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FURTHER RESULTS ON ONE-DIMENSIONAL DIFFUSIONS WITH TIME PARAMETER SET $(-\infty, \infty)$

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Let $p_t, t \geqslant 0$ be the probability transition semigroup of a continuous one-dimensional diffusion. We examine continuous Markov processes ξ_s, defined for $-\infty < s < \infty$, which are governed by p_t. It is shown that the class of such processes, modulo convex combinations and translations, can consist of at most three elements. In addition, it is shown that the first passage times for these processes are related to a previously known existence condition.

1. Notation and definitions. Let I be an interval of real numbers, finite or infinite, with endpoints $r_1 < r_2$, which may or may not belong I. $\mathfrak{B}$ will denote the Borel subsets of I, and C_b will be the set of real-valued, bounded, continuous functions defined on I. $p_t = p_t(x, B)(t \geqslant 0, x \in I, B \in \mathfrak{B})$ is the transition semigroup of a regular, one-dimensional diffusion with continuous sample paths as defined in [4]. X_t, $t \geqslant 0$ will denote the process and P_x, $x \in I$, will be the family of probability measures on the path space which satisfy $P_x[X_0 = x] = 1$ and $P_x[X_{t+s} \in B | X_u, 0 \leqslant u \leqslant s] = p_t(X_s, B)P_x -$ a.e. for $x \in I, s \geqslant 0, t \geqslant 0, B \in \mathfrak{B}$. $\tau_x = \inf\{t \geqslant 0 : X_t = x\}$, and E_x is the expectation operator corresponding to P_x.

As in [2], let Ω be the set of continuous functions from $\mathbb{R}$ to I, $\xi_s(\omega) = \omega_s$ for $s \in \mathbb{R}$, $\mathfrak{F}_s = \sigma[\xi_u, u \leqslant s]$, $\mathfrak{F} = \sigma[\xi_u, u \in \mathbb{R}]$. In addition define $\mathfrak{F}^s = \sigma[\xi_u, u \geqslant s]$. Let $\mathfrak{M} = \{\mathfrak{P} : \mathfrak{P}$ is a probability measure on $(\Omega, \mathfrak{F})$ and $\mathfrak{P}[\xi_{s+t} \in B | \mathfrak{F}_s] = p_t(\xi_s, B)\mathfrak{P} -$ a.e. for $s \in R, t \geqslant 0, B \in \mathfrak{B}\}$.

$\mathfrak{M}$ corresponds to the set of processes ξ_s which are defined for $-\infty < s < \infty$ and move according to p_t. As noted in [2] $\mathfrak{P}$ is actually determined by its marginals $\mathfrak{P}[\xi_s \in du]$ and the semigroup p_t. $\mathfrak{M}$ is convex and $\mathfrak{M}_e$ will denote the extreme points of $\mathfrak{M}$. $\mathfrak{P} \in \mathfrak{M}$ is said to be nontrivial if for some $s, \tilde{s} \in R, B \in \mathfrak{B}$, $\mathfrak{P}[\xi_s \in B] \neq \mathfrak{P}[\xi_{\tilde{s}} \in B]$; otherwise $\mathfrak{P}$ is trivial.

It will be convenient to introduce some notation due to Dynkin used in [3]. For each $s \in R$, $x \in I$ we define a probability measure $\mathfrak{P}^{s,x}$ on $(\Omega, \mathfrak{F}^s)$,

$$\mathfrak{P}^{s,x}\{\omega : \omega_{t_1} \in B_1, \cdots, \omega_{t_n} \in B_n\} = \int_{B_1} \cdots \int_{B_n} p_{t_1 - s}(x, dx_1) \cdots p_{t_n - t_{n-1}}(x_{n-1}, dx_n)$$

where $s \leqslant t_1 \leqslant \cdots \leqslant t_n$, $B_i \in \mathfrak{B}$. $\mathfrak{P}^{s,x}$ corresponds to a process started at time s, position x. Note that $\mathfrak{P}^{s,x}[\xi_{s+t} \in B] = P_x[X_t \in B]$ if $t \geqslant 0$.

