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PREDICTION PROCESSES AND AN AUTONOMOUS GERM-MARKOV PROPERTY

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Let $X(t)$ be a measurable stochastic process on a countably generated space $(E, \mathcal{E})$, and let $G(t) = \cap_{\delta>o}\mathcal{F}^\circ(t, t + \delta)$ be its germ field. By transferring the probabilities to a representation space, we define and analyze the class of such processes which are Markovian relative to $G(t)$ and autonomous, in the sense that they have a stationary transition mechanism. These processes are reduced to Ray processes on an abstract space with a certain weak topology. Five kinds of examples are indicated.

0. Introduction. The present paper is a follow-through to the general nonlinear prediction process developed in [9] and [15]. As such, it may be appropriate to start with a few words on the motivation for that theory. The original motivation for [9] was to make possible a general treatment of the germ-Markov property, as defined for instance in [7]. In the latter a process $X(t)$ is called Markovian relative to the germ fields (or germ-Markov) if for each t the past and future are conditionally independent given the germ field $G(t) = \cap_{\delta>0}\mathcal{F}^\circ(t, t + \delta)$, where $\mathcal{F}^\circ(t, t + \delta)$ denotes the σ-field generated by $\{X(s), t < s < t + \delta\}$.

Such a concept arose previously in the study of Gaussian processes (for example, in [11]) but it is obviously not confined to these. A natural analogy is that of a mechanical system with some random elements present. The future motion depends on the past not only through the instantaneous position, but also through the instantaneous velocity, which can only be defined in terms of a germ field. A familiar recourse in this situation is to introduce the vector Markov process consisting of both position and velocity. However, in the general case obstacles arise for a direct extension of this device. In fact, except in rather special cases, $G(t)$ is not countably generated. Already in the case of Brownian motion this is well known to be true, although of course each $\mathcal{F}^\circ(t, t + \delta)$ is countable generated. Thus in a sense the "present" $G(t)$ is more complicated than the entire process, and it is not quite clear how to simplify the germ-Markov property.

Another (perhaps unrelated) difficulty in studying a germ-Markov property is that there are other definitions of a germ $G(t)$, according to whether $X(t) \in G(t)$ is imposed and/or $\delta < o$ is allowed. These distinctions would become significant in discussing a strong-Markov property of germs, as seen in [7], although they do not affect the class of processes for which the simple germ-Markov property holds.

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Accordingly, in [9] we redefined $\mathcal{F}(t, t + \delta)$ as the σ-field generated by all integrals $\{\int_t^s f(X(\tau))d\tau, t < s < t + \delta\}$ (for all bounded measurable f, and assuming the process $X(t)$ to be measurable). This is equivalent to the former definition if $X(t)$ is right (or left) continuous, but in general it leads to a quite different viewpoint in which the "instantaneous present" has no intrinsic meaning. From this viewpoint the ordinary Markov property is meaningless, and the germ-Markov property replaces it. Consequently, in the framework of [9] the definition of $G(t)$ at least does not depend on inclusion of $X(t)$, although of course $G(t)$ is still in general not countably generated.

In the present paper, we first (Section 1) give a new proof of an important result of Meyer [15] which is needed later. This also provides a review of the main substance of [9] and [15]. Then we make one clarification of the theory, which serves a similar function. For some general applications to nonlinear filtering, we refer to Yor's paper [18].

In Section 2 we develop the germ-Markov property in this framework, and introduce the concept of autonomy for germ-Markov processes, which generalizes the usual concept of time-homogeneity for ordinary Markov processes. Then we study the general autonomous germ-Markov process and show that it reduces to an ordinary Markov process of known type—specifically, to the Ray process of a right process, restricted to its Ray space as in [5], Section 15. A somewhat more elementary description is also given in terms of the framework of [9]. It turns out here that the topology of [9] is weaker than the Ray topology, which makes the two descriptions almost identical. Finally, in Section 3 we give some examples of germ-Markov processes which arise in various ways from stochastic integrals, time changes, etc. We do not study intensively any particular class of such processes, but merely present some of the types which have come to our attention.

Some remark is necessary on why we continue the setup of [9], rather than that of [19]. In the first place, while [19] does permit a simplification of notation, this is bought at the cost of two factors. Namely, it becomes necessary to assume a suitable topology for the state space of $X(t)$, such that the paths are right-continuous with left limits, and it is also necessary to assume that $X(t)$ has the canonical representation on its path space. These assumptions, however, tend to obscure some important distinctions. Two basic consequences of [9] are that it provides a right-continuous process in a natural topology, without any preassigned topology for X itself, and that by transferring the discussion explicitly to an auxiliary space it avoids any assumptions on the probability space of X. In particular, it thus avoids assuming the existence of translation operators.

In the second place, it is not difficult to axiomatize the idea of a "prediction process," and to show that for a given process X all such constructions are roughly equivalent. This was done in [10]. But here it is possibly more relevant to point out that the predictive processes Z_t^μ of [19] are easily obtained from the $Z(t)$ of [9] by forming the inverse images on Ω of $Z(t)$ restricted to the (Borel) image of Ω in Ω'. Consequently, all of the present conclusions transfer in an obvious way to the situation of [19].

We do, however, make one change in the notation of [9]. We replace $Z(t)$ by Z_t^h to make room for $Z_t^h(S)$ as a notation for the (random) measure $Z(t)$ at the set S. Here, as in [19], the superscript h denotes the probability for which Z_t^h gives a conditioning.

1. The general prediction process. We first review briefly the results of Section 1 of [9].[2] Let (Let $(\Omega, \mathcal{F}, P, X(t))$ be a measurable process with values in an abstract space $(E, \mathcal{E})$, where $\mathcal{E}$ is countably generated. We choose a fixed but arbitrary sequence $o \leq h_n \leq 1$ of $\mathcal{E}/\mathcal{B}$-measurable functions which generate $\mathcal{E}$ ($\mathcal{B}$ denotes the Borel field of R), and we map $X(t)$ into the sequential process $Y_n(t) = \int_o^t h_n(X(s))\,ds$, $1 \leq n$. Let $(\Omega', \mathcal{F}'(t_1, t_2), P')$ denote respectively: (a) the space of all sequential paths $(y_n(t))$; $o \leq y_n(t+s) - y_n(t) \leq s$, $o \leq s, t$; $n \geq 1$; (b) the σ-field generated by $y_n(t) - y_n(t_1)$, $t_1 < t < t_2$, $n \geq 1$; and (c) the probability on $\mathcal{F}' = \mathcal{F}'(o, \infty)$ given by $P'(S') = P\{(Y_n) \in S'\}$, $S' \in \mathcal{F}'(o, \infty)$. Since the sets $\{(Y_n) \in S'\}$ for $S' \in \mathcal{F}'(t_1, t_2)$ comprise the σ-field generated by all integrals $\int_{t_1}^s f(X(\tau))d\tau$, $t_1 < s < t_2$, the space $(\Omega', \mathcal{F}', P')$ constitutes a "widesense" approach to the study of P on these σ-fields. We note again that if for a suitable topology $X(t)$ is right-continuous, then by differentiation in x we see that these σ-fields reduce to the usual generated σ-fields of $X(t)$, $t_1 \leq t < t_2$.

