_n x) = 0 \Leftrightarrow \lim_{x\downarrow 0} \liminf_{n\to\infty} n(1 - F(a_n x)) = \infty.$

PROOF. We only prove part 1; part 2 can be proved analogously. We have

$$\liminf_{n\to\infty} F^n(a_n x) = \liminf_{n\to\infty}\left(1 - \frac{n(1 - F(a_n x))}{n}\right)^n$$
$$= \exp\left[-\limsup_{n\to\infty} n(1 - F(a_n x))\right].$$

and (1) follows.

Necessary and sufficient conditions for the stochastic compactness of a sequence of partial maxima are given by the following theorem.

THEOREM 2.2.1. *Let F be a continuous distribution function with $F(0) = 0$, $F(x) < 1$ for all x. The following assertions are equivalent*:

(1) *Let $\{Y_n\}$ be a sequence of* i.i.d. *random variables with common* df F. *The sequence $\{X_n\}$ of sample maxima from $\{Y_n\}$ is stochastically compact.*

(2) $$\lim_{x\to\infty} \limsup_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = 0.$$

(3) $$\exists t_0, x_0 > 0, 0 < M < 1 \textit{ such that } \frac{1 - F(tx)}{1 - F(t)} \leqslant M \textit{ for } t \geqslant t_0, x \geqslant x_0.$$

(4) $$\exists C, \rho > 0 \textit{ such that } \frac{1 - F(tx)}{1 - F(t)} \leqslant Cx^{-\rho} \textit{ for } x \geqslant 1, t \geqslant t_0.$$

(5)
$$\int_1^\infty \frac{1 - F(s)}{s}\, ds < \infty \textit{ and } \liminf_{x\to\infty} \frac{1 - F(x)}{H(x)} > 0 \textit{ with } H(x) = \int_x^\infty \frac{1 - F(s)}{s}\, ds.$$

(6) $$\int_1^\infty \frac{1 - F(s)}{s}\, ds < \infty \textit{ and } \lim_{x\to\infty} \limsup_{t\to\infty} \frac{H(tx)}{H(t)} = 0.$$

REMARK 2.2.2. It is thus necessary for stochastic compactness that $E \ln(1 + Y_1) < \infty$.

PROOF. Here we only prove $1 \Leftrightarrow 2$. The other implications follow from the results of the next section (take $K(x) = 1 - F(x)$).

$1 \Rightarrow 2$. If the sequence $\{X_n\}$ is stochastically compact, then by Lemma 2.2.1

$$\lim_{x\downarrow 0} \liminf_{n\to\infty} n(1 - F(a_n x)) = \infty.$$

Thus, in particular, there is an $x_1 > 0$ such that

$$\liminf_{n\to\infty} n(1 - F(a_n x_1)) > 0.$$

Also, according to Lemma 2.2.1, we have:

$$\lim_{x\to\infty}\limsup_{n\to\infty} n(1 - F(a_n x)) = 0.$$

Define for $t > 0$

$$n(t) = \min\{m \mid a_{m+1} > t\}$$

so that

$$a_{n(t)} \leqslant t < a_{n(t)+1}$$

and

$$\frac{1 - F\left(tx_1 \cdot \dfrac{x}{x_1}\right)}{1 - F(tx_1)} \leqslant \frac{[1 - F(a_{n(t)}x)]n(t)}{[1 - F(a_{n(t)+1}x_1)][n(t) + 1]} \cdot \frac{n(t) + 1}{n(t)}.$$

Because $n(t) \to \infty$ as $t \to \infty$ it follows that

$$\lim_{x\to\infty}\limsup_{t\to\infty}\frac{1 - F(tx)}{1 - F(t)} = 0.$$

$2 \Rightarrow 1$. Define for $n > 1$ the sequence $\{\alpha_n\}$ by:

$$n(1 - F(\alpha_n)) = 1.$$

This is possible because F is continuous. Then:

$$0 \leqslant \lim_{x\to\infty}\limsup_{n\to\infty} n(1 - F(\alpha_n x)) \leqslant \lim_{x\to\infty}\limsup_{t\to\infty}\frac{1 - F(tx)}{1 - F(t)} = 0.$$

Analogously $\lim_{x\downarrow 0}\liminf_{n\to\infty} n(1 - F(\alpha_n x)) = \infty$ follows from $\lim_{x\downarrow 0}\liminf_{t\to\infty} (1 - F(tx))/(1 - F(t)) = \infty$.

