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INFINITE DIVISIBILITY IN STOCHASTIC PROCESSES

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It is shown that infinite divisibility of random variables, such as first passage times in a stochastic process, is often connected with the existence of an imbedded terminating renewal process. The idea is used to prove that for a continuous time Markov chain with two, three or four states all first passage times are infinitely divisible but for more than four states there are first passage times which are not infinitely divisible.

1. Introduction. For a simple continuous time random walk, Feller (1966) showed that the first passage times are infinitely divisible. This was generalized by Miller (1967) to include discrete state Markov processes with skip-free transitions which include, for example, the general birth-death process. Steutel (1973) drew attention to the fact that distributions arising in queueing theory are in some cases infinitely divisible, e.g., waiting times, queue lengths and busy periods. There is no obvious structural property of these processes implying infinite divisibility of passage times which, by contrast, in the case of Brownian motion, for example, is a direct consequence of path continuity and independent increments.

In the present paper it is shown that in many of these cases infinite divisibility is connected with the existence of an imbedded terminating renewal process. The main work of the paper however is to remove the skip-free assumption and to examine whether the first passage times of general continuous time Markov chains are infinitely divisible. The result is given in Theorem 5.2. In a concluding section it is shown that the result for continuous time chains does not in general carry over to discrete time.

2. Terminology and notation. All Markov processes considered will be homogeneous, i.e., the transition probabilities will be assumed to be independent of time.

A Markov process with a countable set of states will be called a Markov chain in continuous time or discrete time, whichever is appropriate.

If $F(x)$ $(x \geqslant 0)$ is a nondecreasing function, then the Laplace-Stieltjes transform

$$f(s) = \int_0^\infty e^{-sx} dF(x)$$

will be referred to simply as the Laplace transform of F and denoted by the corresponding lower case letter. If F is the distribution function (possibly defective) of a random variable X then f will also be referred to as the Laplace transform of X. Also $f(0)$ will always mean $f(0+)$.

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3. The imbedded terminating renewal process. The terminating renewal process is considered by Feller (1971, page 374). Intervals between renewals are nonnegative random variables $X_1, X_2, \cdots$ which are mutually independent with common defective distribution function $G(x)$ where $G(0+) = 0$, $G(\infty) < 1$. The interpretation of the defect $1 - G(\infty)$ is that at each renewal epoch there is a probability $1 - G(\infty)$ that the process terminates. Termination occurs at the nth renewal ($n = 0, 1, \cdots$) with the geometric probability $\{1 - G(\infty)\}\{G(\infty)\}^n$, and the total duration M of the process is finite with probability one and has Laplace transform

$$\frac{1 - G(\infty)}{1 - g(s)} = \frac{1 - g(0)}{1 - g(s)}, \tag{3.1}$$

where $g(s)$ is the Laplace transform of G. Note that M can be zero, i.e., the process may terminate at time zero, with positive probability, namely $1 - G(\infty)$. The Laplace transform (3.1) is that of a compound geometric distribution and is thus infinitely divisible. In this way, if a stochastic process can be shown to have an imbedded terminating renewal process, an infinitely divisible distribution will arise. Some examples follow.

(a) The already known fact that the maximum of a random walk (and the limiting waiting time in the queue $GI/G/1$) is infinitely divisible follows from the above discussion. See Feller (1971) and Spitzer (1956, Theorem 4.1).

(b) *Regenerative events* (Kingman (1964)). A standard, stable regenerative event $\mathcal{E}$ can be regarded as being produced by an alternating renewal process in which the renewal intervals X_{2n-1} ($n = 1, 2, \cdots$) have a common exponential distribution qe^{-qt} ($0 < q < \infty$) whilst the intervals X_{2n} ($n = 1, 2, \cdots$) have an arbitrary, possibly defective distribution $G(x)$ ($G(0+) = 0$). The regenerative event $\mathcal{E}$ is thought of as occurring throughout the currency of the intervals X_{2n-1}, $n = 1, 2, \cdots$. If $g(0) = G(\infty) < 1$ then $\mathcal{E}$ will be transient. The intervals $(X_2 + X_3), (X_4 + X_5), \cdots$ will then form a terminating renewal process of total duration M whose Laplace transform is the compound geometric form

