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LOCAL SAMPLE PATH PROPERTIES OF GAUSSIAN FIELDS

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A zero-one law is derived for a class of Gaussian fields $\{X(t) : t \in R^d\}$ including the generalized multiparameter Brownian motion. Under very general conditions, the joint distribution of the suprema of several Gaussian processes defined over compact metric spaces is shown to be absolutely continuous with a bounded density. Sufficient conditions are given for the existence of proper scaling limits of $\{X(t)\}$. The results are then combined to study local oscillations and local maxima.

1. Introduction. The work described in this paper started with our attempt to prove the result in Section 6 that for a broad class of continuous but nondifferentiable Gaussian fields the local maxima of the sample functions are countable and dense. The proof we found of this result is rather elementary but requires several short pieces of technical preparation. Each piece is of independent interest but they are quite unrelated and just barely fit together. Although as a result the paper lacks uniformity, we find the results interesting.

In Section 2 we consider fields $\{X(t) : t \in R^d\}$ with stationary increments and prove a new zero-one law at $t = 0$ for a fairly large class of such fields. Our result includes McKean's (1963, page 346) zero-one law for Lévy's multiparameter Brownian motion and extends the results of Tutubalin and Friedlin (1962). Examples and extensions of this zero-one law are also given.

In Section 3 we show under very general conditions that the joint distribution of the suprema of several Gaussian processes $\{X^j(t) : t \in T_j\} j = 1, \cdots, n$ is absolutely continuous and has a bounded density. Here the sets T_j are compact metric spaces. For $n = 1$ this is a slight improvement of Ylvisaker's (1968) result but for $n \geqslant 2$ it is, we think, completely new.

In Sections 4 and 5 we consider Gaussian fields $\{X(t) : t \in R^d\}$ with stationary increments and treat the existence of scaling limits of the form

$$c_n^{-1} X(A_n t) \to_{\mathcal{L}} Y(t) \qquad \text{as} \quad n \to \infty$$

where $c_n \to 0$ is a sequence of numbers and $\{A_n\}$ is a sequence of nonsingular $d \times d$ matrices with $\|A_n\| \to 0$. When combined with the zero-one law of Section 2 this gives the striking result (Theorem 4.1) on the local oscillations of Gaussian fields.

Finally, in Section 6 we apply these results to describe the set of local maxima for yet another "very general" class of Gaussian fields.

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2. Zero-one law. In this section, let $\{X(t) : t \in R^d\}$ be a mean zero real or complex valued random field with $X(0) = 0$. We assume that $X(t)$ has stationary increments and a continuous covariance function

$$R(t, s) = EX(t)X(s), \qquad t, s, \in R^d. \tag{2.1}$$

We treat $t = (t_1, \cdots, t_d) \in R^d$ as a column vector. Dot products $\Sigma_1^d t_j\lambda_j$ are written both as $t \cdot \lambda$ and $\langle t, \lambda\rangle$. It is known (see Yaglom (1957)) that $R(t, s)$ has a unique Fourier representation of the form

$$R(t, s) = \int_{R^d}(e^{it\cdot\lambda} - 1)(e^{-is\cdot\lambda} - 1)\Delta(d\lambda) + \langle t, Bs\rangle. \tag{2.2}$$

Here $B = (b_{ij})$ is a positive semidefinite matrix and $\Delta(d\lambda)$ is a nonnegative measure on $R^d - \{0\}$ satisfying

$$\int_{R^d}\frac{|\lambda|^2}{1 + |\lambda|^2}\Delta(d\lambda) < \infty. \tag{2.3}$$

For $T > 0$ let $\mathcal{H}(T)$ be the Hilbert space of random variables obtained by closing up the space of all finite combinations

$$X = a_1X(t_1) + \cdots + a_nX(t_n), \tag{2.4}$$

$|t_j| \leqslant T$ in the L^2 norm $\|X\| = (E|X|^2)^{\frac{1}{2}}$. Set

$$\mathcal{H}(0) = \cap \{\mathcal{H}(T) : T > 0\}. \tag{2.5}$$

We say that $X(t)$ satisfies the (wide sense) zero-one law at $t = 0$ iff $\mathcal{H}(0) = \{0\}$. If $\mathcal{F}(T)$ is the σ-field generated by $\mathcal{H}(T)$ and if $\{X(t) : t \in R^d\}$ is a Gaussian field then the wide sense zero-one law at $t = 0$ is equivalent to the zero-one law for the "tail" σ-field

$$\mathcal{F}(0) = \cap \{\mathcal{F}(T) : T > 0\}.$$

For $\lambda \in R^d$ and $h > 0$ we let

$$B(\lambda, h) = \{x \in R^d : |\lambda - x| \leqslant h\}$$

be the closed ball centered at λ of radius h.

THEOREM 2.1. *If for some* $h > 0$

$$\liminf_{|\lambda|\to\infty}|\lambda|^{d+2}\Delta(B(\lambda, h)) > 0 \tag{2.6}$$

then $X(t)$ *satisfies the zero-one law at* $t = 0$.

PROOF. First, we need some preparation. To each random variable

$$X = \Sigma_{j=1}^n a_jX(t_j) \tag{2.7}$$

we associate the function $f(\lambda) = AX$ given by

$$f(\lambda) = \Sigma_{j=1}^n a_j(e^{it_j\cdot\lambda} - 1). \tag{2.8}$$

It follows directly from (2.2) that

$$E|X|^2 = \int_{R^d}|f(\lambda)|^2\Delta(d\lambda) + \Sigma_{i,j=1}^d \overline{\frac{\partial f}{\partial\lambda_i}}(0)\frac{\partial f}{\partial\lambda_j}(0)b_{ij}. \tag{2.9}$$

Now bring in the weighted L^2 space $L^2(\Delta)$ with norm

$$\|f\|_\Delta = (\int |f(\lambda)|^2 \Delta(d\lambda))^{\frac{1}{2}} \tag{2.10}$$

and the subspaces $H_0(T)$ of $L^2(\Delta)$ of all functions of the form (2.8) with $|t_j| \leqslant T$ for all j, together with the closures $H(T)$ of $H_0(T)$ in $L^2(\Delta)$ and $H(0) = \cap \{H(T) : T > 0\}$.