In the proof of his integral representation theorem in [3], Dynkin used a martingale argument to show that if $\mathfrak{P} \in \mathfrak{M}_e$, $s \in R$, $A \in \mathfrak{F}^s$ for some s, then

$$\lim_{t \to -\infty} \mathfrak{P}^{t, \xi_t}(A) = \mathfrak{P}(A) \quad \mathfrak{P} - \text{a.e.} \tag{1.1}$$

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In general the exceptional null set depends on A. However, standard arguments show that there exists some $\Omega_0 \subset \Omega$, $\mathcal{P}(\Omega_0) = 1$, such that for all $\omega \in \Omega_0$, $s \in R$, $f \in C_b$

$$\lim_{t \to -\infty} \mathcal{E}^{t, \xi_t(\omega)}[f(\xi_s)] = \mathcal{E}[f(\xi_s)], \tag{1.2}$$

where $\mathcal{E}$ is the expectation operator corresponding to $\mathcal{P}$ and $\mathcal{E}^{s,x}$ corresponds to $\mathcal{P}^{s,x}$.

In [2] attention was focused on obtaining necessary and sufficient conditions for $\mathfrak{M}$ to contain nontrivial elements. In particular, Theorem 3.2 in [2] implies that if $\mathcal{P} \in \mathfrak{M}_e$ is nontrivial, then there exists a sequence $z_1, z_2, \cdots$ of elements of I, a sequence of reals $t_n \to -\infty$, and a family of probability measures $\{\Phi_x\}_{x \in I}$ on $\mathbb{R}$ such that

$$\begin{aligned} &\text{(a)} \quad \lim_{n\to\infty} z_n = r, \text{ where } r = r_1 \text{ or } r = r_2,\ r \notin I \\ &\text{(b)} \quad P_{z_n}[\tau_x + t_n \in du] \to_w \Phi_x(du) \\ &\text{(c)} \quad \mathcal{E}[f(\xi_s)] = \textstyle\int_{-\infty}^{s} \Phi_x(du) E_x[f(X_{s-u})] \end{aligned} \tag{1.3}$$

whenever $f \in C_b$ and $f(y) = 0$ for $y \geqslant x$, $r = r_2$; $y \leqslant x$, $r = r_1$. $\to_w$ denotes weak convergence of probability measures. We will refer to (1.3) frequently, and call it a representation for $\mathcal{P}$.

Our present goals are to identify the measures Φ_x above in terms of $\mathcal{P}$ and ξ_s and to understand how "big" $\mathfrak{M}$ can be. To do this we introduce a final definition.

For $s \in \mathbb{R}$ let $\theta_s : \Omega \to \Omega$ be the shift by s, $\theta_s\omega(t) = \omega(s + t)$, and $\theta_s A = \{\omega : \theta_{-s}\omega \in A\}$ for $A \in \mathcal{F}$. The shift induces a map from $\mathfrak{M}$ to $\mathfrak{M}$ given by $(\theta_s\mathcal{P})(A) = \mathcal{P}(\theta_{-s}A)$. Thus $(\theta_s\mathcal{P})[\xi_t \in B] = \mathcal{P}[\xi_{t+s} \in B]$. In addition, if μ is a probability measure on R, define $\theta_s\mu(\Gamma) = \mu(\Gamma + s)$, where $\Gamma + s = \{x : x - s \in \Gamma\}$.

2. The size of $\mathfrak{M}$. It is natural to ask just how large $\mathfrak{M}$ is if we rule out convex combinations and translations induced by the shifts θ_s. The answer is contained in:

(2.1) THEOREM. *$\mathfrak{M}_e$ contains at most three equivalence ($\sim$) classes.*

REMARK. Suitable modifications of the examples in [2] show that each possibility can occur. The three classes represent the trivial or stationary process, the process that "comes in" from r_2, and the one that "comes in" from r_1.

(2.2) LEMMA. *Assume $\mathcal{P} \in \mathfrak{M}_e$ has the representation* (1.3). *Then there is a function $c : I \to R$ such that $c(z_n) = t_n$, $c(z) \to -\infty$ as $z \to r$ (the "r" of* (1.3)), *and $P_z[\tau_x + c(z) \in du] \to_w \Phi_x(du)$ as $z \to r$ for all $x \in I$.*

PROOF. Suppose $r = r_2$. By taking a subsequence if necessary we may assume $z_n \uparrow r_2$. Now fix $x \in I$, eventually $z_n > x$. Since the time to get from z_n to x may be thought as the sum of the time to go from z_n to z_{n-1}, from z_{n-1} to z_{n-2}, etc., and by the strong Markov property these times are *independent*, the convergence in (1.3) suggests that the time to go from z_n to z_{n-1} becomes deterministic as $n \to \infty$. To make this precise, define the concentration functions $q_n(\varepsilon) = \sup_t P_{z_{n+1}}[t \leq \tau_{z_n} \leq t + \varepsilon]$. The convergence in (1.3) implies $q_n(\varepsilon) \to 1$ for all $\varepsilon > 0$. Using these concentration functions it is not difficult to "interpolate" between the z_n to produce a function $c(z)$, with $c(z_n) = t_n$, to satisfy (2.2). □