$$f(s) = \frac{1 - g(0)}{1 - \frac{q}{q + s} g(s)}. \tag{3.2}$$

Thus M is infinitely divisible. Now X_1 and M are independent and $X_1 + M$ is the last epoch of occurrence of $\mathcal{E}$. Since X_1 is infinitely divisible, so is $X_1 + M$. The intervals $X_1, (X_2 + X_3), \cdots, (X_{2n} + X_{2n+1}), \cdots$, may be considered to generate a *delayed* terminating renewal process whose total duration $X_1 + M$ has Laplace transform $qf(s)/(q + s)$. Note that the delayed process cannot in this example terminate at time 0.

(c) *Markov chains in continuous time.* Consider for the sake of simplicity an irreducible recurrent chain with states labelled by the positive integers. States 1 and

2 may be considered as an arbitrary pair of states without loss of generality and let $f_{12}(s)$ denote the Laplace transform of the first passage time from state 1 to state 2. In the notation of example (b) above let $X_1, X_3, X_5, \cdots$ represent the successive intervals during which state 1 is occupied and let $X_2, X_4, \cdots$ represent successive away-times from state 1 but avoiding state 2; the X_{2n} $(n = 1, 2, \cdots)$ thus have a defective distribution $G(x)$. As in example (b), $X_1 + M$ is the duration of a delayed terminating renewal process and represents the last epoch of occupancy of state 1 with state 2 having been a taboo state. Let $K(x)$ be the (defective) distribution function of the first passage time from state 1 to state 2 without a return to state 1. Since the chain is recurrent an exit from state 1 is followed either by an eventual passage to state 2 without returning to state 1 or an eventual return to state 1 while having avoided state 2. Thus,

$$K(\infty) + G(\infty) = k(0) + g(0) = 1, \tag{3.3}$$

and

$$f_{12}(s) = \frac{q}{q+s}k(s) + \frac{q}{q+s}g(s)f_{12}(s), \tag{3.4}$$

whence

$$f_{12}(s) = \frac{q}{q+s}\cdot\frac{k(s)}{1 - \dfrac{q}{q+s}g(s)} \tag{3.5}$$

$$= \frac{q}{q+s}\cdot\frac{k(s)}{k(0)}\cdot\frac{1-g(0)}{1 - \dfrac{q}{q+s}g(s)}, \tag{3.6}$$

$$g(0) < 1, k(0) + g(0) = 1.$$

The representation (3.6) shows that the first passage time from state 1 to state 2 in a recurrent chain may be regarded as the sum of three independent random variables:

(i) a delay X_1 represented by $q/(q + s)$; this is the initial sojourn in state 1;

(ii) a period during which recurrences of state 1, which may number $0, 1, 2, \cdots$, alternate with periods away from state 1 avoiding state 2; this is the duration M of a terminating renewal process and is represented by the last factor in (3.6); and

(iii) a final transit U to state 2 represented by the middle factor in (3.6).

The random variables X_1 and M are infinitely divisible but at this stage nothing can be said about U in this connection.

If the chain is irreducible but transient then $k(0) + g(0) < 1$ instead of (3.3), but the relations (3.4) and (3.5) still hold. However (3.6) now becomes

$$f_{12}(s) = \frac{q}{q+s}\frac{k(s)}{1-g(0)}\cdot\frac{1-g(0)}{1-\dfrac{q}{q+s}g(s)}, \qquad k(0) + g(0) < 1. \tag{3.7}$$

The random variable U, the final transit to state 2 and represented by the middle term in (3.7), is now defective, the defect $1 - k(0)\{1 - g(0)\}^{-1}$ representing the probability of absorption in the set of transient states $(3, 4, 5, \cdots)$ without prior entry to state 2 conditional on no return to state 1.

This example will be developed further in Section 5.