We next introduce the generalized translation operators i_t on Ω' by setting

$$i_t(y_n(s)) = (y_n(t+s) - y_n(t)),$$

and we let $(H, \mathcal{H})$ denote the space of all probability measures on $(\Omega', \mathcal{F}')$. We consider Ω' as a compact metrizable space with the topology uniform convergence for each n and t in compact sets, and we consider $(H, \mathcal{H})$ as a compact metrizable space with the topology of *weak-*convergence of measures* on $(\Omega', \mathcal{F}')$. For each element $h \in H$ (and in particular for P') we construct the prediction process $Z_t^h(S')$, $S' \in \mathcal{F}'$, $t \geq o$, as the unique $(H, \mathcal{H})$-valued process on $(\Omega', \mathcal{F}')$ (up to h-equivalence) with right-continuous paths and satisfying

$$P^h\big(i_t^{-1}S' | \mathcal{F}'(o, t+)\big) = Z_t^h(S'); h\text{-a.s.}, \qquad t \geq o, S' \in \mathcal{F}'.$$

Here we write P^h in place of simply h in order to be consistent with the notation E^h for h-expectation, and $\mathcal{F}'(o, t+) = \cap_{\delta > o}\mathcal{F}'(o, t+\delta)$.

The main results of Section 1 of [9] may now be stated as follows. In the first place, Z_{t-}^h exists for all $t > o$, h-a.s. Next, for any $\mathcal{F}_h'(o, t+)$ stopping time T_1 (respectively, any previsible $\mathcal{F}_h'(o, t+)$ stopping time $T_2 > 0$), where $\mathcal{F}_h'(0, t)$ is

[2]We will point out here some trivial but embarrassing errors in [9]: page 576 line -17, delete h_1, h_2; page 577 line -2, replace $i_t S$ by $i_t S'$; page 581 line -10, replace $E_{Z(s)}g$ by $g(Z(s))$; page 582 line -3 $f \in C^+(\Omega')$; page 584 lines 10, 11 for $C_1 = 0$ and $C_2 = \infty$ we have $E_y \int x P_{Z(t_1)}$.

$\mathfrak{F}'(o, t)$ completed by all h-null sets in $\mathfrak{F}'$, we have respectively

(1.1) $$\text{(a) } P^h\big(i_{T_1}^{-1}S' | \mathfrak{F}_h'(o, T_1 +)\big) = Z_{T_1}^h(S')$$

$$\text{(b) } P^h\big(i_{T_2}^{-1}S' | \mathfrak{F}_h'(o, T_2)\big) = Z_{T_2 -}^h(S'), \qquad S' \in \mathfrak{F}'.$$

Here the σ-field $\mathfrak{F}_h'(0, T_2)$ would customarily be written $\mathfrak{F}_h'(0, T_2 -)$, but since the paths of Ω' are continuous in t the two definitions yield the same object, namely $\{S' \in \mathfrak{F}_h'(0, \infty) : S' \cap (T_2 \leqslant t) \in \mathfrak{F}_h'(0, t)$ for all $t\}$. We note that , since no Markov property is assumed for h, we do not have $\mathfrak{F}_h'(0, t) = \mathfrak{F}_h'(0, t +)$ in general. The second main conclusion of [9] concerns the processes Z_t^h as h varies. There is a Borel transition function $q(t, z, A)$, $A \in \mathcal{H}$, $\mathfrak{B}^+ \times \mathcal{H}$-measurable in (t,z), such that for every $h \in H$ the corresponding Z_t^h is a homogeneous strong-Markov process relative to its generated σ-fields (or equivalently relative to $\mathfrak{F}_h'(0, t +))$ with the same transition function q. Thus, no matter what process X and induced measure P' one has, the corresponding prediction process $Z_t^{P'}$ is a homogeneous Markov process with a given transition function. We can think of $Z_t^{P'}$ as the process of conditional futures of X given the pasts up to time $t +$. Since time $o +$ has, in general, a larger past than time o, we do not have the normality property $P^h\{Z_0^h = h\} = 1$ for all h. An important role is therefore played by the following

Definition 1.1. The set of nonbranching points is $H_o = \{h \in H : q(o, h, \{h\}) = 1\}$.

Since q is measurable, we have $H_o \in \mathcal{H}$. When considering the Z_t^h as a Markov process with transition function q on $(H, \mathcal{H})$, with the canonical representation on the space Ω_H of right-continuous paths, we omit the superscript h, this being preserved in the notation P^v or P^h for the initial distribution or point. The following result of Meyer [15] shows that we can use H_o as a restricted state space for Z_t.

Theorem 1.1. *For any $h \in H$, we have $P^h\{Z_t^h \in H_o$ for all $t \geqslant o\} = 1$. In particular, $P^h\{Z_t^h(S) = 0$ or 1 for all $t \geqslant 0$ and $S \in \mathfrak{F}'(o +)\} = 1$. Restricted to the set H_o, the process Z is a right process in the sense of Meyer* ([14], XIV; [5]).

Proof. Since Z_t^h is right-continuous in H, it follows easily that for each h the set $\{(t, w) : Z_t \in H_o\}$ is optional for the family $\mathfrak{F}_t^h$ of σ-fields generated by Z_s, $s \leqslant t +$, and completed for P^h on Ω_H in the standard manner [14, XIII, 5], this family being right-continuous (see [14], XIII, T 13). Consequently, unless the projection of this set on Ω_H is P^h-null, there would be a stopping time T with $P^h\{Z_T \notin H_o; T < \infty\} > 0$, by the optional section theorem [2, IV, (84)]. Then by the strong-Markov property of Z_T, since Z_T is both past and future of time T, we would have

$$\begin{aligned} 0 &= P^h\{Z_T \neq Z_T\} \\ &= E^h\{P^{Z_T}\{Z_0 \neq Z_T\}\} \\ &= P^h\{Z_T \notin H_o\}, \end{aligned}$$

contradicting the former result and proving the first assertion. For $h \in H_o$ and $S' \in \mathcal{F}'(o+)$, it follows from (1.1) with $T_1 \equiv 0$ that

$$\begin{aligned} I_{S'} &= P^h(S'|\mathcal{F}'(o, o+)) \\ &= Z_0^h(S') \\ &= P^h(S'), \qquad h\text{-a.s.} \end{aligned}$$

Hence $P^h(S') = 0$ or 1, and the second assertion follows from the first. Finally, the properties of a right process on H_o are now clear, except for the right-continuity of excessive functions along the trajectories of Z_t. But this follows, since q is a Borel transition function, by [14, XIV, T11], completing the proof.