(2.3) LEMMA. *Assume* μ, μ_n, $n = 1, 2, \cdots$ *are probability measures on* $\mathbb{R}$, *and* $\mu_n \to_w \mu$ *as* $n \to \infty$. *Let* x_n *be any sequence of real numbers with* $\tilde{\mu}_n = \theta_{x_n}\mu_n$. *Assume* $\tilde{\mu}_n \to_w \tilde{\mu}$ *for some probability measure* $\tilde{\mu}$. *Then* $\lim_{n\to\infty} x_n = x$, *a finite number, and* $\tilde{\mu} = \theta_x \mu$.

PROOF. Elementary. □

PROOF OF THEOREM 2.1. Assume $\mathcal{P}$, $\mathcal{P}' \in \mathfrak{M}_e$ are nontrivial, represented as in (1.3) (with z_n', t_n', Φ_x' for $\mathcal{P}'$). If $r \neq r'$, then $\mathcal{P} \not\sim \mathcal{P}'$, which can be seen by letting $s \to -\infty$ in (1.3). Assume $r = r' = r_2$. We will show that for some $t \in R$, $\Phi_x = \theta_t \Phi_x'$, in which case (1.3) will show $\mathcal{P} = \theta_t \mathcal{P}'$.

Apply Lemma 2.2 to the unprimed representation for $\mathcal{P}$ to obtain $P_z[\tau_x + c(z) \in du] \to_w \Phi_x(du)$ as $z \to r_2$. Letting $\tilde{\mu}_n(du) = P_{z_n'}[\tau_x + c(z_n') \in du]$, this implies $\tilde{\mu}_n(du) \to_w \Phi_x(du)$ as $n \to \infty$. But setting $\mu_n(du) = P_{z_n'}[\tau_x + t_n' \in du]$, we have $\tilde{\mu}_n = \theta_{-t_n' + c(z_n')}\mu_n$ and $\mu_n \to_w \Phi_x'$ as $n \to \infty$. Lemma 2.3 completes the argument. The trivial $\mathcal{P}$, if it exists, forms its own equivalence class. □

3. Hitting times for ξ_s. If $\mathfrak{M}$ is nonempty it makes sense to investigate the properties of the process ξ_s. Interest centers on what happens as $s \to -\infty$. In particular, let $T_x = \inf\{t \in \mathbb{R} : \xi_t = x\}$ be the first hitting time of x for ξ_s, and ${}_aT_x = \inf\{t \geq a : \xi_t = x\}$. Our first result says that if $\mathcal{P} \in \mathfrak{M}_e$ is nontrivial, then T_x is $\mathcal{P}$ – a.e. finite.

(3.1) LEMMA. *Assume* $\mathcal{P} \in \mathfrak{M}_e$ *is nontrivial, with representation* (1.3). *Then for all* $x, y \in I$, $x > y$,

(a)
$$\lim_{t\to-\infty} \mathcal{P}[\xi_t \leq x] = 0, \quad r = r_2$$
$$= 1, \quad r = r_1,$$

(b)
$$\mathcal{P}[|T_x| < \infty] = 1,$$

(c)
$$\mathcal{P}[T_x < T_y] = 1, \quad r = r_2$$
$$= 0, \quad r = r_1.$$

PROOF. Throughout we assume the r in (1.3) is r_2. (a) From (1.3) we see that $\mathcal{E}[f(\xi_s)] \leq (\sup_{y\in I}|f(y)|) \cdot \Phi_x(-\infty, s])$. Let $s \to -\infty$. (b) To handle the possibility $T_x = +\infty$, note that $\{T_x = +\infty\} \subset \{\sup_s \xi_s \leq x\} \cup \{\inf_s \xi_s \geq x\}$. Part (a) above and the regularity of p_t imply each of these sets has $\mathcal{P}$-measure 0. The case $\{T_x = -\infty\}$ is slightly more involved. As noted in [2], since $\mathcal{P}$ is nontrivial, $p_t(x, dy) \to_v \pi(dy)$ as $t \to \infty$ independently of $x \in I$ where $\to_v$ denotes vague convergence and $\pi(I) = 0$ or $\pi(I) = 1$.