4. Some results of mixtures of convolutions of exponentials. The results in this section are of some interest in themselves and with the exception of Lemma 4.4 are required in the sequel. They concern the infinite divisibility of mixtures of convolutions of exponential distributions. Steutel (1967) proved the interesting result that mixtures of exponential distributions are infinitely divisible and extended this in (1973), Theorem 4.2″ to include some special cases of mixtures which permit negative weights. The results of this section, however, appear to be independent of those of Steutel.

A nonnegative random variable is infinitely divisible if and only if its Laplace transform can be written in the form $\exp\{-\psi(s)\}$ (Feller (1971), page 449), where $\psi(s)$ is of the form

$$\psi(s) = \int_0^\infty \frac{1 - e^{-sx}}{x} dP(x). \tag{4.1}$$

Here P is nondecreasing and $\int_1^\infty x^{-1} dP(x) < \infty$.

DEFINITION 4.1. A Laplace transform $f(s)$ will be called infinitely divisible if it is, apart from a positive constant multiple, the Laplace transform of a distribution function, i.e., if

$$f(s) = ce^{-\psi(s)}, \qquad c > 0,$$

where $\psi(s)$ is of the form (4.1).

In particular for the exponential density qe^{-qx} with Laplace transform $q/(q + s)$ the function P in (4.1) is given by $dP(x) = e^{-qx}dx$. The rational infinitely divisible Laplace transform $(q + s)^{-1}$ can therefore be expressed as

$$\frac{1}{q + s} = \frac{1}{q} \exp\left[\int_0^\infty \frac{1 - e^{-sx}}{x} e^{-qx} dx\right].$$

It follows that a rational function $f(s)$, whose zeros and singularities have negative real parts, can be written as

$$f(s) = \frac{(s + a_1) \cdots (s + a_m)}{(s + b_1) \cdots (s + b_n)}, \qquad \operatorname{Re}(a_i) > 0, i = 1, \cdots, m, \quad \operatorname{Re}(b_i) > 0, i = 1, \cdots, n \tag{4.2}$$

$$= c \exp\left[-\int_0^\infty \frac{1 - e^{-sx}}{x} \left(\sum_{i=1}^n e^{-b_i x} - \sum_{i=1}^m e^{-a_i x}\right) dx\right], \tag{4.3}$$

where $c = (a_1 \cdots a_m)/(b_1 \cdots b_n)$. The a_i and b_i are not necessarily distinct. The expression (4.3) is valid provided that $\text{Re}(s) > -\min_{i,j}\{\text{Re}(a_i), \text{Re}(b_j)\}$. A Laplace transform of the form (4.2) will be infinitely divisible if and only if

$$\Sigma_{i=1}^{n} e^{-b_i x} \geqslant \Sigma_{i=1}^{m} e^{-a_i x} \quad \text{for all} \quad x > 0. \tag{4.4}$$

LEMMA 4.1. *Let $a_1 \leqslant a_2$, $b_1 \leqslant b_2$ be positive. Then*

$$(e^{-b_1 x} + e^{-b_2 x}) - (e^{-a_1 x} + e^{-a_2 x}) \geqslant 0 \tag{4.5}$$

for all $x \geqslant 0$ if and only if $a_1 + a_2 \geqslant b_1 + b_2$ and $a_1 \geqslant b_1$.

PROOF. The necessity of the condition $a_1 \geqslant b_1$ can be seen if x is large. Thus assume $a_1 \geqslant b_1$ and consider the sufficiency part of the proof. If $a_2 \geqslant b_2$ then (4.5) is obvious. Suppose therefore that $a_2 < b_2$ and let $\lambda(x)$ denote the l.h.s. of (4.5). Then

$$e^{b_2 x}\lambda(x) = e^{(b_2 - a_1)x}\{e^{(a_1 - b_1)x} - 1\} - \{e^{(b_2 - a_2)x} - 1\}. \tag{4.6}$$