REMARK. It is worth emphasizing again that $\mathcal{F}'(0, 0+)$ is not countably generated. Hence it does not follow from the fact that $Z_t(S) = 0$ or 1 on $\mathcal{F}'(0, 0+)$ that Z_t^h for each (t, w') is concentrated on an atom of $\mathcal{F}'(0, 0+)$. Otherwise, if $h = P'$ were induced by a Brownian motion $X(t)$, then since $X(t)$ can be identified with its own prediction process in an obvious way, it would follow that $X(t)$ is a pure jump process in the sense of [8]. In short, an atom of $\mathcal{F}'(0, 0+)$ specifies the trajectory in a time interval of positive length (depending on the sample path). For a fuller discussion, see Blackwell and Dubins (1975).

Our second general result concerns the connection of Z_t as a Markov process on H_o with the set H of probabilities on Ω'. It does not quite hold true that every initial distribution for Z_t on H_o defines a process equivalent in distribution to some particular Z_t^h, $h \in H$. The reason is clear from the following example (based on Example 1.4.1 of [9]). Suppose that $X(t)$ has only two paths, each having probability $\frac{1}{2}$, and that they coincide for $0 \leqslant t \leqslant 1$ but differ for $1 < t$. Then the induced conditional future process Z_t^h will have the same general behavior. However, if we consider the two paths as two separate deterministic processes X_1 and X_2, with induced processes $Z_t^{(1)}$ and $Z_t^{(2)}$, and form the process Z_t for P^v where v is the initial distribution on H_o assigning probability $\frac{1}{2}$ to $Z_0^{(1)}$ and to $Z_0^{(2)}$, Z_t will not correspond to any Z_t^h, $h \in H$. Indeed, by (1.1) with $T_1 \equiv 0$ we have for any h

$$P^h(S'|\mathcal{F}'(0, 0+)) = Z_0^h(S'), \qquad S' \in \mathcal{F}'.$$

In particular, for $S' \in \mathcal{F}'(0, 0+)$ this becomes $I_{S'} = Z_0^h(S')$, h-a.s. But for P^v, Z_0 is a.s. constant (0 or 1) at each $S' \in \mathcal{F}'(0, 0+)$ (in fact, on $\mathcal{F}'(0, 1)$). If Z and Z^h were equivalent in distribution it would follow that $I_{S'}$ is constant h-a.s., hence h would have only values 0 or 1 on $\mathcal{F}'(0, 0+)$. But then, since $\mathcal{F}'$ is countably generated, Z_0^h would h-a.s. coincide with h, while in fact Z_0 for P^v has two values each of probability $\frac{1}{2}$.

Consequently, we need the following

DEFINITION 1.2. Let $\mathcal{H}_o$ denote the restriction of $\mathcal{H}$ to H_o. A probability v on $(H_o, \mathcal{H}_o)$ is called *resoluble on* Ω' if for some $h \in H$ the prediction process Z_t^h has the same joint distributions as Z_t for P^v. The measure h (which is unique) is called the *resultant* of v.

DEFINITION 1.3. A function $g : \Omega' \to H_o$, $\mathcal{F}'(0, 0+)/\mathcal{H}_o$-measurable, is called *$\Omega'$-consistent* if for each S' in the (countable generated) σ-field induced by g on Ω' we have $g(w'; S') = I_{S'}(w')$ (where $g(w'; S')$ is the measure $g(w')$ of the set $S' \in \mathcal{F}'(0, 0+)$).

The principal result connecting these definitions is

THEOREM 1.2. *A probability v on $(H_o, \mathcal{H}_o)$ is resoluble on Ω' if and only if it is the image of a probability on $\mathcal{F}'(0, 0+)$ by some Ω'-consistent function g.*

PROOF. We suppose first that v is resoluble on Ω', and let h and Z^h denote the resultant measure and prediction process on Ω'. Then by (1.1) with $T_1 \equiv 0$, and Theorem 1.1, Z_0^h is $\mathcal{F}_h'(0, 0+)/\mathcal{H}_o$-measurable. Since $\mathcal{H}_o$ is countable generated, we may define an h-equivalent $\underline{Z}_0^h$ which is $\mathcal{F}'(0, 0+)/\mathcal{H}_o$-measurable.

Let (A_n) be a countable field generating $\mathcal{H}_o$, and $S_n' = \{w' : \underline{Z}_0^h \in A_n\}$ be the induced sequence in $\mathcal{F}'(0, 0+)$. Then we have

$$\begin{aligned} \underline{Z}_0^h(S_n') &= P^h(S_n' | \mathcal{F}'(0, 0+)) \\ &= I_{S_n'} \text{ for all } n,\ h\text{-a.s.} \end{aligned}$$

Consequently $\underline{Z}_0^h(S') = I_{S'}$ for all S' in the σ-field induced by $\underline{Z}_0^h$ on Ω' except for w' in an h-null set N in the induced σ-field. If N is nonvoid, let h^* be any element of H_o with $h^*(N) = 1$ (for example, a unit measure concentrated at $w' \in N$), and define

$$\begin{aligned} g(w') &= \underline{Z}_0^h \quad \text{on } \Omega' - N \\ &= h^* \quad \text{on } N. \end{aligned}$$

Then the σ-field generated by g is that jointly generated by N and $\{S_n' \cap (\Omega' - N); 1 \leq n\}$, and since $\underline{Z}_0^h(S_n' \cap (\Omega' - N)) = 1$ on $S_n' \cap (\Omega' - N)$ we see that g is Ω'-consistent. Also, since $\underline{Z}_0^h = g$ except on an h-null set, the image on $\mathcal{H}_o$ of h on $\mathcal{F}'(0, 0+)$ by g is the same as that by Z_0^h, namely v.

Conversely, suppose that v is the image of a probability h_o on $\mathcal{F}'(0, 0+)$ by an Ω'-consistent function g. We consider the probability measure

$$\begin{aligned} h(S') &= \int g(w'; S') h_0(dw') \\ &= \int g(w'; S') h(dw'), \qquad S' \in \mathcal{F}', \end{aligned}$$

where the second expression follows since, by consistency, $h = h_o$ on the σ-field generated by g. To show that v is resoluble on Ω', it suffices to show that

$$P^h(S' | \mathcal{F}'(0, 0+)) = g(w'; S'),\ S' \in \mathcal{F}',\ h\text{-a.s.} \tag{1.2}$$

Now for $S_o \in \mathcal{F}'(0, 0+)$ we have

$$\begin{aligned} \int_{S_o} g(w'; S') h(dw') &= \int_{S_o} g(w'; S') \int_{\Omega'} g(w''; dw') h(dw'') \\ &= \int_{\Omega'} \left(\int_{S_o} g(w'; S') g(w''; dw') \right) h(dw''). \end{aligned} \tag{1.3}$$