REMARK 2.2.3. The assumption that the df F is continuous plays an important role in this proof. It is possible, however, to replace this assumption by a slightly weaker one, namely the condition that there are an $x_1 > 0$ and an n_0 such that for $n \geqslant n_0$

(2.2.4) $$\liminf_{n\to\infty} n(1 - F(\beta_n x_1)) > 0$$

with

$$\beta_n = \inf\left\{x \mid 1 - F(x) \leqslant \frac{1}{n}\right\} \qquad n = 1, 2, \cdots.$$

Since in case the sequence $\{X_n\}$ is stochastically compact there are $x_1, x_2 > 0$ and an n_0 such that for $n \geqslant n_0$

$$n(1 - F(a_n x_1)) > 1$$
$$n(1 - F(a_n x_2)) \leqslant 1,$$

it follows that

$$x_1 < \frac{\beta_n}{a_n} \leqslant x_2.$$

Thus we can replace the sequence of constants $\{a_n\}$ from the definition of stochastic compactness by the sequence $\{\beta_n\}$. It is now not difficult to show that (2.2.4) together with (2) of Theorem 2.2.1 is equivalent to stochastic compactness for df's which are possibly not continuous.

3. The class of asymptotically decreasing functions. In this section we investigate the class of nonincreasing functions which satisfy assertion 3 of Theorem 2.2.1.

DEFINITION 3.1. Let K be a positive and nonincreasing function defined on $\mathbb{R}^+$. The function K is *asymptotically decreasing* if there are $x_0, t_0 > 0$ and $0 < M < 1$ such that

$$\frac{K(tx)}{K(t)} \leqslant M \quad \text{for} \quad x \geqslant x_0, t \geqslant t_0.$$

REMARK 3.1.
1. From this definition it follows that $\lim_{x\to\infty} K(x) = 0$.
2. If we define $K^*(x) = \sup\{K(y) \mid y > x\}$ it can easily be shown that K^* is right-continuous and that K is asymptotically decreasing if K^* is asymptotically decreasing. Thus we can suppose without loss of generality that K is right-continuous.
3. The sequence of sample maxima $\{X_n\}$ is stochastically compact iff $1 - F$ is asymptotically decreasing.
4. The above-defined concept is a one-sided generalisation of the concept of regular variation.

In the definition of R-O-variation (a two-sided generalisation of regular variation) two inequalities appear: A positive and measurable function K is said to be R-O-varying if there are $a > 1$, $t_0 > 0$, $m > 0$ and $M < \infty$ such that

$$m \leqslant \frac{K(tx)}{K(t)} \leqslant M \quad \text{for} \quad 1 \leqslant x \leqslant a \text{ and } t \geqslant t_0$$

(see, e.g., Seneta (1976)).

Another (different) generalisation of regular variation is the concept of dominated variation which is used by Feller (1965). A non-increasing, positive function K is dominatedly varying if there are $C, \rho, t_0 > 0$ such that

$$\frac{K(tx)}{K(t)} \geqslant Cx^{-\rho} \quad \text{for} \quad t \geqslant t_0, x \geqslant 1$$

(cf. assertion 3 of Theorem 3.1)

5. It should be noted that in Definition 3.1 we have assumed that the function K is nonincreasing. This assumption plays an important role in the derivation of the necessary and sufficient conditions for a function to be asymptotically decreasing. It is however possible to define a similar concept for measurable but not necessarily nonincreasing functions. In that case the necessary and sufficient conditions are different from those given in Theorem 3.1 (see Section 4).