The functions $f(\lambda)$ in $H_0(T)$ are all restrictions to R^d of entire analytic functions $f(z)$ of the d complex variables $z = (z_1, \cdots, z_d) \in \mathbb{C}^d$. Pitt (1975) has shown that under the condition (2.6) there is a $T_0 > 0$ so that whenever $0 < T \leqslant T_0$ the function

$$L(T, z) \equiv \sup \{|f(z)| : f \in H_0(T) \text{ and } \|f\|_\Delta \leqslant 1\} \tag{2.11}$$

is finite for all $z \in \mathbb{C}^d$ and satisfies an inequality of the form

$$L(T, z) \leqslant M(T)e^{(T+\varepsilon)|z|} \qquad \text{for each } \varepsilon > 0. \tag{2.12}$$

Let $L(T) = \sup \{L(T, z) : z \in \mathbb{C}^d \text{ and } |z| \leqslant 1\}$. Since each function $f(z)$ in $H_0(T)$ is analytic and satisfies $f(0) = 0$, it follows from Schwartz's lemma for analytic functions that $f(z)$ satisfies

$$|f(z)| \leqslant L(T)\|f\|_\Delta |z|, \qquad \text{for } |z| \leqslant 1. \tag{2.13}$$

From (2.13) we see that each f in $H_0(T)$ satisfies

$$\left|\frac{\partial f}{\partial \lambda_j}(0)\right| \leqslant L(T)\|f\|_\Delta; \qquad 1 \leqslant j \leqslant d. \tag{2.14}$$

Now let X and $f(\lambda) = AX$ be given by (2.7) and (2.8) with $|t_j| \leqslant T_0$ for all j. Since $B = (b_{ij})$ is positive semidefinite it follows from (2.9) and (2.14) that there exists a finite constant $c \geqslant 1$ with

$$\|AX\|_\Delta \leqslant \|X\| \leqslant c\|AX\|_\Delta. \tag{2.15}$$

Thus the map $X \to AX = f(\lambda)$ extends by continuity to a continuous invertible map of $\mathcal{H}(T)$ onto $H(T)$ for $T \leqslant T_0$ and $\mathcal{H}(0) = \{0\}$ iff $H(0) = \{0\}$. Theorem 2.1 is thus reduced to the following analytic result.

THEOREM 2.2. *If* (2.6) *is satisfied for some* $h > 0$ *then* $H(0) = \{0\}$.

PROOF. It follows from (2.12) (see Pitt (1975)) that each function $f(\lambda)$ in $H(0)$ is the restriction to R^d of an entire function $f(z)$, $(z \in \mathbb{C}^d)$ of minimal exponential type in the sense that $|f(z)| = 0(e^{\varepsilon|z|})$ as $|z| \to \infty$ for each $\varepsilon > 0$. Moreover from (2.13) we see that $f(0) = 0$.

The proof will be completed by showing:

$$\text{each } f(\lambda) \text{ in } H(0) \text{ is a polynomial} \tag{2.16}$$

and

$$\text{each polynomial in } L^2(\Delta) \text{ is a constant.} \tag{2.17}$$

To prove (2.16) we need a lemma from Pitt (1975), see also Lin (1965). Here

$\hat{\varphi}(\lambda) = \int e^{it\cdot\lambda}\varphi(t)\, dt$ is the Fourier transform and $L^2(B(0, T)) = \{\varphi(t) \in L^2(R^d, dt) : \varphi(t)$ vanishes off $B(0, T)\}$.

LEMMA 2.1 (Pitt (1975), page 306). *If $\hat{\Delta}(d\lambda)$ is a positive (not finite) measure on R^d and if for some $h > 0$,*

$$\liminf_{|\lambda|\to\infty} \hat{\Delta}(B(\lambda, h)) > 0, \tag{2.18}$$

then there exist $c > 0$ and $T > 0$ with

$$c\int_{R^d} |\hat{f}(\lambda)|^2 d\lambda \leqslant \int_{R^d} |\hat{f}(\lambda)|^2 \hat{\Delta}(d\lambda) \tag{2.19}$$

for all $f \in L^2(B(0, T))$.

Now set $\hat{\Delta}(d\lambda) = |\lambda|^{d+2}\Delta(d\lambda)$. Then $\hat{\Delta}(d\lambda)$ satisfies (2.18). Let $T > 0$ be a number for which (2.19) holds. We set $S = \frac{1}{2}T$ and choose $\varphi(t) \in L^2(B(0, S))$ such that $\hat{\varphi}(\lambda) > 0$ for all $\lambda \in R^d$ and such that

$$0 < \liminf_{|\lambda|\to\infty} \frac{\hat{\varphi}(\lambda)^2}{1 + |\lambda|^{d+2}} \leqslant \limsup_{|\lambda|\to\infty} \frac{\hat{\varphi}(\lambda)^2}{1 + |\lambda|^{d+2}} < \infty. \tag{2.20}$$

Note, for any $f(\lambda) = \Sigma a_j(e^{it_j\cdot\lambda} - 1) \in H_0(S)$ that $f(\lambda)\hat{\varphi}(\lambda) = \hat{\Psi}(\lambda)$ with $\Psi(s) = \Sigma a_j(\varphi(s - t_j) - \varphi(s))$ and $\Psi(s) \in L^2(B(0, T))$. By (2.20) and Lemma 2.1 we have

$$\begin{aligned} \int_{R^d} |f(\lambda)\hat{\varphi}(\lambda)|^2 d\lambda &\leqslant \text{const.} \int_{R^d} |f(\lambda)\hat{\varphi}(\lambda)|^2 \hat{\Delta}(d\lambda) \\ &\leqslant \text{const.} \int_{R^d} |f(\lambda)|^2 \Delta(d\lambda). \end{aligned} \tag{2.21}$$

Thus $\hat{\Psi}(\lambda) = f(\lambda)\hat{\varphi}(\lambda) \in L^2$ and $\|\hat{\Psi}\|_2 \leqslant \text{const.} \|f\|_\Delta$. Moreover, since Ψ is supported on $B(0, T)$ we have by Schwartz's inequality

$$\begin{aligned} |\hat{\Psi}(\lambda)| &\leqslant (\text{Vol.}\,(B(0, T)))^{\frac{1}{2}} \|\Psi\|_2 \\ &\leqslant \text{const.} \|f\|_\Delta. \end{aligned}$$

In particular, $\hat{\Psi}(\lambda)$ is bounded and by (2.20) we see that

$$|f(\lambda)| \leqslant \text{const.}\,(1 + |\lambda|^{(d/2)+1})\|f\|_\Delta. \tag{2.22}$$

Since (2.22) holds for all $f \in H_0(S)$, by continuity, it holds for all $f(\lambda) \in H(S)$. We conclude that each $f(\lambda)$ in $H(0)$ is an entire function of minimal exponential type with polynomial growth on R^d. It follows easily from the Paley-Wiener theorem (Stein and Weiss (1971), page 108) that $f(\lambda)$ must be a polynomial and thus (2.16) follows.

To prove (2.17) and thus complete the proof we simply observe that (2.20) and (2.21) imply that

$$\int_{|\lambda|>1} \frac{|f(\lambda)|^2}{|\lambda|^{d+2}} d\lambda < \infty$$

for each $f \in H(S)$. But elementary estimates show this is impossible if f is a nonconstant polynomial.

Examples and extensions. We conclude this section with several examples.

The first three are covered by Theorem 2.1. The fourth falls outside the domain of Theorem 2.1 but the theorem is easily modified to cover it. The last example satisfies the zero-one law at $t = 0$ but condition (2.6) fails so spectacularly that there is no hope of getting at it by the techniques of this section.