Suppose $\pi(I) = 1$, $-\infty < a < b < s$, and $K \subset I$ is a compact continuity interval for π. Then

$$\mathcal{P}[\xi_s \in K] \geq \mathcal{P}[\xi_s \in K, {}_aT_x \leq b]$$
$$= \int_a^b \mathcal{P}[{}_aT_x \in du] p_{s-u}(x, K)$$
$$\geq \mathcal{P}[{}_aT_x \leq b](\inf_{t \geq s-b} p_t(x, K)).$$

Let $a\downarrow -\infty$, $b\downarrow -\infty$ to obtain $\mathcal{P}[\xi_s \in K] \geq \pi(K) \cdot \mathcal{P}[T_x = -\infty]$. In view of (a) of (3.1) this is impossible unless $\mathcal{P}[T_x = -\infty] = 0$. If $\pi(I) = 0$, a similar argument completes the proof. (c) Using path continuity and the preceding parts of the lemma we can write $\mathcal{P}[T_x < T_y] = \lim_{s\to-\infty} \mathcal{P}[T_x < T_y, T_y > s, \xi_s > x] = \lim_{s\to-\infty} \mathcal{P}[T_y > s, \xi_s > x] = 1$. □

Using the strong Markov property and Lemma 3.1 it is possible to replace $\Phi_x(du)$ in part (c) of (1.3) with $\mathcal{P}[T_x \in du]$, suggesting $\mathcal{P}[T_x \in du] = \Phi_x(du)$. This is in fact true.

(3.2) THEOREM. *Assume* $\mathcal{P} \in \mathfrak{M}_e$ *is nontrivial with representation* (1.3). *Then*

(*a*) $\lim_{t\to-\infty} \xi_t = r \quad \mathcal{P}$ – a.e. (*the* "r" *of* (1.3))

(*b*) $P_{\xi_t}[\tau_x + t \in du] \to_w \Phi_x(du) \quad \mathcal{P}$ – a.e. *as* $t \to -\infty$, $x \in I$,

(*c*) $\mathcal{P}[T_x \in du] = \Phi_x(du)$ *for all* $x \in I$.

PROOF. Let Ω_0 be the set in (1.2). For fixed $\omega \in \Omega_0$, $x \in I$, the family of probability measures $P_{\xi_t(\omega)}[\tau_x + t \in du]$, $t \leq 0$, is tight, as shown in the proof of Theorem 3.2 in [2]. Consider any convergent subsequence, say $t_n^\omega \to -\infty$ and $P_{\xi_{t_n^\omega}(\omega)}[\tau_x + t_n^\omega \in du] \to_w \varphi_x^\omega(du)$. Lemma 3.3 in [2] shows, by taking a further subsequence if necessary, $\xi_{t_n^\omega}(\omega) \to r^\omega$ where $r^\omega = r_1$ or r_2. It is also a consequence of this lemma that there must be a family φ_y^ω, $y \in I$ of probability measures on R such that $P_{\xi_{t_n^\omega}(\omega)}[\tau_y + t_n^\omega \in du] \to_w \varphi_y^\omega(du)$. Hence the representation (1.3) is valid with φ_x^ω replacing Φ_x. Using (a) of (3.1) we see $r^\omega = r$ (the "r" of (1.3)). Now we will show $\varphi_x^\omega = \Phi_x$.

Let $t_n' = t_n^\omega$, $z_n' = \xi_{t_n'}(\omega)$, $r' = r^\omega = r$, $\Phi_x' = \varphi_x^\omega$, and $\mathcal{P}' = \mathcal{P}$. The argument used in the proof of Theorem 2.1 shows $\Phi_x' = \theta_t \Phi_x$ some $t \in R$. Using this fact in (1.3) and changing variables gives

$$\mathcal{E}[f(\xi_s)] = \int_{-\infty}^s \Phi_x(du) E_x[f(X_{s-u})]$$
$$= \int_{-\infty}^s \Phi_x'(du) E_x[f(X_{s-u})]$$
$$= \int_{-\infty}^{s-t} \Phi_x(du) E_x[f(X_{s+t-u})]$$
$$= \mathcal{E}[f(\xi_{s+t})].$$

Iterating gives $\mathcal{E}[f(\xi_o)] = \mathcal{E}[f(\mathcal{E}_{nt})]$ for all $n \in Z$. In view of (a) of (3.1), we must have $t = 0$. This means $\Phi_x' = \varphi_x^\omega = \Phi_x$. This holds for all $\omega \in \Omega_0$, independent of the choice of t_n^ω.