Now, if $a_1 + a_2 \geqslant b_1 + b_2$ then $a_1 - b_1 \geqslant b_2 - a_2 > 0$, and (4.6) implies that

$$\begin{aligned} e^{b_2 x}\lambda(x) &\geqslant e^{(b_2 - a_1)x}\{e^{(b_2 - a_2)x} - 1\} - \{e^{(b_2 - a_2)x} - 1\} \\ &= \{e^{(b_2 - a_1)x} - 1\}\{e^{(b_2 - a_2)x} - 1\} \\ &> 0 \end{aligned}$$

for all $x > 0$. Hence the condition $a_1 + a_2 \geqslant b_1 + b_2$ is sufficient for (4.5). It is clearly also necessary since $\lambda(0) = 0$ and $\lambda'(0) = (a_1 + a_2) - (b_1 + b_2)$; if $\lambda'(0) < 0$ then by continuity $\lambda(x) < 0$ in an interval to the right of $x = 0$.

LEMMA 4.2. *Let $a_1 \leqslant a_2$, $b_1 \leqslant b_2$ be positive. Then*

$$f(s) = \frac{(a_1 + s)(a_2 + s)}{(b_1 + s)(b_2 + s)}, \tag{4.7}$$

is an infinitely divisible Laplace transform if and only if $a_1 + a_2 \geqslant b_1 + b_2$ and $a_1 \geqslant b_1$.

PROOF. The result follows from Lemma 4.1 and the condition (4.4).

LEMMA 4.3. *Suppose $b_1 > 0$, $b_2 > 0$, a is complex and $\bar{a}$ is the complex conjugate of a. Then*

$$f(s) = \frac{(a + s)(\bar{a} + s)}{(b_1 + s)(b_2 + s)} \tag{4.8}$$

is an infinitely divisible Laplace transform if and only if $\text{Re}(a) \geqslant \frac{1}{2}(b_1 + b_2)$.

PROOF. Let $a = \alpha + i\eta$ where α and η are real and let

$$\lambda(x) = e^{-b_1 x} + e^{-b_2 x} - e^{ax} - e^{\bar{a}x} = e^{-b_1 x} - e^{-b_2 x} - 2e^{-\alpha x}\cos \eta x.$$

Now, $\lambda(x) \geqslant 0$ if and only if $\alpha \geqslant \frac{1}{2}(b_1 + b_2)$ for by Lemma 1

$$\lambda(x) \geqslant e^{-b_1 x} + e^{-b_2 x} - 2e^{-\alpha x} \geqslant 0$$

if $2\alpha \geqslant b_1 + b_2$. Also $\lambda'(0) = 2\alpha - (b_1 + b_2)$ and $\lambda'(0) \geqslant 0$ is a necessary condition for $\lambda(x) \geqslant 0$ $(x > 0)$. Again the condition (4.4) implies the result.

LEMMA 4.4. *Suppose* $a_1 > 0, a_2 > 0, b_1 > 0, b_2 > 0, \alpha > 0, \beta > 0$ *and let*

$$f(s) = \frac{\alpha}{(a_1 + s)(a_2 + s)} + \frac{\beta}{(b_1 + s)(b_2 + s)}. \tag{4.9}$$

Then $f(s)$ *is infinitely divisible* (i.e., *a mixture of two convolutions each of two exponentials is infinitely divisible*).

PROOF. Suppose without loss of generality that $a_1 + a_2 \leqslant b_1 + b_2, a_1 \leqslant a_2, b_1 \leqslant b_2$. Let $a_1' = \min(a_1, b_1), b_1' = \max(a_1, b_1)$. Then

$$f(s) = (\alpha + \beta)\frac{(s_1 + s)(s_2 + s)}{(a_1' + s)(b_1' + s)(a_2 + s)(b_2 + s)}. \tag{4.10}$$

If s_1 and s_2 are real then $\min(s_1, s_2) \geqslant a_1'$ since the Laplace transform of a nondecreasing function can have no real zeros to the right of the abscissa of convergence. Further, for real or complex s_1, s_2

$$s_1 + s_2 = \frac{\alpha}{\alpha + \beta}(b_1 + b_2) + \frac{\beta}{\alpha + \beta}(a_1 + a_2).$$

Hence $s_1 + s_2$ is an average of $(a_1 + a_2)$ and $(b_1 + b_2)$ and so $s_1 + s_2 \geqslant a_1 + a_2 \geqslant a_1' + a_2$. It now follows from Lemma 4.2 or Lemma 4.3 that $(s_1 + s)(s_2 + s)/\{(a_1' + s)(a_2 + s)\}$ is infinitely divisible and so therefore is $f(s)$.