The conditions are given by the following theorem.

THEOREM 3.1. *The following assertions are equivalent:*

(1) *The function K is asymptotically decreasing.*

(2) $$\lim_{x\to\infty}\limsup_{t\to\infty}\frac{K(tx)}{K(t)} = 0.$$

(3) $$\exists C, \rho, t_0 > 0 \text{ such that } \frac{K(tx)}{K(t)} \leqslant Cx^{-\rho} \text{ for } x \geqslant 1, t \geqslant t_0.$$

(4) $$\int_1^\infty \frac{K(s)}{s}\,ds < \infty \text{ and } \limsup_{x\to\infty}\frac{H(x)}{K(x)} < \infty \text{ with } H(x) = \int_x^\infty \frac{K(s)}{s}\,ds.$$

(5) $$\int_1^\infty \frac{K(s)}{s}\,ds < \infty \quad \text{and } H \text{ is asymptotically decreasing.}$$

PROOF. $1 \Rightarrow 3$ (cf. Feller (1965), page 385). If K is asymptotically decreasing, there are $0 < M < 1$, $x_0 > 1$ and $t_0 > 0$ such that

$$\frac{K(tx_0)}{K(t)} \leqslant M \text{ for } t \geqslant t_0.$$

Thus with induction

$$\frac{K(tx_0^n)}{K(t)} \leqslant M^n = x_0^{-n\rho} \text{ with } \rho = \frac{-\ln M}{\ln x_0} > 0.$$

Then for $x_0^{n-1} \leqslant x < x_0^n$ $(n = 1, 2, \cdots)$ and $t \geqslant t_0$

$$\frac{K(tx)}{K(t)} \leqslant \frac{K(tx_0^{n-1})}{K(t)} \leqslant x_0^{-(n-1)\rho} \leqslant Cx^{-\rho}$$

with $C = x_0^\rho$.

$3 \Rightarrow 2$ and $2 \Rightarrow 1$. The implications are trivial.

Next we prove that if K is asymptotically decreasing then $H(1) < \infty$. Assertion 3 of Theorem 3.1 implies that there is a $t_0 > 0$ and $x_1 > 1$ such that

$$\frac{K(tx)}{K(t)} \leqslant \left(\frac{x}{x_1}\right)^{-\rho} \text{ for } x \geqslant 1, t \geqslant t_0.$$

Let $A > \max\{x_1, t_0\}$ and define for $n = 2, 3, \cdots$

$$I_n = \int_{A^n}^{A^{n+1}} \frac{K(s)}{s}\,ds.$$

Then

$$I_n = \int_{A^{n-1}}^{A^n} \frac{K(sA)}{s}\,ds \leqslant \int_{A^{n-1}}^{A^n} \frac{K(s)}{s}\left(\frac{A}{x_1}\right)^{-\rho} ds = \left(\frac{A}{x_1}\right)^{-\rho} I_{n-1}.$$

Thus

$$I_n = \left(\frac{A}{x_1}\right)^{-(n-2)\rho} I_2$$

and hence $H(1) < \infty$.

We now prove the other implications.

3 ⇒ 4. For $t \geqslant t_0$

$$\frac{H(t)}{K(t)} = \int_1^\infty \frac{K(tx)}{K(t)}\frac{dx}{x} \leqslant C\int_1^\infty x^{-\rho-1}dx = \frac{C}{\rho} < \infty.$$

4 ⇒ 2. For $x \geqslant 1$ and $b > 1$

$$\infty > a \geqslant \frac{\int_x^\infty \frac{K(t)}{t}dt}{K(x)} = \frac{\int_1^\infty K(tx)\frac{dt}{t}}{K(x)} \geqslant \frac{\int_1^b K(bx)\frac{dt}{t}}{K(x)} = \frac{K(bx)}{K(x)} \text{ lnb}.$$