We have now shown that for all $\omega \in \Omega_0$, $x \in I$, $P_{\xi_t(\omega)}[\tau_x + t \in du] \to_w \Phi_x(du)$ and $\xi_t(\omega) \to r$ as $t \to -\infty$, where as usual Φ_x and r are as in (1.3). It remains to show $\Phi_x(du) = \mathcal{P}[T_x \in du]$.

Fix $x \in I$. Let D be a countable dense subset of R which contains no discontinuity point of Φ_x. Applying Dynkin's result (1.2) there exists $\Omega_1 \subset \Omega_0$, $\mathcal{P}(\Omega_1) = 1$, such that for all $a, b \in D$, $\omega \in \Omega_1$, $\lim_{t\to-\infty} \mathcal{P}^{t,\xi_t(\omega)}[{}_aT_x \leqslant b] = \mathcal{P}[{}_aT_x \leqslant b]$. Thus,

$$\begin{aligned}|\mathcal{P}[T_x \leqslant b] - \Phi_x((-\infty, b])| &\leqslant |\mathcal{P}[T_x \leqslant b] - \mathcal{P}[{}_aT_x \leqslant b]| \\ &\quad + |\mathcal{P}[{}_aT_x \leqslant b] - \mathcal{P}^{t,\xi_t(\omega)}[{}_aT_x \leqslant b]| + |\mathcal{P}^{t,\xi_t(\omega)}[{}_aT_x \leqslant b] \\ &\quad - \mathcal{P}^{t,\xi_t(\omega)}[{}_tT_x \leqslant b]| + |\mathcal{P}^{t,\xi_t(\omega)}[{}_tT_x \leqslant b] - \Phi_x((-\infty, b))|.\end{aligned}$$

Let $t\downarrow -\infty$, then $a\downarrow -\infty$ to obtain the desired conclusion. □

Remark. The Markov chain analog (discrete time and state space) of part (c) of (3.2) was established in [1] by F. Spitzer and H. Kesten using Fourier analytic random walk techniques.

Remark. The fact that $\xi_t \to r$ a.s. $\mathcal{P}$ as $t \to -\infty$ can also be proved using Lemma (3.1) and the fact that if $\mathcal{P} \in \mathfrak{M}_e$ is nontrivial, then $\{\lim_{t\to-\infty} \xi_t \to r\}$ has $\mathcal{P}$-measure 0 or 1 by Dynkin's theorem quoted in [2].

4. Boundary classification. Theorem 3.2 in [2] asserts that $\mathfrak{M}$ contains a nontrivial element if and only if (a) and (b) of (1.3) hold for some (and hence all) $x \in I$. This is really a *boundary condition*; it depends only on how the process behaves near r. For convenience, let us call r a *-boundary point if there exist z_n, t_n, $x \in I$, and Φ_x which satisfy (a) and (b) of (1.3).

Let m and S be the speed measure and scale function of p_t (see [4]). If r is a *-boundary point, then

$$|S(r)| = +\infty \tag{4.1}$$

and

$$\left|\int_x^r \int_y^r dm(u)\, dS(y)\right| = +\infty, \tag{4.2}$$

which indicates r is a *natural* boundary point in the usual terminology (see [5]). The validity of (4.1) was established in [2]. (4.2) holds because the integral is simply $\lim_{b\to r} E_b[\tau_x]$, and if this limit is finite, Φ_x cannot be a probability measure. However, not all natural boundary points are *-boundary points (see Example (a) of [2]).

A *sufficient* condition, also introduced in [2], for r to be a *-boundary point is that (4.2) and (4.3) hold, where

$$\left|\int_x^r \int_y^r \left(\int_z^r dm(u)\right)^2 dS(z)\, dS(y)\right| < \infty. \tag{4.3}$$

If $J \subset I$ is an interval with one endpoint in the interior of I and the other endpoint r, (4.3) implies $m(J) < \infty$. This is a severe restriction, yet we have no example for which this fails but r is a *-boundary point.

In view of these comments we pose two questions:

(4.4) Is there a semigroup p_t such that r is a *-boundary point and $m(J) = +\infty$ whenever J is an interval contained in I with r as one endpoint?

(4.5) What are necessary conditions on m and $\dot{S}$ near r for r to be a *-boundary point?

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