EXAMPLE 2.1. Let $\{X(t) : t \in R^d\}$ have the covariance

$$R(t, s) = \tfrac{1}{2}\{|t|^\alpha + |s|^\alpha - |t - s|^\alpha\}, \qquad \text{with } 0 < \alpha < 2. \tag{2.23}$$

The special case $\alpha = 1$ is the Lévy Brownian motion investigated by McKean (1963). It is elementary to check that $R(t, s)$ has the representation

$$R(t, s) = c(\alpha, d)\int_{R^d}(e^{it\cdot\lambda} - 1)(e^{-is\cdot\lambda} - 1)\frac{d\lambda}{|\lambda|^{d+\alpha}} \tag{2.24}$$

where $c(\alpha, d)$ is a constant. Condition (2.6) is trivially satisfied here and thus $X(t)$ satisfies the zero-one law at $t = 0$.

EXAMPLE 2.2. Let $\{Y(t) : t \in R^d\}$ be a stationary process with covariance function

$$EY(t)Y(s) = e^{c|t-s|^\alpha} \qquad \text{with} \quad c > 0 \quad \text{and} \quad 0 < \alpha < 2, \tag{2.25}$$

and let $X(t) = Y(t) - Y(0)$. Starting from (2.24) it is an elementary exercise in Fourier transforms to show that the covariance $R(t, s) = EX(t)X(s)$ has a Fourier representation of the form (2.2) with an absolutely continuous measure $\Delta(d\lambda) = \Delta(\lambda)d\lambda$ where $\Delta(\lambda)$ satisfies

$$\liminf_{|\lambda|\to\infty}|\lambda|^{d+\alpha}\Delta(\lambda) > 0. \tag{2.26}$$

Again (2.6) is trivially satisfied and $X(t)$ satisfies the zero-one law at $t = 0$.

EXAMPLE 2.3. Condition (2.6) does not require that $\Delta(d\lambda)$ be absolutely continuous. In fact, if $\Delta(d\lambda)$ is a discrete measure concentrated on the integer lattice of R^d with $\Delta(\{n\}) = \Delta_n$ then

$$R(t, s) = \int(e^{it\cdot\lambda} - 1)(e^{-is\cdot\lambda} - 1)\Delta(d\lambda)$$

is the covariance of a periodic field $X(t)$. Condition (2.6) holds for each $h > 1$ iff

$$\liminf_{|n|\to\infty}|n|^{d+2}\Delta_n > 0, \tag{2.27}$$

and under this condition $X(t)$ satisfies the zero-one law at $t = 0$.

EXAMPLE 2.4. Let $d = 2$ and $\Delta(\lambda) = \Pi_{j=1}^2(1/(1 + |\lambda_j|^3))$. If we set $\lambda_1 = \lambda_2 = x$ and $\lambda = (x, x)$, then letting $|x| \to \infty$ we see $|\lambda|^{d+2}\Delta(B(\lambda, h)) \approx \text{const. } 1/|\lambda|^2$ as $|x| \to \infty$. Condition (2.6) is thus violated. There is however an easy way out in this case.

The key observation here is that except for the very last step, the proof of Theorem 2.1 holds under the weaker condition:

For some $N > 0$ and some $h > 0$

$$\liminf_{|\lambda|\to\infty}|\lambda|^N\Delta(B(\lambda, h)) > 0. \tag{2.28}$$

With only notational changes in the proof of Theorem 2.1 it can be shown that condition (2.28) implies:

Each nonzero $f(\lambda) \in H(0)$ is a polynomial of degree less than $\frac{1}{2}(N - d)$ with $f(0) = 0$, and a given polynomial $f(\lambda)$ is in $H(0)$ iff $f \in L^2(\Delta)$.

Moreover,

$$\dim \mathcal{H}(0) = \dim H(0). \tag{2.29}$$

Thus we can state

THEOREM 2.3. *If condition* (2.28) *is satisfied, then* $H(0)$ *is the finite dimensional space of all polynomials* $f(\lambda) \in L^2(\Delta)$ *satisfying* $f(0) = 0$. *If in addition each polynomial* $f(\lambda) \in L^2(\Delta)$ *is a constant then* $X(t)$ *satisfies the zero-one law at* $t = 0$.

Theorem 2.3 is an extension of the result of Tutubalin and Friedlin (1962) and is easily seen to apply to Example 2.4. It is, however, far from being the complete story and the next example shows that there are processes for which the zero-one law is satisfied but for which no condition like (2.28) is satisfied.

EXAMPLE 2.5. Let $d = 1$ and

$$\Delta(\lambda) = \frac{1}{|\lambda|^2} \cos^2(\log |\lambda|). \tag{2.30}$$

One shows easily that

$$\int_{-\infty}^{\infty} \frac{|\log \Delta(\lambda)|}{1 + |\lambda|^2} d\lambda < \infty. \tag{2.31}$$

Starting with (2.31) it follows directly from a result of Levinson and McKean (1964, pages 113–114) that the subspace $H(0)$ of $L^2(\Delta)$ consists exactly of all entire functions $f(\lambda)$ in $L^2(\Delta)$ which are of minimal exponential type and that

$$M(z) = \sup \{|f(z)| : f \in H(0) \text{ and } \|f\|_\Delta \leqslant 1\}$$

is a finite continuous function of $z \in \mathbb{C}^1$.

We will now show for the $\Delta(\lambda)$ given by (2.30) that $M(z) \equiv 0$ and hence $H(0) = \{0\}$.

To see this note that for any $\alpha = e^{n\pi}$ with $n = 0, \pm 1, \pm 2, \cdots$; $\Delta(\alpha\lambda) = \alpha^{-2}\Delta(\lambda)$. Thus for any $f(\lambda) \in L^2(\Delta)$ we have $\|\alpha^{-\frac{1}{2}} f(\alpha\lambda)\|_\Delta = \|f(\lambda)\|_\Delta$. But if $f(\lambda)$ is of minimal exponential type so is $\alpha^{-\frac{1}{2}} f(\alpha\lambda)$ and hence $f(\lambda) \to \alpha^{-\frac{1}{2}} f(\alpha\lambda)$ is a unitary map of $H(0)$ onto itself. We conclude that

$$M(\lambda) = e^{n\pi/2} M(e^{-n\pi}\lambda) \qquad \text{for all} \quad \lambda \in R^1 \quad \text{and all } n. \tag{2.32}$$

But the argument leading to (2.13) shows that

$$M(\lambda) = 0(|\lambda|) \qquad as\ |\lambda| \downarrow 0. \tag{2.33}$$

Letting $n \to +\infty$ in (2.32) and using (2.33) we get $M(\lambda) \equiv 0$, and hence $H(0) = 0$.

3. Joint distribution of the suprema. We now consider the joint distribution of the suprema of several separable jointly Gaussian processes, $\{X^j(t) : t \in T_j\}, j = 1, \cdots, n$.

Our assumptions are:

(3.1) The sets T_j are compact metric spaces.