By letting in turn $a_2 \to \infty$, $a_1 \to \infty$ and $b_2 \to \infty$ in (4.9) the following is obtained.

COROLLARY. *Laplace transforms of the following forms*

$$\frac{\alpha}{a_1 + s} + \frac{\beta}{(b_1 + s)(b_2 + s)}, \tag{4.11}$$

$$\alpha + \frac{\beta}{(b_1 + s)(b_2 + s)}, \tag{4.12}$$

$$\alpha + \frac{\beta}{(b_1 + s)}, \tag{4.13}$$

are infinitely divisible provided $a_1 > 0, b_1 > 0, b_2 > 0, \alpha \geqslant 0, \beta \geqslant 0$.

It may be noted that the result (4.13) follows by a passage to the limit from Steutel's theorem on mixtures of exponentials.

LEMMA 4.5. *Let*

$$f(s) = \alpha + \beta\left[\frac{(a + s)}{(b_1 + s)(b_2 + s)}\right] \tag{4.14}$$

where $\alpha \geqslant 0, \beta > 0$ *and* a, b_1 *and* b_2 *are all positive. Then* $f(s)$ *is infinitely divisible provided that the function in square brackets is the Laplace transform of a bounded nondecreasing function*, i.e., *provided that* $a \geqslant \min(b_1, b_2)$.

PROOF. The function $f(s)$ is rational and provided $\alpha > 0$ may be written as

$$f(s) = \alpha \cdot \frac{(s + s_1)(s + s_2)}{(s + b_1)(s + b_2)} \tag{4.15}$$

where $-s_1$ and $-s_2$ are the zeros of the numerator and satisfy

$$s_1 + s_2 = b_1 + b_2 + \beta\alpha^{-1} \geqslant b_1 + b_2. \tag{4.16}$$

If the function in square brackets in (4.14) is the Laplace transform of bounded nondecreasing function then so also is $f(s)$. This means that if s_1 and s_2 are real then $\min(s_1, s_2) \geqslant \min(b_1, b_2)$, which, together with (4.16) and Lemma 4.2, implies that $f(s)$ is infinitely divisible. If s_1 and s_2 are complex conjugates then the result follows from (4.16) and Lemma 4.3. The result for $\alpha = 0$ follows by taking a limit as $\alpha \to 0$.

The next lemma shows that the infinite divisibility of mixtures of the type so far considered does not extend in general to cases where more than two exponential factors are involved.

LEMMA 4.6. *Let $b_i > 0\,(i = 1, 2, 3), p > 0, q > 0, p + q = 1$, and let*

$$f(s) = p + q\frac{b_1 b_2 b_3}{(b_1 + s)(b_2 + s)(b_3 + s)}. \tag{4.17}$$

Then it is possible to choose values of p, q and b_1, b_2, b_3 such that $f(s)$ is not infinitely divisible.

PROOF. A numerical case demonstrates the result. Let $p = 99/1170$, $q = 1071/1170$, $b_1 = 1$, $b_2 = 9$, $b_3 = 11$. Then

$$f(s) = \frac{(s + 18)\{(s + 1.5)^2 + 62.75\}}{(s + 1)(s + 9)(s + 11)} \cdot \frac{99}{1170}.$$

The roots of the numerator are thus -18 and $-1.5 \pm i(62.75)^{\frac{1}{2}}$. Let $\lambda(x) = e^{-x} + e^{-9x} + e^{-11x} - 2e^{-1.5x}\cos \eta x - e^{-18x}$ where $\eta = (62.75)^{\frac{1}{2}}$. Then if $x_0 = 2\pi/\eta$ a numerical calculation will show that $\lambda(x_0) < 0$ from which it follows by the condition (4.4) that $f(s)$ is not infinitely divisible.