Let $S'(n, k) = \{w' : k2^{-2} < g(w'; S') \leqslant (k + 1)2^{-n}\}, 0 \leqslant k, 0 \leqslant n$. Then the integrand of (1.3) becomes

$$\int_{S_o} g(w'; S')g(w''; dw') = \lim_{n\to\infty}\Sigma_k k2^{-n}g(w''; S_o \cap S'(k, n)) \\ = g(w''; S')g(w''; S_o) \tag{1.4}$$

since, by consistency, all of the summands vanish except the one for which $w'' \in S'(k, n)$, and for it $g(w''; S'(k, n)) = 1$. If $g(w''; S_o) = 0$ then so does $g(w''; S_o \cap S')$, while if $g(w''; S_o) = 1$ then $g(w''; S_o \cap S') = g(w''; S')$. Hence in either case the right side of (1.4) equals $g(w''; S' \cap S_o)$. Since $g(w'', \cdot) \in H_o$, by Theorem 1.1 $g(w''; S_o)$ takes on only values 0 or 1. Hence the last expression in (1.3) becomes

$$\int_{\Omega'} g(w'', S' \cap S_o)h(dw'') = h(S' \cap S_o),$$

proving (1.2) and the theorem.

Since every $h \in H$ defines a process Z_t^h equivalent Z_t for some P^v, namely that in which v is the distribution of Z_0^h on $\mathcal{H}_o$, it is no real loss of generality to consider Z_t only as a Markov process on $(H_o, \mathcal{H}_o)$. In the sequel, we will assume that Z_t has the canonical representation on the space of all right-continuous paths on H_o, with left limits in H for $t > o$.

2. An autonomous germ-Markov property. We continue to treat our processes X as probabilities on the auxiliary space $(\Omega', \mathcal{F}')$ since this is easier and more general. It is possible to specialize to the subset of such probabilities induced by all processes X on any reasonable space $(E, \mathcal{E})$, such as a U-space as defined in [5], and this was done in [9, Section 2]. Here, however, we will proceed on the more general level, and leave it to the reader to translate the definitions in terms of an underlying process X.

DEFINITION 2.1. Let $G^+(t) = \cap_{\varepsilon>0}\mathcal{F}'(t, t + \varepsilon)$ and $G^-(t) = \cap_{\varepsilon>0}\mathcal{F}'(t - \varepsilon, t)$. We say that a probability $h \in H$ is germ-Markov at time t if for each $S' \in \mathcal{F}'$,

$$P^h\big(i_t^{-1}S'|\mathcal{F}'(0, t +)\big) = P^h\big(i_t^{-1}S'|G^+(t)\big). \tag{2.1}$$

We say that $h \in H$ is a Markov process of germs, or has the "germ-Markov property" if (2.1) holds for all $t \geqslant 0$.

REMARK. We note that even though the left side of (2.1) is given by $Z_t^h(S')$ in accordance with (1.1), and Z_t^h is itself a homogeneous Markov process, there is as yet no assumption of homogeneity on the germ. Thus, any inhomogeneity is covered up in the construction of Z_t^h, as can be understood by considering a case in which h is induced by a right-continuous but inhomogeneous Markov process X (and hence satisfies (2.1)).

We will prove one general result on germ-Markov processes without assuming any kind of additional homogeneity.

THEOREM 2.1. (a) *h has the germ-Markov property if (and only if) it is germ-Markov at a countable dense set of t.*

(b) *h has the germ-Markov property if and only if for all* $t > 0$

$$P^h(i_t^{-1}(S')|\mathcal{F}'(0, t)) = P^h(i_t^{-1}S'|G^-(t)); \qquad S' \in \mathcal{F}'. \tag{2.2}$$

PROOF. Suppose that h is germ-Markov on the dense set (t_n). Then, with a slight abuse of notation, we have for any $S' \in \mathcal{F}'(\varepsilon, \infty)$, $\varepsilon > 0$, and any $t \geqslant 0$,

$$\begin{aligned} p^h(i_t^{-1}S'|\mathcal{F}'(0, t+)) &= \lim_{t+\varepsilon>t_n\downarrow t} P^h(i_t^{-1}S'|\mathcal{F}'(0, t_n+)) \\ &= \lim_{t+\varepsilon>t_n\downarrow t} P^h(i_t^{-1}S'|\mathcal{F}'(t, t_n+)) \\ &= P^h(i_t^{-1}S'|G^+(t)), \end{aligned} \tag{2.3}$$

where we used the fact that $i_t^{-1}S' \in \mathcal{F}'(t+\varepsilon, \infty)$, and then martingale convergence of conditional expectations. Letting $\varepsilon \to 0$, (2.3) holds on a field generating $\mathcal{F}'(0, \infty)$. Since the left side equals $Z_t^h(S')$, which is a measure in S', extension of measures extends (2.3) to $\mathcal{F}'(0, \infty)$, proving (a).

Turning to (b), we assume first that h has the germ-Markov property.Then for $t > 0$

$$\begin{aligned} P^h(i_t^{-1}(S')|\mathcal{F}'(0, t)) &= \lim_{t-\varepsilon<t_n\uparrow t} P^h(i_t^{-1}(S')|\mathcal{F}'(0, t_n+) \vee \mathcal{F}'(t_n, t)) \\ &= \lim_{t-\varepsilon<t_n\uparrow t} P^h(i_t^{-1}(S')|G^+(t_n) \vee \mathcal{F}'(t_n, t)) \\ &= \lim_{t-\varepsilon<t_n\uparrow t} P^h(i_t^{-1}(S')|\mathcal{F}'(t_n, t)) \\ &= P^h(i_t^{-1}(S')|G^-(t)), \end{aligned} \tag{2.4}$$

using the conditional independence of $\mathcal{F}'(0, t_n+)$ and $\mathcal{F}'(t_n, t)$ given $G^+(t_n)$. Conversely, since $G^+(0) = \mathcal{F}'(0, 0+)$ the germ-Markov property at $t = 0$ always holds. Moreover, assuming (2.2) and $S' \in \mathcal{F}'(\varepsilon, \infty)$, $\varepsilon > 0$, we have

$$\begin{aligned} P^h(i_t^{-1}S'|\mathcal{F}'(0, t+)) &= \lim_{t+\varepsilon>t_n\downarrow t} P^h(i_t^{-1}S'|\mathcal{F}'(0, t_n)) \\ &= \lim_{t+\varepsilon>t_n\downarrow t} P^h(i_t^{-1}S'|G^-(t_n)) \\ &= \lim_{t+\varepsilon>t_n\downarrow t} P^h(i_t^{-1}S'|\mathcal{F}'(t, t_n)) \\ &= P^h(i_t^{-1}S'|G^+(t)). \end{aligned} \tag{2.5}$$

Letting $\varepsilon \to 0$ extends (2.5) to $S' \in \mathcal{F}'$ as in part (a), completing the proof. Of course, the expression (2.2) equals $Z_{t-}^h(S')$ by (1.1), h-a.s.