(3.2) The covariance matrices

$$\bar{R}(t_1, \cdots, t_n) = \mathrm{Cov}\left(X^1(t_1), \cdots, X^n(t_n)\right), \qquad t_j \in T_j$$

are nonsingular and jointly continuous in the t_j.

$$P\left\{\max_j \sup_{t \in T_j} |X^j(t)| < \infty\right\} = 1. \tag{3.3}$$

We set $S^j = \sup\{X^j(t) : t \in T_j\}$ for $1 \leqslant j \leqslant n$, and we prove the

THEOREM 3.1. *If conditions* (3.1), (3.2) *and* (3.3) *hold then* $(S^1, \cdots, S^n)$ *has an absolutely continuous joint distribution with a bounded density.*

PROOF. Let T be the Cartesian product $T_1 \times \cdots \times T_n$ and denote points in T by $\mathbf{t} = (t_1, \cdots, t_n)$. R^n denotes the space of n-dimensional real column vectors $\bar{x} = (x^1, \cdots, x^n)^*$. Here $*$ is the transpose. In R^n we will use the norm $|\bar{x}| = \max\{|x^1|, \cdots, |x^n|\}$ and for any $n \times n$ matrix $A = (a_{ij})$, we will use the corresponding matrix norm

$$\|A\| = \sup_i \Sigma_j |a_{ij}|.$$

Let I be the $n \times n$ identity matrix.

We introduce the Gaussian vector process

$$\bar{X}(\mathbf{t}) = \left(X^1(t_1), \cdots, X^n(t_n)\right); \; \mathbf{t} = (t_1, \cdots, t_n) \in T. \tag{3.4}$$

Let $\bar{R}(\mathbf{t}, \mathbf{s})$ be the covariance matrix function

$$\bar{R}(\mathbf{t}, \mathbf{s}) = \mathrm{Cov}\left(\bar{X}(\mathbf{t}), \bar{X}^*(\mathbf{s})\right) = \mathrm{Cov}\left(X^i(t_i) X^j(t_j)\right). \tag{3.5}$$

Note that (3.2) implies $\bar{R}(\mathbf{t}, \mathbf{s})$ is jointly continuous and that $\bar{R}(\mathbf{t}, \mathbf{t}) = \bar{R}(t_1, \cdots, t_n)$ is nonsingular.

As a reduction of the theorem we prove

LEMMA 3.1. *Without loss of generality it may be assumed that for all* $\mathbf{t}$ *and* $\mathbf{s}$ *in* T,

$$\|\bar{R}(\mathbf{s}, \mathbf{t}) \bar{R}^{-1}(\mathbf{t}, \mathbf{t}) - I\| \leqslant \tfrac{1}{2}. \tag{3.6}$$

PROOF. By the compactness of the T_j's and the continuity of $\bar{R}(t, s)$ we may cover each T_j with open sets $U_j^1, \cdots, U_j^N$ such that for each set U of the form $U = U_1^{i_1} \times \cdots \times U_n^{i_n}$ we have

$$\|\bar{R}(\mathbf{s}, \mathbf{t}) \bar{R}^{-1}(\mathbf{t}, \mathbf{t}) - I\| \leqslant \tfrac{1}{2} \qquad \text{for all } \mathbf{t} \text{ and } \mathbf{s} \text{ in } U. \tag{3.7}$$

Define the variables $S^j(U) = \sup\{X^j(\mathbf{t}) : \mathbf{t} \in U\}$ and note that if the theorem holds under the assumption (4.6) then $(S^1(U), \cdots, S^n(U))$ has a bounded density

$g_U(\bar{s})$. If we also note that for any set $B \subset R^n$ the inclusion $(S^1, \cdots, S^n) \in B$ implies $(S^1(U), \cdots, S^n(U)) \in B$ for some U, we may conclude that $(S^1, \cdots, S^n)$ has a density $g(\bar{s})$ which is bounded by $\Sigma_U g_U(\bar{s})$ and hence is bounded.

This is the only place where the continuity of $\bar{R}(\mathbf{t})$ and the topology of the T_j's is used. The separability of the $\{X^j(t) : t \in T_j\}$ is now used to guarantee the existence of a sequence $\{\mathbf{t}_k; k \geqslant 1\}$ in T with $S_k^j \equiv \max \{X^j(\mathbf{t}_i) = 1 \leqslant i \leqslant k\} \uparrow S^j$ a.e. The distribution of $(S_k^1, \cdots, S_k^n)$ then converges weakly to the distribution of $(S^1, \cdots, S^n)$ and the theorem will follow directly from

PROPOSITION 3.1. *Let $\bar{X}(\mathbf{t}) = (X^1(\mathbf{t}) \cdots X^n(\mathbf{t}))^* : \mathbf{t} \in T\}$ be an n-variate Gaussian process defined on some nonempty finite set $T = \{\mathbf{t}_1, \cdots, \mathbf{t}_k\}$. If $\bar{R}(\mathbf{t}, \mathbf{t})$ is nonsingular for all $\mathbf{t} \in T$ and if condition* (3.6) *holds, then the variables $S^j = \max \{X^j(\mathbf{t}) : \mathbf{t} \in T\}$ have an absolutely continuous distribution with a bounded density $g(\bar{s})$ satisfying*

$$|g(\bar{s})| \leqslant \left(\frac{2}{\pi}\right)^{n/2} \left(\det \bar{R}(\mathbf{t}_1, \mathbf{t}_1)\right)^{-\frac{1}{2}}. \tag{3.8}$$

PROOF. Introduce the matrix function

$$C(\mathbf{t}) = (c_{ij}(\mathbf{t})) \equiv R(\mathbf{t}, \mathbf{t}_1)R^{-1}(\mathbf{t}_1, \mathbf{t}_1) \text{ and set } \bar{X} = \bar{X}(\mathbf{t}_1).$$

We define the process $\bar{Y}(\mathbf{t}) = X(\mathbf{t}) - C(\mathbf{t})\bar{X}$ and write

$$\bar{X}(\mathbf{t}) = C(\mathbf{t})\bar{X} + \bar{Y}(\mathbf{t}).$$

One can check directly that Cov $(\bar{X}, \bar{Y}(\mathbf{t})) = 0$ for all $\mathbf{t} \in T$. Since $\{\bar{X}(\mathbf{t})\}$ is Gaussian we may conclude that $\bar{X}$ is independent of the process $\{\bar{Y}(\mathbf{t})\}$.