Hence

$$\frac{K(bx)}{K(x)} \leqslant \frac{a}{\ln b}.$$

4 ⇒ 5. Let $a(t) = \dfrac{K(t)}{H(t)}$. Then for $x \geqslant x_1 > 0$

$$\int_{x_1}^x \frac{a(t)}{t}\,dt = \int_{x_1}^x \frac{K(t)}{t\int_t^\infty \frac{K(s)}{s}\,ds}\,dt = -\ln H(x) + \ln H(x_1).$$

Thus

$$H(x) = \beta \exp\left(-\int_{x_1}^x \frac{a(t)}{t}\,dt\right) \text{ for } x \geqslant x_1 \text{ with } \beta = H(x_1).$$

Then for $x \geqslant 1$

$$\limsup_{t\to\infty} \frac{H(tx)}{H(t)} = \limsup_{t\to\infty} \exp\left(-\int_t^{tx}\frac{a(s)}{s}\,ds\right)$$

$$\leqslant \exp\left(-\int_1^x \frac{1}{s}\liminf_{t\to\infty} a(st)\,ds\right) \leqslant x^{-\eta}$$

because $\liminf_{s\to\infty} a(s) = \eta > 0$.

5 ⇒ 4. There is an $x_0 > 1$ such that

$$0 < \text{lin inf}_{t\to\infty} \frac{H(t) - H(tx_0)}{H(t)} = \liminf_{t\to\infty} \int_1^{x_0}\frac{K(ts)}{H(t)}\frac{ds}{s} \leqslant \liminf_{t\to\infty}\frac{K(t)}{H(t)}\int_1^{x_0}\frac{ds}{s}.$$

Hence (4) holds.

Assertion (5) states that the function H is asymptotically decreasing. Thus lim $\sup_{t\to\infty} H(tx)/H(t) \leqslant Cx^{-\rho}$ for $C, \rho > 0$. However, it is shown above that one can take $C = 1$. This is also true for functions which are possibly not integrals involving a monotone K-function, as stated in the following theorem (the proof is omitted).

THEOREM 3.2. *Let L be a nonincreasing function and let the derivative of L exist. The function L is asymptotically decreasing with* $C = 1$ i.e., $\exists t_0, \eta > 0$ *such that* $L(tx)/L(t) \leqslant x^{-\eta}$ *for* $t \geqslant t_0$, $x \geqslant 1$ iff $\lim\sup_{x\to\infty} L_*'(x) < 0$ *with* $L_*(x) = \ln L(e^x)$.

4. An alternative definition of stochastic compactness of sample maxima. Until now we have assumed that the common df F of the Y_n's which appear in the definition of the sequence of sample maxima $\{X_n\}$ is continuous. This requirement played an important role in the proof of Theorem 2.2.1. It is, however, possible to drop the continuity requirement. As a consequence we will have to exclude in the definition a class of limit distributions different from that excluded in Definition 2.1.2. We define:

DEFINITION 4.1. The sequence of sample maxima $\{X_n\}$ is *stochastically compact*[1] if there is a sequence of positive constants $\{a_n\}$ such that every convergent subsequence of $\{X_n/a_n\}$ converges weakly to a distribution with df G (depending on the subsequence) which satisfies the following requirements:

1. $G(\infty) = 1$
2. $G(x) < 1$ for all $x \in \mathbb{R}$ (cf. Remark 2.1.1).

The requirement that $G(\infty) = 1$ is equivalent to the assertion that $\lim_{x\to\infty} \liminf_{n\to\infty} F^n(a_n x) = 1$; the requirement that $G(x) < 1$ for all x is equivalent to $\limsup_{n\to\infty} F^n(a_n x) < 1$ for all $x > 0$. Necessary and sufficient conditions for the stochastic compactness[1] of the sequence $\{X_n\}$ are given by the following theorem.

THEOREM 4.1. *The following assertions are equivalent*:

(1) *The sequence of partial maxima* $\{X_n\}$ *is stochastically compact*[1].