The main content of the present section lies perhaps more in the *definition* of an autonomous germ-Markov process, rather than in any one theorem. To explain Definition 2.2, we first obtain an equivalent form of the germ-Markov property at time t.

THEOREM 2.2. *A probability $h \in H$ is germ-Markov at time t if and only if there is an $H_t \in \mathcal{H}_o$ such that* (a) $P^h\{Z_t^h \in H_t\} = 1$, *and* (b) *the trace of $\mathcal{H}_o$ on H_t coincides with the σ-field on H_t generated by $\{z(S); S \in \mathcal{F}'(0, 0+)\}$ as a set of mappings of $H \to R^+$.*

REMARK. Since $\mathcal{H}_o$ is countably generated, its trace on H_t is likewise. But $\mathcal{F}'(0, 0+)$ is not countably generated, hence neither is the σ-field generated by $\{z(S); S \in \mathcal{F}'(0, 0+)\}$ on H_o. The restriction on H_t imposed by (b) is equivalent to the requirement that $h \in H_t$ be determined uniquely by its values on $\mathcal{F}'(0, 0+)$ and that the σ-field generated on H_t by $\{z(S); S \in \mathcal{F}'(0, 0+)\}$ be countable generated (see [2], III. 26).

PROOF. If h is germ-Markov at time t, then by (2.1) and (1.1) the corresponding Z_t^h generates a subfield of the completion of $G^+(t)$ for h in $\mathcal{F}'$. Then if (A_n) is a countable field generating $\mathcal{H}_o$, by discarding an h-null set in $\mathcal{F}'$ we can arrange that each $\{w' : Z^h(w', \cdot) \in A_n\}$ and consequently Z_t^h itself, is $G^+(t)$-measurable. Setting $S_n^+ = \{Z_t^h \in A_n\} \in G^+(t)$, we have

$$P^h(S_n^+ | \mathcal{F}'(0, t+)) = Z_t^h(i_t(S_n^+))$$
$$= I_{S_n^+} \quad ,h\text{-a.s.}$$

where $i_t(S_n^+) = \{i_t w'; w' \in S_n^+\} \in G^+(0)$, and we used the fact that $S_n^+ \in \mathcal{F}'(0, t+) \cap i_t^{-1}(\mathcal{F}')$. Accordingly, the random variables $Z_t^h(i_t(S_n^+))$, $n \geq 1$, generate the same σ-field as Z_t^h up to h-null sets in $\mathcal{F}'$, and again by discarding such a set we can make these generated σ-fields identical. Let $\Omega_t' \in \mathcal{F}'$, $h(\Omega_t') = 1$, be the complement of such a set, where we will assume without loss of generality that $Z_t^h \in H_o$ on all of Ω_t' (Theorem 1.1). Since $Z_t^h(i_t(S_n^+))$ has only values 0 or 1, the σ-field generated by Z_t^h on Ω_t' is the inverse of the product σ-field in $\times_{n=1}^{\infty}\{0, 1\}$ under the mapping $Z_t^h(i_t(S_n^+))$, $1 \leq n$. It follows easily that there is a $\times_{n=1}^{\infty}\{0, 1\}/(H_o, \mathcal{H}_o)$-measurable function j on $\times_1^{\infty}\{0, 1\}$ such that $Z_t^h = j(Z_t^h(i_t(S_n^+)))$ on Ω_t' (the usual proof, as in Doob (*Stochastic Processes*, Supplement, Theorem 1.5), assumes that Z_t^h is real-valued, but since the compact metric space $(H, \mathcal{H})$ is measure-isomorphic to a real Borel set with its Borel field, this entails no difficulty). We now restrict j to the measurable subset B of $\times_{n=1}^{\infty}\{0, 1\}$ on which, for $x = (x_n) \in B$, $j(x)(i_t(S_n^+)) = x_n$, $1 \leq n$. Then plainly j is one-to-one on B, and $P\{Z_t^h \in j(B)\} = 1$. Consequently, $j(B)$ is an element of $\mathcal{H}_o$ [2, III. 21], and on $j(B)$ the σ field $\mathcal{H}_o$ coincides with that generated by $z(i_t(S_n^+))$, $1 \leq n$. Setting $H_t = j(B)$, we have proved (a) and (b) of the theorem.

Conversely, assuming (a) and (b), since $\mathcal{H}_o$ is countably generated there is a sequence $S_n \in \mathcal{F}'(0, 0+)$ such that $h(S_n)$, $1 \leq n$, generates the same σ-field as $\mathcal{H}_o$ on H_t. Since $P^h\{Z_t^h \in H_t\} = 1$, and

$$Z_t^h(S_n) = P^h(i_t^{-1}S_n | \mathcal{F}_h'(0, t+))$$
$$= I_{i_t^{-1}S_n}, \qquad h\text{-a.s.},$$

where $i_t^{-1}S_n \in G^+(t)$, it follows that Z_t^h is measurable over the completion of $G^+(t)$ in $\mathcal{F}'$ for h. This implies (2.1), completing the proof.

We may regard the set H_t of Theorem 2.2 as determining the transition from the germ to the entire conditional future at time t through the correspondence between Z_t^h on $\mathcal{F}'(0, 0+)$ and Z_t^h on all of $\mathcal{F}'$. The idea intended by the word "autonomy" is that the same transition mechanism is valid for all t. Hence we propose the following

DEFINITION 2.2. A probability $h \in H$ has the autonomous germ-Markov property if there is a $K_o \in \mathcal{K}_o$ such that (a) the trace of $\mathcal{K}_o$ on K_o coincides with the σ-field generated by $\{z(S); S \in \mathcal{F}'(0, 0+)\}$ on K_o, and (b) $P^h\{Z_t^h \in K_o$ for all $t > 0\} = 1$.

REMARK. By reinterpreting P^h as a measure on the paths of Z_t, we can omit the superscript in Z_t^h. In view of Theorem 2.1(a) it is not implausible that existence of such a K_o might follow from the weaker hypothesis that, for a different K_o, (b) holds only for a countable dense set of t. But this seems hard to prove. The reason that $t = 0$ is excluded in (b) is that we consider h as an entrance law for the process, in the same sense as for ordinary Markov processes.

A more serious issue is the absence in Definition 2.2 of any assumption on Z_{t-}^h. Indeed, since only the germs $G(t-) = \cap_{\delta>0}\mathcal{F}^o(t-\delta, t)$ are observable in a usual scientific sense, it might seem preferable methodologically to replace Z_t^h in (b) by Z_{t-}^h. However, this turns out to be the wrong approach (it eliminates the possibility of branching points). Fortunately, we recover the process Z_{t-}^h without further assumptions in Theorem 2.2(d) below.