Setting $\bar{S} = (S^1, \cdots, S^n)$ we see that for any Borel set $B \subset R^n$ the conditional probability that $\bar{S} \in B$ given the σ-field $\mathfrak{F}(\bar{Y})$ generated by the process $\{\bar{Y}(\mathbf{t})\}$ is

$$P\{\bar{S} \in B | \bar{Y}\} = P\{\bar{X} \in \bar{F}^{-1}(B)\} \tag{3.9}$$

where $\bar{F}(\mathbf{x})$ is the random vextor function with components

$$F^i(\bar{x}) = \max \{\Sigma_{j=1}^n c_{ij}(\mathbf{t})x^j + Y^i(\mathbf{t}) : \mathbf{t} \in T\}. \tag{3.10}$$

But (3.6) shows that $\Sigma_j |c_{ij}(\mathbf{t}) - \delta_{ij}| \leqslant \frac{1}{2}$ for all $i = 1, \cdots, n$ and $\mathbf{t} \in T$. From this and (3.10) we see that $\bar{F}(\bar{x})$ has the form $\bar{F}(\bar{x}) = \bar{x} + \bar{\Psi}(\bar{x})$ where $\bar{\Psi}(\cdot)$ satisfies the contraction condition $|\bar{\Psi}(\bar{x}) - \bar{\Psi}(\bar{y})| \leqslant \frac{1}{2}|\bar{x} - \bar{y}|$ for $\bar{x}$ and $\bar{y}$ in R^n. Applying the most elementary inverse function theorem (see, e.g., Schwartz (1969), page 14), we conclude that $\bar{F}(\bar{x})$ maps R^n one-one and onto R^n and that the inverse map $\bar{J}(\bar{s}) = \bar{F}^{-1}(\bar{s})$ satisfies the Lipschitz condition

$$|\bar{J}(\bar{s}_1) - \bar{J}(\bar{s}_2)| \leqslant 2|\bar{s}_1 - \bar{s}_2|, \bar{s}_1 \text{ and } \bar{s}_2 \text{ in } R^n. \tag{3.11}$$

The map $\bar{J}(\bar{s})$ is differentiable a.e. (in fact it is piecewise linear) and the Jacobi matrix $(\partial J^i/\partial s^j)$ satisfies

$$\Sigma_j \left|\frac{\partial J^i}{\partial s_j}\right| \leqslant 2.$$

Thus

$$\Sigma_j \left| \frac{\partial J^i}{\partial s_j} \right|^2 \leqslant 4$$

and by the Hadamard determinant inequality we see that

$$\Delta(\bar{s}) = \left| \det \left(\frac{\partial J^i}{\partial s^j} \right) \right| \leqslant 2^n \quad \text{a.e.} \tag{3.12}$$

Combining this with (3.9) we see that when conditioned on $\mathcal{F}(\bar{Y})$, $\bar{S}$ has the conditional density

$$g(\bar{s}|\bar{Y}) = w(\bar{J}(\bar{s}))\Delta(\bar{s}),$$

where $w(\bar{x})$ is the density of $\bar{X}$. Since $\bar{X}$ is normal with covariance matrix $\bar{R}(\mathbf{t}_1, \mathbf{t}_1))$ we have

$$w(\bar{x}) \leqslant \left((2\pi)^n \bar{R}(\mathbf{t}_1, \mathbf{t}_1)\right)^{-\frac{1}{2}}.$$

This and (3.12) give the inequality

$$g(\bar{s}, \bar{Y}) \leqslant \left(\frac{2}{\pi}\right)^n \left(\bar{R}(\mathbf{t}_1, \mathbf{t}_1)\right)^{-\frac{1}{2}}.$$

Thus $\bar{S}$ has the density $g(\bar{s}) = Eg(\bar{s}|\bar{Y})$ which satisfies (3.8).

REMARK. The topological assumptions in the theorem can be removed by assuming the sets T_j are countable and equiping T_j with the metric $d_j(t, s) = (\mathrm{Var}\,(X^j(t) - X^j(s)))^{\frac{1}{2}}$, and then completing T_j in this metric. It is easy to show that condition (3.3) implies the completed spaces $\bar{T}_j$ are compact and we may easily deduce

PROPOSITION 3.2. *If the sets* $\{T_j, j = 1, \cdots, n\}$ *are countable and if condition* (3.3) *holds then the condition*

$$\inf\left\{|\det(\bar{R}(\mathbf{t}))|; \mathbf{t} \in T_1 \times \cdots \times T_n\right\} > 0 \tag{3.13}$$

implies that $\bar{S}$ *has an absolutely continuous distribution with a bounded density.*

Condition (3.13) cannot be removed. In fact, if $\{B(t) : 0 < t \leqslant 1\}$ is the standard Brownian motion and $f(t) = 1 + 2\log_3(t)/\log_2(t)$ it can be shown using Kolmogorov's test (see Itô and McKean (1964), page 33) that

$$X(t) = B(t)/\left(2t \log_2(1/t) f(1/t)\right)^{\frac{1}{2}}$$

satisfies

$$P\{\sup\{X(t) : 0 < t \leqslant 1\} = 1\} > 0.$$

In conclusion we mention that the techniques used to prove Proposition 3.2 can be combined with the results of Itô and Nisio (1968) to show for any sequence $\{X_n\}$ of mean zero Gaussian variables with $M = \sup X_n < \infty$ a.e. that either M has a density $p(x)$ with $p(x) > 0$ for all x or there is a number $a \geqslant 0$ such that M

has an absolutely continuous distribution concentrated on (a, ∞) with the possible exception, illustrated with the above example, that $P\{M = a\} > 0$ can occur.

4. Scaling limits and oscillations of Gaussian fields. Let $\{X(t) : t \in R^d\}$ be a real mean zero Gaussian field with continuous sample functions, stationary increments and $X(0) = 0$. We will call $X(t)$ proper if for all sufficiently small $\varepsilon > 0$ and all real x,

$$P\{\sup_{|t|=\varepsilon} X(t) < -x\} > 0. \tag{4.1}$$

The process $X(t)$ is proper unless the spectral measure introduced in (2.2) is supported on some proper linear subspace of R^d. To see this we observe by Theorem 4 of Kallianpur (1971) that a sufficient condition for the properness of $X(t)$ is that for each small $\varepsilon > 0$ the reproducing kernel Hilbert space $\mathcal{K}(X)$ of $\{X(t)\}$ contains a function $\varphi(t)$ satisfying $0 < \varphi(t)$ for all t with $|t| = \varepsilon$.

Now set $\varphi(t) = \int (e^{it\cdot\lambda} - 1)f(\lambda)\Delta(d\lambda)$ where $f(\lambda) \in L^2(\Delta)$ is even and satisfies $f(\lambda) < 0$ for all λ. It is known that $\varphi(t) \in \mathcal{K}(X)$. Since $\Delta(d\lambda) \geqslant 0$ is even we have

$$\varphi(t) = \int (\cos t \cdot \lambda - 1)f(\lambda)\Delta(d\lambda) \geqslant 0 \tag{4.2}$$

with $\varphi(t) > 0$ unless $\cos t \cdot \lambda = 1$ for each λ in the support of $\Delta(d\lambda)$.