(2)

$$\lim_{x\to\infty}\limsup_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = 0 \text{ and } \liminf_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} > 0 \text{ for all } x > 0.$$

(3)

$$\exists x_0 < x_1, t_0, 0 < m \leqslant M < 1 \text{ such that}$$

$$m \leqslant \frac{1 - F(tx)}{1 - F(t)} \leqslant M \text{ for } x_0 \leqslant x \leqslant x_1, t \geqslant t_0.$$

(4)

$$\exists t_0, C_1, C_2, \rho, \tau > 0 \text{ such that}$$

$$C_1 x^{-\tau} \leqslant \frac{1 - F(tx)}{1 - F(t)} \leqslant C_2 x^{-\rho} \text{ for } x \geqslant 1, t \geqslant t_0.$$

(5)
$$\int_1^\infty \frac{1-F(s)}{s}\,ds < \infty, \ \liminf_{x\to\infty} \frac{1-F(x)}{H(x)} > 0 \textit{ and } \limsup_{x\to\infty} \frac{1-F(x)}{H(x)} < \infty$$

$$\textit{with as before } H(x) = \int_x^\infty \frac{1-F(s)}{s}\,ds.$$

(6)
$$\int_1^\infty \frac{1-F(s)}{s}\,ds < \infty \textit{ and } \exists t_0, \eta, \theta > 0 \textit{ such that}$$

$$x^{-\theta} \leqslant \frac{H(tx)}{H(t)} \leqslant x^{-\eta} \textit{ for } t \geqslant t_0, x \geqslant 1.$$

PROOF. First we prove $4 \Rightarrow 1$. From assertion 4 it follows that

$$\frac{1-F(tx)}{1-F(t)} \leqslant \frac{1}{C_1} x^{-\tau} \text{ for } t \geqslant t_0/x, 0 < x \leqslant 1.$$

Define for $n = 2, 3, \cdots$

$$a_n = \inf\left\{x \,|\, 1 - F(x) \leqslant \frac{1}{n}\right\}.$$

From the right-continuity of F it follows that $1 - F(a_n) \leqslant 1/n$. We also have that $1 - F(a_n -) \geqslant \frac{1}{n}$.

Then for $0 < \varepsilon < \frac{1}{2}$

$$\frac{1-F(t(1-\varepsilon))}{1-F(t)} \leqslant \frac{1}{C_1}(1-\varepsilon)^{-\tau} \text{ for } t \geqslant 2t_0.$$

In particular:

$$1 - F(a_n(1-\varepsilon)) \leqslant \frac{1}{C_1}(1-\varepsilon)^{-\tau}(1-F(a_n)) \text{ for } n \geqslant n_0.$$

Letting $\varepsilon \downarrow 0$ we conclude that $1/n \geqslant 1 - F(a_n) \geqslant C_1/n$ for $n \geqslant n_0$. Now it can easily be shown (extend Lemma 2.2.1 for this case) that $\{X_n\}$ is stochastically compact[1].

For the implications $1 \Rightarrow 2$, $2 \Rightarrow 4$, $3 \Leftrightarrow 4$, $4 \Rightarrow 5$, $5 \Leftrightarrow 6$ the reader is referred to the proof of Theorem 3.1. No particular difficulties arise from the appearance of left-hand inequalities in the assertions.

Finally we prove $5, 6 \Rightarrow 2$. Assertion 6 gives $\lim_{x\to\infty}\limsup_{t\to\infty}(H(tx)/H(t)) = 0$. Using assertion 5 we find

$$0 \leqslant \lim_{x\to\infty}\limsup_{t\to\infty}\frac{1-F(tx)}{1-F(t)}$$
$$\leqslant \limsup_{t\to\infty}\frac{1-F(t)}{H(t)} \lim_{x\to\infty}\limsup_{t\to\infty}\frac{H(tx)}{H(t)} \limsup_{t\to\infty}\frac{H(t)}{1-F(t)} = 0.$$

Analogously one proves

$$\liminf_{t\to\infty}\frac{1-F(tx)}{1-F(t)} > 0 \text{ for all } x > 0.$$

REMARK 4.1.