For a first type of example, suppose that h is induced by a realization $X(t)$ of a right-continuous Markov process (in the sense of Dynkin [4]) on a Lusin space $(E, \mathcal{E})$, relative to its generated σ-fields $\mathcal{F}^o(t+)$, and with homogeneous transition function $p(s, x, A)$. Then h is autonomous germ-Markov. In fact, we can use as K_o the image of E in H under the mapping from x to the future $h(x) \in H$ induced by $p(\cdot, x, \cdot)$ and its iterates, namely $h(x)(S') = P^x\{(Y_n) \in S'\}$, as in Section 1. Condition (a) then follows because x is a measurable function of the values of $h(x)$ on $\mathcal{F}'(0, 0+)$. Indeed, $h(x)$ assigns probability 1 to the sequences $(y_n(t))$ with $(d^+/dt^+)y_n(0) = h_n(x)$ (we are assuming without loss of generality that the h_n of Section 1 are chosen continuous). Then for $L \in \mathcal{K}_0$ there is an $E_1 \in \mathcal{E}$ such that if $B_\infty = \{(h_n(x)) : x \in E_1\}$ we have

$$\begin{aligned} K_0 \cap L &= K_0 \cap \{h(x) : x \in E_1\} \\ &= K_0 \cap \{h(x) : (h_n(x)) \in B_\infty\} \\ &= K_0 \cap \left\{z \in H_0 : z\left\{\left(\frac{d^+}{dt^+}y_n(0)\right) \in B_\infty\right\} = 1\right\}. \end{aligned}$$

This expresses $K_0 \cap L$ as generated by

$$z(S),\ S = \{((d^+/dt^+)y_n(0)) \in B_\infty\} \in \mathcal{F}(0, 0+),$$

as required. Property (b) is obvious in the present case.

The essential result concerning Definition 2.2 is that, given h and K_o, Z_t^h is simply a realization of a Ray process on a certain subset of H. But before turning to this, we draw a simpler consequence as follows.

THEOREM 2.3. *Let T be an $\mathcal{F}'(0, t+)$-stopping time, and let $G+(T) = \bigcap_{\varepsilon>0}\mathcal{F}'(T, T+\varepsilon)$ where $\mathcal{F}'(T, T+\varepsilon)$ is generated by $(y_n(T+s) - y_n(T), 0 < s < \varepsilon)$, $1 \leqslant n$. Then any h satisfying Definition* 2.2 *has the* strong germ-Markov property:

$$P^h(i_T^{-1}(S)|\mathcal{F}'(0, T+)) = P^h(i_T^{-1}(S)|G^+(T))$$
$$= Z_T^h(S),\ h\text{-a.s. } on\ \{T < \infty\}.$$

PROOF. The first and third terms are equal by (1.1), hence we have only to prove the second equality. Replacing T by $T \wedge t$, we may assume that $P^h\{T < \infty\} = 1$. We want to show that $Z_T^h(S)$ is measurable over the completion of $G^+(T)$ for h. Over $\{T = 0\} G^+(T)$ reduces to $\mathcal{F}'(0, 0+)$, hence there is no difficulty. Over $\{T > 0\}$ we have $Z_T^h \in K_o$, h-a.s., and hence can express Z_T^h as a measurable function of $Z_T^h(S_n)$, $1 \leqslant n$, for a sequence $S_n \in \mathcal{F}'(0, 0+)$, outside an h-null set. Then over $\{T > 0\}$ we have

$$Z_T^h(S_n) = P^h(i_T^{-1}S_n|\mathcal{F}'(0, T+)$$
$$= I_{i_T^{-1}S_n}, \qquad h\text{-a.s.} \quad \text{for all } n,$$

since $i_T^{-1}(S_n) \in G^+(T) \subset \mathcal{F}'(0, T+)$. This shows that $Z_T^h(S_n)$ is measurable over $G^+(T)$ completed for h. Hence the same holds for $Z_T^h(S)$, as required.

We turn now to our main structure theorem for autonomous germ-Markov processes. The intuitive meaning will be mentioned following the proof. For the definition and properties of Ray processes we refer to [5]. Note that by Theorem 1.1 the condition $h_o \in H_o$ is not restrictive.

THEOREM 2.4. *Let $h_o \in H_o$ and $K_o \in \mathcal{K}_o$ satisfy Definition* 2.2. *Then there is a universally measurable set $K \subset H_o$ such that*:

(a) *for every $h \in K$, $P^h\{Z_t \in K$ for all $t \geqslant 0\} = 1$; moreover $h_o \in K$.*

(b) *Z_t is a right process on K.*

(c) *every probability on K is resoluble on Ω' (see Definition* 1.1) *by a resultant measure $h \in H$ having the autonomous germ-Markov property determined by K_o.*

(d) *there is a one-to-one continuous mapping $h(z)$ of the Ray space R_K of K onto a subset of the resultant measures of* (c) *which preserves Z_t.*

PROOF. We begin by setting

$$K = \{z \in H_o : P^z\{Z_t \in K_o \text{ for all } t > 0\} = 1\}.$$

In the terminology of [5], Section 12, we have $K = \{z \in H_o : P^\alpha_{H_o - K_o}1(z) = 0\}$ for

$\alpha > 0$, in which $T_{H_o - K_o}$ is the hitting time of $H_o - K_o$ and $P^{\alpha}_{H_o - K_o} f(z) = E^z\{e^{-\alpha T} f(Z_T); T < \infty\}$ for $T = T_{H_o - K_o}$. Since 1 is α-excessive, we know that $P^{\alpha}_{H_o - K_o} 1$ is α-excessive. By definition of a right process (see [5], pages 53 and 79) this implies that $\{z \in H_o : P^{\alpha}_{H_o - K_o} 1(z) = 0\}$ is "well-measurable," in such a way that for each $z \in K$, $I_K(Z_t)$ is P^z-indistinguishable from a well-measurable process of the σ-fields of Z_t completed for P^z. Then for $z \in K$, *unless* $P^z\{Z_t \in K$ for all $t \geq 0\} = 1$, we can apply the optional section theorem [2, IV, 84] to obtain a stopping time T of these σ-fields with $P^z\{Z_T \notin K; T < \infty\} > 0$. But then, by the strong Markov property, we would have

$$\begin{aligned} 1 &> E^z\{P^{Z_T}\{Z_t \in K_o \text{ for all } t > 0\}\} \\ &= P^z\{Z_t \in K_o \text{ for all } t > T\}, \end{aligned}$$

contradicting the definition of K.

Again by [5], page 79, K is a U-space in H_o, hence we can define $q(t, x, A \cap K)$ for $A \in \mathcal{H}_o$. It follows from the above that for each initial distribution v on K, there is a Markov process Z^v on K with transition function q. Moreover, we may assume that this process has right-continuous paths, so that the first axiom $HD\,1$ of right processes [5, Section 9] is satisfied. To see that $HD\,2$ is also satisfied (namely, that excessive functions are right-continuous along the paths), we observe that if f_K is α-excessive for the resolvent on K, then setting

$$\begin{aligned} f(x) &= f_K(x) \text{ for } x \in K, \\ &= \infty \text{ for } x \notin K, \end{aligned}$$

we obtain an α-supermedian function whose α-excessive regularization [5, Section 2] coincides with f_K on K. Thus $HD\,2$ on K follows from Theorem 1.1, and we have proved (b).