The set $V = \{t : \varphi(t) = 0\}$ is a closed additive subgroup of R^d. If $X(t)$ is not proper, $\varphi(t)$ will have arbitrarily small zeros, and the identity component V_0 of V will be a linear subspace of R^d with $V_0 \neq \{0\}$. It then follows from (4.2) and a continuity argument that Δ is supported on the subspace $V_0^{\perp} = \{\lambda : t \cdot \lambda = 0$ for all $t \in V_0\}$. Since $V_0 \neq \{0\}$, $V_0^{\perp} \subsetneq R^d$. The proof is complete.

We note that Lévy's Brownian motion and all the examples in Section 2 are proper.

It is also worth noting that when Δ is supported on some subspace $V \subsetneq R^d$ it follows from (2.2) that $R(s, t) = \sum_{i,j} s_i b_{ij} t_j$ for each s perpendicular to V. From this it is elementary to show that for s perpendicular to V we have $E|X(s)|^2 = E|X(-s)|^2 = -EX(s)X(-s)$. By the converse of Schwartz's inequality $X(-s) = -X(s)$ and the process $\{X(t)\}$ is not proper.

To summarize this discussion in a neat way we introduce the matrix $M_\Delta = \{m_{ij}\}$ of weighted moments

$$m_{ij} = \int_{R^d} (\lambda_i \lambda_j) \frac{|\lambda|^2}{1 + |\lambda|^4} \Delta(d\lambda).$$

The measure $\Delta(d\lambda)$ is concentrated on a proper subspace $V \subset R^d$ iff M_Δ is singular, or what is the same, iff $\det M_\Delta = 0$. Thus we can state

PROPOSITION 4.1. *A mean-zero Gaussian process $\{X(t) : t \in R^d\}$ with stationary increments, continuous sample paths and covariance function $R(t, s)$ given in (2.2) is proper iff $\det M_\Delta \neq 0$.*

We now come to the main idea in this section. This is a new regularity condition that $R(t, s)$ is assumed to satisfy and which we formulate in terms of the existence of certain proper scaling limits of $X(t)$. The utility of this condition is illustrated in

Theorem 4.1. Section 5 contains a general set of technical conditions for the existence of scaling limits, together with several examples.

Let $C = C(R^d)$ be the space of real valued continuous functions on R^d with the topology of uniform convergence on compact sets. For a fixed process $X(t)$ (or $Y(t)$) we write $\mu_X(d\omega)$ for the probability measure on C induced by $X(t)$, i.e., $\mu_X\{\omega : \omega(t_i) \leqslant a_i, 1 \leqslant i \leqslant n\} = P\{X(t_i) \leqslant a_i, 1 \leqslant i \leqslant n\}$.

We will say $Y(t)$ is a scaling limit of $X(t)$ if there exist sequences $c_n > 0$ of numbers and $\{A_n\}$ of invertible $d \times d$ matrices which satisfy

(4.3) $$c_n \to 0 \qquad \text{as } n \to \infty,$$

(4.4) $$\|A_n\| \to 0 \qquad \text{as } n \to \infty,$$

(4.5) $$\text{for } Y_n(t) = c_n^{-1} X(A_n t)$$
$$\mu_{Y_n} \to \mu_Y \text{ weakly on } C \text{ as } n \to \infty.$$

If $\{Y(t)\}$ is both proper and a scaling limit of $\{X(t)\}$ we say that $\{X(t)\}$ admits a proper scaling limit.

The next theorem, which concerns the local oscillations of $\{X(t)\}$, illustrates the point that the existence of proper scaling limits for $\{X(t)\}$ reflects a considerable amount of regularity in the local behavior of the sample functions of $\{X(t)\}$.

Let $\{X(t) : t \in R^d\}$ be a real Gaussian field with $X(0) = 0$, stationary increments and continuous sample functions. We say $\{X(t)\}$ *oscillates infinitely often near* $t = 0$ if there exists a sequence of nondegenerate ellipsoids $E_n \subset R^d$ which are centered at $t = 0$ and which shrink to 0 as $n \to \infty$ with

(4.6) $$P\{\inf X(t) : t \in E_n) > 0 \text{ for infinitely many } n\} = 1$$

(4.7) $$P\{(\sup X(t) : t \in E_n) < 0 \text{ for infinitely many } n\} = 1.$$

THEOREM 4.1. *If* $\{X(t) : t \in R^d\}$ *satisfies the zero-one law at* $t = 0$ *and if* $\{X(t)\}$ *has a proper scaling limit* $\{Y(t)\}$ *then* $X(t)$ *oscillates infinitely often near* $t = 0$.

PROOF. Since $Y(t)$ is proper we may find an $\varepsilon > 0$ with
$$P\{\sup_{|t|=\varepsilon} Y(t) < 0\} > 0.$$
Now let E_n denote the ellipsoid $\{s = A_n t : |t| = \varepsilon\}$. But for $Y_n(t) = c_n^{-1}X(A_n t)$ we know that μ_{Y_n} converges weakly to μ_Y on C.
Thus
$$\begin{aligned} P\{[\sup_{t\in E_n} X(t) < 0] \text{ infinitely often}\} &\geqslant \liminf P\{\sup_{|t|=\varepsilon} Y_n(t) < 0\} \\ &\geqslant P\{\sup_{|t|=\varepsilon} Y_n(t) < 0\} \\ &> 0. \end{aligned}$$
Since E_n shrinks to 0 and since $X(t)$ is assumed to satisfy the zero-one law at $t = 0$ we see that (4.7) holds. The proof of (4.6) is obtained by reversing the appropriate inequalities.

5. Existence of scaling limits. In this section we present general sufficient conditions for the existence of proper scaling limits. These conditions are stated in terms of the spectral measure $\Delta(d\lambda)$ appearing in (2.2). The methods are based on the work of Belayev (1960) and Garsia (1972). The end result, Theorem 5.1, is quite sharp but also is often very difficult to apply. At the end of this section we give two less general but easily applied criteria for $\{X(t)\}$ to admit a proper scaling limit.

Starting with a continuous mean zero Gaussian field $\{X(t)\}$ with covariance function $R(t, s)$ given by (2.2) and an invertible matrix A we observe that the covariance function of the process $X(At)$ is given by

$$R_A(t, s) = \int (e^{it\cdot\lambda} - 1)(e^{-is\cdot\lambda} - 1)\Delta_A(d\lambda) + \langle t, A^*BAs\rangle \tag{5.1}$$

where A^* denotes the transpose of A and Δ_A is the measure $\Delta_A\{E\} = \Delta\{(A^*)^{-1}E\}$.

In the remainder of this section $f(x) \geqslant 0$ will denote a fixed increasing function on $[0, \infty)$ satisfying the conditions:

(5.2)
(a) $F(x) = x$ for $0 \leqslant x \leqslant 1$ and $\lim_{x\uparrow\infty} f(x) = +\infty$.
(b) For all sufficiently large x, $[f(x)/x]$ is decreasing.
(c) $\int_1^\infty \frac{dx}{xf(x)(\log x)^{\frac{1}{2}}} < \infty$.

Any such function $f(x)$ will be called a dominating function.