1. With a similar argument as in Remark 2.2.2 one can show that, without loss of generality, the constants a_n in Definition 4.1 can be chosen as $a_n = \inf\{x | 1 - F(x) \leqslant 1/n\}$.
2. The sequence $\{X_n\}$ is stochastically compact[1] iff $1 - F$ is R-O-varying at infinity with constant $M < 1$ (cf. Remark 3.1).
3. The sequence $\{X_n\}$ is stochastically compact iff the inverse function of $1/(1 - F)$ is $R.O.$-varying at infinity. This is true even if in the definition of stochastic compactness we do not require the continuity of F.

5. Stochastic compactness and convergence. We have investigated the stochastic compactness of the sequence of partial maxima $\{X_n\}$. This sequence can however show a more precise limiting behaviour. It is for instance possible that there exists a sequence of positive constants $\{a_n\}$ such that $\{X_n/a_n\} \to_w C$. The weak law of large numbers gives necessary and sufficient conditions for this case (Gnedenko (1943)). Another possibility is that there is a positive sequence $\{a_n\}$ such that $\{X_n/a_n\} \to_w X$ (X nondegenerate). The conditions for the stochastic compactness, for the stochastic compactness[1], and for the weak convergence of the sequence $\{X_n\}$ are compared in the following scheme (remember $H(x) = \int_x^\infty 1 - F(s)/s\, ds$):

I. Equivalent are

(a) $\{X_n\}$ is stochastically compact (Definition 2.2.1).

(b) $\exists C, \rho, t_0 > 0$ such that

$$\frac{1 - F(tx)}{1 - F(t)} \leqslant Cx^{-\rho} \text{ for } x \geqslant 1, t \geqslant t_0.$$

(c) $\exists c > 0$ such that

$$\frac{1 - F(x)}{H(x)} \geqslant c \text{ for all } x > 0.$$

II. Equivalent are

(a) $\{X_n\}$ is stochastically compact[1] (Definition 4.1).

(b) $\exists C_1, C_2, \tau, \rho, t_0 > 0$ such that

$$C_1 x^{-\tau} \leqslant \frac{1 - F(tx)}{1 - F(t)} \leqslant C_2 x^{-\rho} \text{ for } t \geqslant t_0, x \geqslant 1.$$

(c) $\exists C_1 > 0, C_2 < \infty$ such that

$$C_2 \geqslant \frac{1 - F(x)}{H(x)} \geqslant C_1 \text{ for all } x > 0.$$

III. Equivalent are

(a) There are constants $\{a_n\}$ such that $\{X_n/a_n\} \to_w X$ with $P\{0 < X < \infty\} = 1$.

(b) $\exists 0 < \alpha \leqslant \infty$ such that

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha} \qquad \text{for } x > 0.$$

(c) $\lim_{x\to\infty} \dfrac{1 - F(x)}{H(x)} = \alpha \quad (0 < \alpha \leqslant \infty).$

Note that the case $\alpha = \infty$ corresponds to the weak law of large numbers.

A derivation of the necessary and sufficient conditions for weak convergence can be found in Gnedenko (1943), de Haan (1970).

6. Stochastic compactness of sample maxima and sums. In this section we investigate the relation between the stochastic compactness of partial maxima and partial sums. Let $\{Y_n\}$ be a sequence of i.i.d. random variables with common df F_0; suppose that the distribution is symmetric about zero. Define for $n = 1, 2, \cdots$

$$S_n = Y_1 + \cdots + Y_n.$$

In Feller (1965) the following definition of the stochastic compactness of the sequence $\{S_n\}$ is given.

DEFINITION 6.1. The sequence of sample sums $\{S_n\}$ is *stochastically compact* if there is a sequence of positive constants $\{a_n\}$ such that every subsequence of $\{S_n/a_n\}$ contains a further subsequence which weakly converges to a nondegenerate and nondefective distribution.