Turning to (c), let t_n be a decreasing sequence with limit o, and let v be a probability on K. Since K is universally measurable, we may as well consider v concentrated on an $\mathcal{H}_o$-subset of K. Let μ_n denote the P^v-distribution of Z_{t_n}. By definition of K, μ_n is concentrated on K_o. Now by Definition 2.2 and the fact that $\mathcal{H}_o$ is countably generated, there is a sequence $S_n \in \mathcal{F}'(0, 0+)$ such that $\{z(S_n), n \geq 1\}$ generates on K_o the trace of $\mathcal{H}_o$. We will show that μ_n is resoluble on Ω'. Referring to Definition 1.3, we define an Ω'-consistent function $g : \Omega' \to H_o$ as follows:

$$\begin{aligned} g(w') &= z \text{ if } z \in K_o \text{ and } z(S_n) = I_{S_n}(w') \text{ for all } n \\ &= \delta_{w''} \text{ if there is no such } z, \end{aligned}$$

where $\delta_{w''}$ is the unit mass concentrated at a fixed point w'' not covered for any z under the first case (if any).

Since the set of w' covered under the first case has the form

$$\lim_{N \to \infty} \{w' : \text{for some } z \in K_o,\ z(S_n) = I_{S_n}(w'),\ 1 \leq n \leq N\}$$

where there are at most 2^N nonvacuous possibilities (0 or 1's) for $z(S_n)$, $1 \leq n \leq N$, we see that the set covered by the first case is in $\mathcal{F}'(0, 0+)$, and from this it is easy to see that g is $\mathcal{F}(0, 0+)/\mathcal{K}_o$ measurable. To obtain the Ω'-consistency it is easiest to observe that each $z \in K_o$ concentrates its measure on a unique atom of the σ-field generated by the S_n, and w' is in that atom if and only if $g(w') = z$. Thus $g(w'; S') = I_{S'}(w')$ is clear for w' in the union over K_o of these atoms, while in the complementary set we have $g(w'; S') = \delta_{w''}(S') = I_{S'}(w'') = I_{S'}(w')$.

Introducing, now, the probability

$$h_n(S') = \int_K \int_{\Omega'} g(w'; S') z(dw') \mu_n(dz)$$
$$= \int_K z(S') \mu_n(dz), \qquad S' \in \mathcal{F}',$$

we have

$$P^{h_n}(S' | \mathcal{F}'(0, 0+)) = g(w'; S')$$
$$= Z_0^{h_n}(S'), \qquad h_n\text{-a.s.}$$

Hence μ_n is resoluble on Ω', with resultant measure h_n. Clearly, h_n is germ-Markhov.

Recalling that μ_n is the P^v-distribution of Z_{t_n}, we consider the measures h'_n on $i_{t_n}^{-1}(\mathcal{F}')$ given by $h'_n(i_{t_n}^{-1}S) = h_n(S)$. It will be shown that the measures h'_n are consistent, in the sense needed for the Kolmogorov extension theorem. This does not depend on any germ-Markov property, but only on the fact that Z_t has q as transition function. Thus for $n > m$ and $S \in \mathcal{F}'$ we have

$$h'_m\left(i_{t_m}^{-1}S\right) = h_m(S)$$
$$= E^{h_m}\left[E^{h_m}(S | \mathcal{F}'(0, 0+))\right]$$
$$= E^{h_m} Z_0^{h_m}(S)$$
$$= E^v \int q\left(t_m - t_n, Z_{t_n}, dz\right) z(S)$$
$$= E^{h_n}\left[E^{h_n}\left(Z_{t_m - t_n}^{h_n}(S) | \mathcal{F}'(0, 0+)\right)\right]$$
$$= E^{h_n}\left[E^{h_n}\left(i_{t_m - t_n}^{-1}(S) | \mathcal{F}'(0, 0+)\right)\right]$$
$$= h_n\left(i_{t_m - t_n}^{-1}(S)\right)$$
$$= h'_n\left(i_{t_m}^{-1}(S)\right).$$

From this consistency, by extension of the measures there is a unique $h \in H$ reducing to h'_n on $i_{t_n}^{-1}(\mathcal{F}')$ for each n. Then by martingale convergence and the germ-Markov property of the h_m, for $o < t_n < t$ we have

$$Z_t^h(S) = E^h\left(i_t^{-1}S | \mathcal{F}'(0, t+)\right)$$
$$= \lim_{m\to\infty} E^h\left(i_t^{-1}S | \mathcal{F}'(t_m, t+)\right)$$
$$= \lim_{m\to\infty} E^{h'_m}\left(i_t^{-1}S | \mathcal{F}'(t_m, t+)\right)$$
$$= E^{h'_m}\left(i_t^{-1}S | G^+(t)\right)$$
$$= E^h\left(i_t^{-1}S | G^+(t)\right).$$

The last equality shows that Z_t^h has distribution $\int_{K_o} q(t - t_n, z, A)\, \mu_n(dz)$, which is the distribution of Z_t for P^v. As $t \to 0+$, this determines the P^v-distribution of Z_0 by right continuity. Hence h is the resultant measure for v. Of course, we have $P^v\{Z_0 \in K\} = 1$ since $K \subset H_o$, hence by definition of K we see that Z_t^h has the autonomous germ-Markov property determined by K_o.

It remains to prove (d). Perhaps the chief advantage of the Ray space R_K is that it does not really depend on K being embedded in H, as shown in [5], Section 15. We review quickly the basic ingredients of the "Ray-Knight compactification" of K, and of the Ray space R_K. First of all, we form the minimal convex cone of bounded positive functions on K containing $U^\alpha C^+$ for $\alpha > 0$, and closed under the operations $U^\alpha f$ and $f \wedge g$, where C^+ denotes the restrictions to K of continuous nonnegative functions on H, and U^α denotes the resolvent of Z_t on K. Next, we form the compactification $\bar{K}$ of K in the uniformity determined by this cone: since there is a countable uniformly dense set, $\bar{K}$ is a compact metrizable space. We extend U^α to $\bar{K}$ by setting $\bar{U}^\alpha \bar{f} = \overline{U^\alpha f}$, where f is any element of the cone, extended to $\bar{f}$ on $\bar{K}$. Then $\bar{U}^\alpha(\bar{f} - \bar{g}) = \bar{U}^\alpha \bar{f} - \bar{U}^\alpha \bar{g}$ defines a Ray resolvent on $\bar{K}$ in the new (Ray) topology, and K is a U-space in $\bar{K}$. We define the Ray space of K by