Associated with the measure Δ_A and each dominating function $f(x)$ we associate the number

$$c^2(A) = \int_{R^d} f^2(|\lambda|)\Delta_A(d\lambda) + \operatorname{trace}(A^*BA), \tag{5.3}$$

and the $d \times d$ matrix with entries

$$m_{ij} = \int_{R^d} \frac{\lambda_i\lambda_j|\lambda|^2}{1 + |\lambda|^4}\Delta_A(d\lambda),$$

and determinant

$$D(A) = \det(m_{ij}). \tag{5.4}$$

Note that $c^2(A)$ may well be infinite but that condition (2.3) implies that the matrix (m_{ij}) is well defined. Also note that (m_{ij}) is positive definite so $D(A) \geqslant 0$. With the convention that $D(A)/[c^2(A)]^d = 0$ if $c(A) = +\infty$ we can state

THEOREM 5.1. *A sufficient condition that $\{X(t) : t \in R^d\}$ admits a proper scaling limit is that there exists a dominating function $f(x)$ for which*

$$\lim_{\varepsilon\downarrow 0}\sup_{\|A\|\leqslant\varepsilon}\left\{D(A)/\left[c^2(A)\right]^d\right\} = l > 0. \tag{5.5}$$

The proof will be broken into several pieces. First we let A_n be a sequence of matrices with $\|A_n\| \to 0$ and $l = \lim_{n\to\infty}\{D(A_n)/[c(A_n)]^d\} > 0$. Set $c_n^2 = c^2(A_n)$ and $Y_n(t) = c_n^{-1}X(A_nt)$. Denoting the covariance function of $Y_n(t)$ with

$$R_n(t, s) = c_n^{-2}R(A_nt, A_ns),$$

we claim the sequence $\{R_n(t, s)\}$ has a subsequence which converges uniformly on bounded sets. To see this, note that $Y_n(t)$ has stationary increments so

$$R_n(t, s) = \tfrac{1}{2}\{R_n(t, t) + R_n(s, s) - R_n(t - s, t - s)\}.$$

Setting $S_n^2(t) = R_n(t, t)$, it suffices by Ascoli's theorem to show $\{S_n^2(t)\}$ is locally uniformly bounded and equicontinuous. But $S_n^2(0) = 0$ and $|S_n(t) - S_n(s)| = |\|Y_n(t)\| - \|Y_n(s)\|| \leq S_n(t - s)$. Thus we only require a function $\psi^2(u) \geq 0$ on $[0, 1]$ with $\psi^2(0+) = 0$ and

$$|S_n^2(t)| \leq \psi^2(|t|) \text{ for all } n \text{ and } t \text{ with } |t| \leq 1. \tag{5.6}$$

To obtain the bound (5.6) we observe that (5.1) gives

$$S_n^2(t) = c_n^{-2}\{2\int(1 - \cos t \cdot \lambda)\Delta_{A_n}(d\lambda) + \langle t, A_n^* B A_n t\rangle. \tag{5.7}$$

Setting $\Delta_n(d\lambda) = c_n^{-2}\Delta_{A_n}(d\lambda)$ we observe that the definition of c_n^2 gives both

$$\|c_n^{-2}A_n^* B A_n\| \leq 1, \tag{5.8a}$$

$$\int f^2(|\lambda|)\Delta_n(d\lambda) \leq 1. \tag{5.8b}$$

The next lemma is a slight modification of a result of Belyaev (1960); see also Cramér and Leadbetter (1967), page 181.

Lemma 5.1. *Let $f(x)\uparrow$ satisfy condition* (a) *and* (b) *of* (5.2), *and suppose that $\boldsymbol{\Delta}(d\lambda)$ is an even measure satisfying*

$$\int f^2(|\lambda|)\boldsymbol{\Delta}(d\lambda) \leq 1.$$

Then for some constant c that is independent of $\boldsymbol{\Delta}$,

$$2\int(1 - \cos t \cdot \lambda)\Delta(d\lambda) \leq [c/f^2(1/|t|)], \qquad t \in R^d. \tag{5.9}$$

Substituting in $\Delta_n(d\lambda)$, (5.8) *and Belyaev's lemma give*

$$S_n^2(t) \leq \Psi^2(|t|) \equiv \frac{c}{f^2\left(\frac{1}{|t|}\right)} + |t|^2. \tag{5.10}$$

This gives the desired estimate (5.6).

By choosing a subsequence if necessary we may assume that $R_n(t, s)$ converges locally uniformly. It follows that the finite dimensional distributions of $\{Y_n(t)\}$ converge to those of some limiting process $\{Y(t)\}$. To show that the measures μ_{Y_n} converge weakly to the measure μ_Y on C we must check that

(5.11) the measures $\{\mu_{Y_n}\}$ are tight on C.

We will then show that

(5.12) $\rho(t, s) = \lim_{n\to\infty} R_n(t, s)$ is the covariance function of a proper process $Y(t)$.

The question of tightness follows directly from (5.10) and Theorem 1 of Garsia (1972). In fact, Garsia shows that if $p(u)$ is increasing on $[0, 1]$ with $p(0+) = 0$ and

$S_n^2(t-s) = E|Y_n(t) - Y_n(s)|^2$ satisfies

$$S_n^2(|t-s|) \leqslant p(u) \text{ for all } |t-s| \leqslant d^{\frac{1}{2}}u,$$

then for any cube I in R^d with edges of length 1 we have

(5.13)

$$|Y_n(t) - Y_n(s)| \leqslant 16\left\{(\log B_n)^{\frac{1}{2}} p(|t-s|) + (2d)^{\frac{1}{2}} \int_0^{|t-s|} (\log(1/u))^{\frac{1}{2}}\, dp(u)\right\},$$

where

$$B_n = \int_I \int_I \exp \tfrac{1}{4}\left\{\left(Y_n(t) - Y_n(s)/p\left(|s-t|/d^{\frac{1}{2}}\right)\right\}^2 dt ds.$$

By Fubini's theorem one checks that $EB_n \leqslant 4 \cdot 2^{\frac{1}{2}}$. It then follows from (5.13) that the $\{\mu_{Y_n}\}$ are tight on C provided only that

$$\int_0^1 (\log(1/u))^{\frac{1}{2}}\, dp(u) < \infty. \tag{5.14}$$

Integration by parts shows that (5.14) is equivalent to

$$\int_0^1 \frac{p(u)}{u(\log 1/u)^{\frac{1}{2}}}\, du < \infty. \tag{5.15}$$

By (5.10) we may take $p(u) = (\text{const.})/f(d^{\frac{1}{2}}/u)$, and a simple change of variable argument shows (5.15) is equivalent to (5.2c).