Necessary and sufficient conditions for the stochastic compactness of the sequence $\{S_n\}$ are given in the following theorem. Here F is the df of $|Y_1|$.

THEOREM 6.1 (Feller (1965)). *The following assertions are equivalent*:

(1) *The sequence of sample sums* $\{S_n\}$ *is stochastically compact.*

(2) $\exists 0 < \varepsilon \leqslant 2$ *such that*

$$\limsup_{t\to\infty} \frac{G(tx)}{G(t)} \leqslant x^{2-\varepsilon} \textit{ for } x \geqslant 1 \textit{ with } G(x) = \int_0^x s[1 - F(s)]\, ds.$$

(3)

$$\liminf_{x\to\infty} \frac{G(x)}{x^2[1 - F(x)]} > \tfrac{1}{2} \textit{ with } G(x) = \int_0^x s[1 - F(s)]\, ds.$$

REMARK. The formulation of the theorem is different in Feller's paper. Using partial integration one sees that (3) is equivalent to Feller's (9.1). The equivalence of (2) and (3) follows as in the proof of Theorem 2.2.1.

The relation between the stochastic compactness of partial maxima and partial sums is given by the following theorem.

THEOREM 6.2. *For a* df F *the following implication holds*:

$$\lim_{x\to\infty} \limsup_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = 0 \Rightarrow \liminf_{x\to\infty} \frac{G(x)}{x^2[1 - F(x)]} > \frac{1}{2}.$$

PROOF. It is given that

$$\limsup_{t\to\infty}\frac{1-F(tx)}{1-F(t)} = \varphi(x) \text{ with } \lim_{x\to\infty}\varphi(x) = 0.$$

Hence for $0 < x < 1$

$$\liminf_{t\to\infty}\frac{1-F(tx)}{1-F(t)} = \frac{1}{\varphi\left(\frac{1}{x}\right)} = k(x) \geqslant 1.$$

We then have:

$$\liminf_{x\to\infty}\frac{G(x)}{x^2[1-F(x)]} \geqslant \int_0^1 sk(s)\,ds.$$

Because $\lim_{s\downarrow 0}k(s) = \infty$ there is an $0 < s_0 \leqslant 1$ such that if $s \leqslant s_0$ then $k(s) \geqslant M > 1$. Hence:

$$\int_0^1 sk(s)\,ds = \int_0^{s_0} sk(s)\,ds + \int_{s_0}^1 sk(s)\,ds \geqslant \int_0^{s_0} sM\,ds + \int_{s_0}^1 s\,ds$$
$$= \tfrac{1}{2}s_0^2(M-1) + \tfrac{1}{2} > \tfrac{1}{2}.$$

Theorem 6.2 shows that the stochastic compactness or the stochastic compactness[1] of a sequence of sample maxima implies the stochastic compactness of the corresponding sequence of sample sums. The converse however is not true (see Example 7.3). Note that the convergence of the entire sequence $\{X_n/a_n\}$ is equivalent to the convergence of $\{S_n/a_n\}$ in the case of a nondegenerate limit distribution. The well-known technique of considering the partial maxima as functionals on the partial sums process, successful in the case of convergence of the sequence, cannot be used here.

REMARK 6.1. A similar relation between partial maxima and partial sums exists in the case of the weak law of large numbers (W.L.L.N.). If F is a df and $\{X_n\}$ the corresponding sequence of partial maxima and $\{S_n\}$ the corresponding sequence of partial sums, then

(1) W.L.L.N. for sample sums: There are constants $\{a_n\}$ such that $\lim_{n\to\infty}P(|(S_n/a_n) - 1| > \varepsilon) = 0$ for all $\varepsilon > 0$ iff

$$\lim_{x\to\infty}\frac{\int_0^x[1-F(s)]\,ds}{x[1-F(x)]} = 0$$

(Feller (1971)).