5. Markov chains in continuous time. Consider a continuous time Markov chain and let the states be labelled by the positive integers. All states are assumed to be stable; thus instantaneous transitions are excluded and the duration of a sojourn in state j has the exponential density $q_j e^{-q_j t}$ $(0 < q_j < \infty; j = 1, 2, \cdots)$. For any set A of states let ${}_A f_{jk}(s)$ be the Laplace transform of the first passage time density from state j to state k (or if $j = k$ the first return time density to state j) maintaining a taboo on the states in the set A. Even if $j \in A$, the first passage time is defined to include the initial sojourn in state j, and in general the first return time also includes this initial sojourn in state j. Note that the process need not be irreducible or recurrent.

Further let p_{jk} be the transition probabilities of the imbedded discrete time Markov chain. Thus $p_{jk} q_j dt + o(dt)$ is the probability of a transition from state j to state k in the interval $(t, t + dt)$ conditional on state j being occupied at time t.

The p_{jk} satisfy

$$p_{jj} = 0 \qquad\qquad j = 1, 2, \cdots,$$
$$\Sigma_k p_{jk} = 1,$$

and $p_{jk} = q_{jk}/q_j$ where q_{jk} is the transition intensity from state j to state k.

THEOREM 5.1.

(i) *The following relations hold*:

$$f_{jk}(s) = \frac{{}_jf_{jk}(s)}{1 - {}_kf_{jj}(s)}, \tag{5.1}$$

$${}_jf_{jk}(s) = \frac{q_j}{q_j + s}\{p_{jk} + \Sigma_{i;i\neq k}\, p_{ji\,j}f_{ik}(s)\}. \tag{5.2}$$

(ii) *If ${}_jf_{jk}(s)$ is infinitely divisible, so is $f_{jk}(s)$.*

PROOF. Suppose first that $p_{jk} > 0$ for all j, k $(j \neq k)$. Then the chain is irreducible and in the notation of example 3(c),

$${}_1f_{12}(s) = \frac{q_1}{q_1 + s}k(s),$$

and

$${}_2f_{11}(s) = \frac{q_1}{q_1 + s}g(s).$$

Since the states 1 and 2 were arbitrary in example 3(c), (5.1) follows from (3.5). The relation (5.2) follows by considering all the possible passages from j to k without returning to j, following the first transition out of state j.

The infinite divisibility statement (ii) follows from the fact $\{1 - {}_kf_{jj}(s)\}^{-1}$ is of the compound geometric form.

If some p_{jk} are zero the results follow by suitable passages to the limit from the case $p_{jk} > 0$.

THEOREM 5.2. *For a Markov chain with n states all first passage time distributions are infinitely divisible if $n \leqslant 4$, but if $n \geqslant 5$ there are first passage distributions which are not infinitely divisible.*

PROOF. It is supposed for convenience in the following proof that $p_{jk} > 0$ for all $j, k, j \neq k$. The general result will follow by suitable passages to the limit.

(i) $n = 2$. The result here is obvious since there are only two first passage time distributions and they are both exponential.

(ii) $n = 3$. Consider without loss of generality the formula (5.1) for $j = 1$, $k = 3$; the numerator as given by (5.2) is

$${}_1f_{13}(s) = \frac{q_1}{q_1 + s}\{p_{13} + p_{12\,1}f_{23}(s)\}.$$

Now ${}_1f_{23}(s)$ represents a first passage time in the subchain consisting of two states