Turning now to the properness of $Y(t)$ we observe that condition (5.8b) enables us to choose a subsequence for which the measures $\Delta_n(d\lambda)$ converge weakly to some finite measure $\boldsymbol{\Delta}(d\lambda)$ on $R^d - \{0\}$ satisfying (2.3) and that for some matrix $\mathbf{B}$

$$\begin{aligned} \rho(t, s) &= \lim R_n(t, s) \\ &= \int (e^{it\cdot\lambda} - 1)(e^{-is\cdot\lambda} - 1)\boldsymbol{\Delta}(d\lambda) + \langle t, \mathbf{B}s \rangle. \end{aligned}$$

The matrix $M_{\boldsymbol{\Delta}}$ of Proposition 4.1 satisfies $M_{\boldsymbol{\Delta}} = \lim M_{\Delta_n}$. Thus det $M_{\boldsymbol{\Delta}} = \lim$ det M_{Δ}. But det $M_{\Delta_n} = D(A_n)/[c^2(A_n)]^d \to l \neq 0$. Hence det $M_{\boldsymbol{\Delta}} \neq 0$ and by Proposition 4.1 we see that $Y(t)$ is proper.

REMARK. The methods used to prove Theorem 5.1 can be used to prove tightness of Gaussian measures in a far broader context than the context of the theorem indicates. In particular, see Garsia's (1972) paper and also note that in the special case when $f^2(x) = [\log(1+x)]^{1+\varepsilon}$, (5.9) and (5.13) give Hunt's (1950) classical condition for the continuity of stationary Gaussian processes.

Theorem 5.1 and the arguments which lead to it can easily be modified to give elementary criteria for the existence of proper scaling limits. We now state two such.

Let $S^2(t) = R(t, t)$ be fixed and let A be a nonsingular matrix satisfying

$$\|A\| < 1. \tag{5.16}$$

Suppose also, $\Psi(t) > 0$ is a function on $R^d - \{0\}$ satisfying

$$\int_0^1 \frac{p(u)}{u\,(\log 1/u)^{\frac{1}{2}}}\,du < \infty \tag{5.17}$$

where $p(u) \equiv \sup\{\Psi(t) : |t| \leq d^{\frac{1}{2}}u\}$, and

$$\liminf_{|t|\to 0} \frac{\Psi(t)}{|t|} = +\infty. \tag{5.18}$$

CRITERION 1. *Suppose* (5.16), (5.17) *and* (5.18) *are satisfied and that there exist positive constants a, b and* $c < 1$ *with*

$$\Psi(At) = c\Psi(t) \qquad \text{for } t \in R^d - \{0\}, \tag{5.19}$$

and

$$a\Psi(t) \leq S(t) \leq b\Psi(t) \qquad \text{for } |t| \leq 1. \tag{5.20}$$

Then $X(t)$ *admits a proper scaling limit.*

PROOF. Let $Y_n(t) = c^{-n}X(A^n t)$. Then $S_n^2(t) = c^{-2n}S^2(A^n t)$. From (5.19) and (5.2) we deduce

$$a^2\Psi^2(t) \leq S_n^2(t) \leq b^2\Psi^2(t), \qquad |t| \leq 1. \tag{5.21}$$

From (5.13), (5.17) and (5.21) we conclude the measures $\{\mu_{Y_n}\}$ are tight. Some subsequence $\{\mu_{Y_n'}\}$ will then converge to a limiting measure $\{\mu_Y\}$ and $\mathbf{S}^2(t) = EY^2(t)$ will satisfy $a^2\Psi^2(t) \leq \mathbf{S}^2(t) \leq b^2\Psi^2(t)$, $|t| \leq 1$. From this and (5.18) we may conclude that $Y(t)$ is proper.

EXAMPLE 5.1. The examples (3.1) and (3.2) with covariances given by $S^2(t) = |t|^\alpha$; and $S^2(t) = 2(1 - e^{-|t|})^\alpha$ respectively both have as proper scaling limits the process $Y(t)$ with covariance

$$\tfrac{1}{2}\{|t|^\alpha + |s|^\alpha - |t - s|^\alpha\}, \qquad 0 < \alpha < 2.$$

An example that is not even close to radial occurs with $d = 2$ and $S^2(t) = |t|^{\alpha_1} + |t|^{\alpha_2}$ where $t = (t_1, t_2)$ and $0 < \alpha_1, \alpha_2 < 2$.

We now state a spectral version of Criterion 1 that is easily deduced from Theorem 5.1. We leave the proof to the reader and comment that it may be easily generalized using the methods of Section 2.

CRITERION 2. *Let* $\{X(t)\}$ *be proper and have covariance* $R(t, s)$ *given by* (2.2). *Suppose that there exists an invertible matrix A satisfying* (5.16) *and positive constants a and b such that for each Borel set* $E \subset R^d$, $a\Delta(E) \leq \Delta(A^{-1}E) \leq b\Delta(E)$. *Then* $\{X(t)\}$ *admits a proper scaling limit.*

COMMENT. Examples which exhibit nearly every imaginable type of scaling behavior can be constructed. This can be done using the methods to construct pathologies common to the theory of domains of partial attraction. See, e.g., Feller (1971), pages 554–558.

6. Local maxima.

DEFINITION 6.1. $X(\cdot, \omega)$ has a local maximum at s if there is an open set $0 \subset R^d$ with $s \in 0$ such that $X(t) \leqslant X(s)$ for $t \in 0$. This local maximum is called strict if $X(t) < X(s)$ for all $t \in 0$ with $t \neq s$.

THEOREM 6.1. *Let $\{X(t) : t \in R^d\}$ be a Gaussian field with stationary increments and continuous sample functions*:

(i) *if $\{X(t)\}$ satisfies the zero-one law at $t = 0$ and has a proper scaling limit $\{Y(t)\}$ then* a.s. *the set of local maxima is dense in R^d*;

(ii) *if in addition $\overline{R}(t, s)$ is nonsingular for $t \neq s$ then* a.s. *the set of local maxima is also countable and each local maximum is strict.*

PROOF. Part (i) is an immediate consequence of Theorem 4.1 and the stationary increment property of $X(t)$. We now prove part (ii).

Let S and T be two disjoint nondegenerate rectangles in R^d. Then Theorem 4.1 implies that

$$P\{(\sup X(t) : t \in S) = (\sup X(t) : t \in T)\} = 0.$$

The intersection of $\{(\sup X(t) : t \in S) = (\sup X(t) : t \in T)\}$ over all rectangles S, T the vertices of which are rational is a subset of the set of ω such that each local maximum is strict. Following the same line of argument as in Freedman (1971), page 38, it is easy to show that the set of local maxima is countable.

EXAMPLE 6.1. Let $\{X(t) : t \in R^d\}$ have the covariance $R(t, s) = \frac{1}{2}\{|t|^\alpha + |s|^\alpha - |t - s|^\alpha\}$ with $0 < \alpha < 2$. This process was discussed in Examples 2.1 and 5.1. It satisfies the conditions of Theorem 6.1.

REMARK. Recently Tran (1976) has shown that for the two-parameter Wiener process, a.s. the set of local maxima is dense and each local maxima is strict. However his method of argument was based mainly on the independent increment property of this process and does not extend to any process considered here.

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