(2) W.L.L.N. for sample maxima: There are constants $\{a_n\}$ such that $\lim P_{n\to\infty}(|(X_n/a_n) - 1| > \varepsilon) = 0$ for all $\varepsilon > 0$ iff

$$\lim_{t\to\infty}\frac{1-F(tx)}{1-F(t)} = \infty \qquad 0 < x < 1$$
$$= 0 \qquad x > 1$$

(Gnedenko (1943)).

If the sequence $\{X_n\}$ satisfies the condition of the W.L.L.N. for partial maxima then

$$\lim_{x\to\infty}\frac{\int_0^x[1-F(s)]\,ds}{x[1-F(x)]} \geqslant \int_0^1 \liminf_{x\to\infty}\frac{1-F(sx)\,ds}{1-F(x)} = \infty.$$

We conclude that $\{S_n\}$ satisfies the condition of the W.L.L.N. for partial sums.

7. Examples and counterexamples. In this section we give some examples of df's for which the corresponding sequence of partial maxima is not stochastically compact. Moreover we show that the converse of Theorem 6.2 is not true. Finally we prove that the sequence of partial maxima from the geometric distribution is stochastically compact, though it is not weakly convergent.

EXAMPLE 7.1. Let

$$\begin{aligned} F(x) &= 0 & x &< e \\ &= 1 - \frac{1}{\ln x} & x &\geqslant e. \end{aligned}$$

Both the sequence of sample maxima and that of sample sums are not stochastically compact.

EXAMPLE 7.2. Let

$$\begin{aligned} F(x) &= 0 & x &< e \\ &= 1 - \exp^{-\alpha}(2^{\frac{1}{2}} + \sin \ln \ln x)\ln x & x &\geqslant e. \end{aligned}$$

Then for every sequence $\{t_k\}$ with $t_k \to \infty$ we have:

$$\lim_{k\to\infty}\frac{1-F(t_k x)}{1-F(t_k)} = \varphi(x) \Leftrightarrow \lim_{k\to\infty} L_*(s_k + y) - L_*(s_k) = \ln \varphi(e^y)$$

with $L_*(x) = \ln(1 - F(e^x))$, $y = \ln x$ and $s_k = \ln t_k$.

Because $s_k\{\sin(\ln s_k + \ln(1 + y/s_k)) - \sin \ln s_k\} - y \cos \ln s_k \to 0$ as $k \to \infty$ we have:

$$\lim_{k\to\infty} L_*(s_k + y) - L_*(s_k) = -\alpha 2^{\frac{1}{2}} x - \alpha x \lim_{k\to\infty}(\sin \ln(s_k + y) + \cos \ln s_k).$$

The limit points of $(1 - F(tx))/(1 - F(t))$ as $t \to \infty$ are thus given by

$$\varphi(x) = x^{-C} \text{ with } C \in [0, 2\alpha 2^{\frac{1}{2}}].$$

Hence:

$$\lim_{x\to\infty}\limsup_{t\to\infty}\frac{1-F(tx)}{1-F(t)} = 1.$$

EXAMPLE 7.3. The converse of Theorem 6.2 is not true. Consider namely Example 7.2 with $\alpha = 6$. Then the distribution has a finite variance and according to the central limit theorem the sequence $\{S_n/a_n\}$ converges to a normal distribution. The sequence of partial maxima is, however, not stochastically compact.

EXAMPLE 7.4 (Petersburg game). Let

$$\begin{aligned} F(x) &= 1 - e^{-[\log x]} \qquad & x \geqslant 0 \\ &= 0 & x < 0 \end{aligned}$$

It is easy to see that $\lim_{x\to\infty} \limsup_{t\to\infty}(1 - F(tx))/(1 - F(t)) = 0$.

Because F is not continuous, we must show that there are positive C_1, C_2 and a sequence $\{a_n\}$ such that

$$\frac{C_1}{n} \leqslant 1 - F(a_n) \leqslant \frac{C_2}{n} \text{ for } n \geqslant n_0.$$

With $a_n = n$ we have

$$\frac{1}{n} \leqslant 1 - F(a_n) < \frac{e}{n}.$$

Hence F satisfies the first compactness property.

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