(2, 3). Hence ${}_1f_{23}(s) = p_{23}q_2/(q_2 + s)$ and

$${}_1f_{13}(s) = \frac{q_1}{q_1 + s}\left(p_{13} + \frac{p_{12}p_{23}q_2}{q_2 + s}\right),$$

which is infinitely divisible by (4.13). Thus $f_{13}(s)$ is infinitely divisible by Theorem 5.1(ii) and is given by

$$f_{13}(s) = \frac{q_1}{q_1 + s}\left(p_{13} + \frac{p_{12}p_{23}q_2}{q_2 + s}\right)\{1 - {}_3f_{11}(s)\}^{-1} \tag{5.3}$$

where

$${}_3f_{11}(s) = p_{12}p_{21}\frac{q_1q_2}{(q_1 + s)(q_2 + s)}. \tag{5.4}$$

(iii) $n = 4$. In this case consider (5.2) with $j = 1$, $k = 4$. Then

$${}_1f_{14}(s) = \frac{q_1}{q_1 + s}\{p_{14} + p_{12}\,{}_1f_{24}(s) + p_{13}\,{}_1f_{34}(s)\}. \tag{5.5}$$

The two terms ${}_1f_{24}(s)$ and ${}_1f_{34}(s)$ represent passage times within the 3-state subchain (2, 3, 4) and may be expressed in forms similar to (5.3); for these two terms the denominators are respectively $1 - {}_{14}f_{22}(s)$ and $1 - {}_{14}f_{33}(s)$ and these are equal since

$${}_{14}f_{22}(s) = {}_{14}f_{33}(s) = p_{23}p_{32}\frac{q_2q_3}{(q_2 + s)(q_3 + s)}.$$

So, (5.5) becomes

$${}_1f_{14}(s) = \frac{q_1}{q_1 + s}\left[p_{14} + R(s)\{1 - {}_{14}f_{22}(s)\}^{-1}\right] \tag{5.6}$$

where

$$R(s) = \frac{p_{12}q_2}{q_2 + s}\left(p_{24} + \frac{p_{23}p_{34}q_3}{q_3 + s}\right) + \frac{p_{13}q_3}{q_3 + s}\left(p_{34} + \frac{p_{32}p_{24}q_2}{q_2 + s}\right). \tag{5.7}$$

Note in (5.6) that $R(s)\{1 - {}_{14}f_{22}(s)\}^{-1}$ is the Laplace transform of a defective probability distribution and that it reduces to the form

$$\beta(a + s)/\{(b_1 + s)(b_2 + s)\}$$

where $\beta > 0$, $a > 0$ and $-b_1$, $-b_2$, the zeros of $1 - {}_{14}f_{22}(s)$, are real and satisfy $0 < b_1 \leqslant \min(q_2, q_3) \leqslant \max(q_2, q_3) \leqslant b_2$. Lemma 4.5 may therefore be applied. Hence the square-bracketed term in (5.6) is infinitely divisible and so in turn are ${}_1f_{14}(s)$ (from (5.6)) and $f_{14}(s)$ (from Theorem 5.1(ii)).

(iv) $n = 5$. Consider the following example of a 5-state chain. Let $p_{12} > 0, p_{15} > 0$ $(p_{12} + p_{15} = 1), p_{23} = 1, p_{34} = 1, p_{45} = 1, p_{5k}$ arbitrary $(k = 1, \cdots, 4)$. Then

$$f_{15}(s) = \frac{q_1}{q_1 + s}\left[p_{15} + \frac{p_{12}q_3q_4q_5}{(q_3 + s)(q_4 + s)(q_5 + s)}\right]$$

$$= \frac{q_1}{q_1 + s}g(s), \text{ say.}$$

Now $g(s)$ is of the form (4.17) and for suitably chosen values of the constants is not infinitely divisible by Lemma 4.6. Nor for these values can $f_{15}(s)$ be infinitely divisible for all q_1, for if it were so also would $g(s)$ be by letting $q_1 \to \infty$. The theorem is therefore proved.

It is perhaps worth noting that the theorem may be rephrased as follows. If a subchain of a Markov chain in continuous time consists of one, two or three states then all first passage times out of the subchain are infinitely divisible, but not all such passage times are infinitely divisible if the subchain consists of more than three states.

6. Markov chains in discrete time. The results of Section 5 do not carry over to the discrete time case. As will be seen below, a perfectly ordinary three-state chain, for example, can have first passage times which are not infinitely divisible. A slight qualification is necessary in discussing integer-valued random variables. Let X be a nonnegative integer-valued random variable with probability generating function

$$g(z) = \Sigma_{n=0}^{\infty} p_n z^n.$$

A necessary and sufficient condition that X (or $g(z)$) be infinitely divisible is that $g(z)$ should have the compound Poisson form

$$g(z) = \exp[\lambda\{h(z) - 1\}] \qquad \lambda > 0 \tag{6.1}$$

where $h(z)$ is a probability generating function of a nonnegative random variable. Thus it is necessary that $g(0) = p_0 > 0$. Now a passage time N in a Markov chain is necessarily positive, i.e., $N \geqslant 1$, so that its probability generating function will always have a factor z. Thus, strictly, N can never be infinitely divisible into integer-valued factors but $N - 1$ can be and as shown by Miller (1967), the first passage times for skip-free chains are infinitely divisible if the passage time is defined as $N - 1$. It is in this sense that passage times are considered in this section.

The formulae of Theorem 5.1 carry over, in an obvious notation, to the discrete time case. Thus

$$f_{jk}(z) = \frac{{}_jf_{jk}(z)}{1 - {}_kf_{jj}(z)}, \tag{6.2}$$

$${}_jf_{jk}(z) = p_{jk}z + \Sigma_{i;i\neq j,\, k}\, p_{ji}\, {}_jf_{ik}(z). \tag{6.3}$$

For a three-state chain

$${}_1f_{13}(z) = p_{13}z + \frac{p_{12}p_{23}z^2}{1 - p_{22}z}, \tag{6.4}$$

$${}_3f_{11}(z) = p_{11}z + \frac{p_{12}p_{21}z^2}{1 - p_{22}z}. \tag{6.5}$$

Now consider the numerical example

$$\mathbf{P} = (p_{jk}) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} \frac{5}{20} & \frac{14}{20} & \frac{1}{20} \\ \frac{5}{30} & \frac{14}{30} & \frac{11}{30} \\ \cdot & \cdot & \cdot \end{array}\right] \end{array},$$

where p_{3k} $(k = 1, 2, 3)$ may be arbitrary. In this case (6.2) becomes, for $j = 1$, $k = 3$,

$$f_{13}(z) = \frac{{}_1f_{13}(z)}{1 - {}_3f_{11}(z)} = \frac{z}{20} \frac{1 + (14/3)z}{1 - (43/60)z} \tag{6.6}$$

after some calculation. The function (6.6), ignoring the factor z, cannot be an infinitely divisible probability generating function because it has a zero at $z = -(3/14)$ and a function of the form (6.1) is analytic and zero free in the region of the complex plane $|z| < 1$. Thus the results of Section 5 do not carry over to discrete time.

Nor indeed do the lemmas of Section 4 have straightforward counterparts for generating functions. Note first that a probability generating function $g(z)$ is infinitely divisible if and only if $\log\{g(z)/g(0)\}$ is a power series in z with nonnegative coefficients and convergent for $|z| \leqslant 1$. This can be seen from (6.1). Thus the rational function

$$\frac{(1 - a_1z)(1 - a_2z)}{(1 - b_1z)(1 - b_2z)}, \quad a_i \text{ real}, |a_i| < 1, 0 < b_i < 1, i = 1, 2, \tag{6.7}$$

which corresponds to (4.7), will (ignoring a normalization constant) be an infinitely divisible probability generating function if and only if

$$b_1^n + b_2^n \geqslant a_1^n + a_2^n \qquad n = 1, 2, \cdots. \tag{6.8}$$

Now it is not difficult to prove that for given b_1, b_2 $(0 < b_1 < 1, 0 < b_2 < 1)$ a necessary and sufficient condition for (6.8) is that a_1 and a_2 should lie on or inside the octagon in the (a_1, a_2) − plane with vertices at $(\pm b_1, \pm b_2)$ and $(\pm b_2, \pm b_1)$. This condition is rather more complicated than that of Lemma 4.2.

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