$$R_K = \{z \in \bar{K} : \alpha \bar{U}^\alpha(z, K) = 1\}.$$

Then $K \subset R_K$, and further, if $\bar{P}_t$ denotes the transition function of the Ray process $\bar{Z}_t$ on $\bar{K}$ with resolvent $\bar{U}^\alpha$, then by [5, 15.6] we have $\bar{P}_t(z, K) = 1$ for all $z \in R_K$ and $t > 0$ (but not in general for $t = 0$). For $z \in K$, we have $\bar{P}_t(z, A) = q(t, z, A)$, in such a way that $\bar{Z}_t$ and Z_t coincide. (This is explained in detail in [5], Section 11; since the Ray topology does not coincide with the H-topology, we must restrict the path space to the intersection of the respective path spaces. However, the universally measurable sets in either topology coincide.) Finally, we remark that for $\bar{Z}_t$ the set $\bar{K} - R_K$ is "useless" (inutile) in the sense that for initial distributions v on R_K neither $\bar{Z}_t$ nor $\bar{Z}_{t-}$ ever reach $\bar{K} - R_K$, $\bar{P}^v$-a.s. (by [17], Theorem 10, Corollary 2). Thus we can consider $\bar{Z}_t$ as a process on R_K.

Returning to the proof of (d), let us first make precise the sense in which the asserted mapping $h(z)$ preserves Z_t. By this we mean that for every $z \in R_K$ the process $\bar{Z}_t$ is actually a process on K for $t \geqslant 0$, and that as such it has the same joint distributions as Z_t^h for the corresponding $h = h(z)$. In particular, the points of $R_K - K$ are all branching points of $\bar{Z}_t$. To see this, note again that for $z \in R_K$ since $\alpha \bar{U}^\alpha(z, K) = 1$ there are $0 < t_n \downarrow 0$ such that $\bar{P}^z\{\bar{Z}_{t_n} \in K\} = 1$. Then $\bar{P}^z\{\bar{Z}_t \in K \text{ for all } t > 0\} = 1$, and further, we can proceed exactly as for (c) to construct the resultant measure $h(z)$. It is immediate from the definition of K that the right limits Z_0^h are in K, h-a.s. Hence if we show that the function $h(z)$ is one-to-one and continuous on R_K, since $h(z) = z$ on K we will have $\bar{Z}_0 \equiv Z_0^h \in K$, hence Z_t is preserved by $h(z)$.

To this effect, we first note that $h(z)$ is one-to-one since (as there are clearly no "degenerate branching points" [17]) processes equivalent for $t > 0$ have the same z. Suppose now that $z_n \to z$ in R_K, with $z_n \in K$. Then by definition of the Ray topology, we have for $f \in C^+$

$$\lim_{n\to\infty} U^\alpha f(z_n) = \overline{U^\alpha f}(z).$$

In particular, let $f(z) = E^z g$ for $0 \leqslant g$ continuous on Ω'. Then

$$U^\alpha f(z_n) = E^{z_n} \int_0^\infty e^{-\alpha s} E^{Z_s^{z_n}} g \, ds$$
$$= E^{z_n} \int_0^\infty e^{-\alpha s} g o i_s \, ds,$$

by definition of $Z_s^{z_n}$ as a conditional probability given $\mathcal{F}'(0, s+)$. But, as shown in Lemma 1.3.2 of [9], convergence of these $U^\alpha f(z_n)$ for all $\alpha > 0$ and g is equivalent with convergence at $s = 0$ alone: i.e., of $E^{z_n} g$ (here one uses the uniform continuity of goi_s in s, uniformly on Ω'). Hence the measures z_n converge in H to a unique limit h. If $z \in K$, then

$$\overline{U^\alpha f}(z) = U^\alpha f(z)$$
$$= E^h \int_0^\infty e^{-\alpha s} g o i_s \, ds.$$

Thus $z = h = h(z)$, and so $h(z)$ is continuous on K (in other words, the Ray topology is stronger than the H-topology on K). If $z \in R_K - K$, then

$$\overline{U^\alpha f}(z) = \bar{E}^z \int_0^\infty e^{-\alpha s} E^{\bar{Z}_s} g \, ds$$
$$= E^{h(z)} \int_0^\infty e^{-\alpha s} E^{Z_s^{h(z)}} g \, ds$$
$$= E^{h(z)} \int_0^\infty e^{-\alpha s} g o i_s \, ds,$$

since $\bar{Z}_s \in K$ for $s > 0$. Comparison of this with $U^\alpha f(z_n)$ shows that $h = h(z)$. To complete the proof of continuity, it only remains to observe that since K is dense in R_K, whenever $z_n \to z$ in R_K we can choose $z_n' \in K$ with $z_n' \to z$ and the H-metric $d_H(z_n, z_n') < n^{-1}$. Then $h(z_n') \to h(z)$ implies that $h(z_n) \to h(z)$, completing the proof of (d).

REMARK. The content of Theorem 2.4 is essentially that an autonomous germ-Markov process carries with it a space K on which we have a corresponding process in the sense of Dynkin, complete with Borel transition function q. Moreover, by Theorem 2.2(d) the process is right-continuous in a natural topology, with left limits identified either as points of K or as branching points into K (i.e., as probabilities on K). The "true state space" can be viewed as the set of atoms of the trace of $\mathcal{H}$ on $K \cap K_0$. Since this trace is countably generated while also generated by $\{z(S), S \in \mathcal{F}'(0, 0+)\}$, observations reduce to checking a countable number of 0 or 1 germ probabilities, i.e., a countable number of corresponding germ sets. Realistically speaking, of course, these germs are not observable, being in the future. However, at any previsible stopping time $T > 0$ one can observe $Z_{T-}^h = \lim_{n\to\infty} Z_{T_n}^h$, where T_n increase to T, which still involves only countably many fixed sets (and hence can be finitely approximated). Then one can rely on the moderate Markov property of (1.1)(b). In the case of simple Markov processes, these atoms are identified with points of the state space by the correspondence of points with futures, hence of infinitesimal futures with futures, generated by the given transition function. It is worth noting here that this method can be used to improve a transition function, under essentially the same hypotheses as [16]. In short, we do not require that the semigroup property hold for every x in order to define the above correspondence for a particular process $X(t)$, in the sense of giving conditional futures for every t. An "X-polar" exceptional set can be permitted. Since $q(t, x, A)$ satisfies the semigroup property identically, it then can be used to generate a complete Markov process consistent with $X(t)$.

3. Examples of autonomous germ-Markov processes. The present section is not intended as a justification of the previous one, and in fact it is quite separate. We will describe certain types of examples which seem intriguing, but we will not use the representation space $(\Omega', \mathfrak{F}', P')$. This is because our examples are either Gaussian or have right-continuous paths. In such cases it is easiest just to describe the processes $X(t)$ which induce the measures h of Section 2, and to indicate the set of X-futures whose image on $(\Omega', \mathfrak{F}')$ is to be the set K_o of Definition 2.2. We will classify five types of examples, as follows:

1. Germ-deterministic processes.
2. Gaussian processes.
3. Examples based on Poisson processes.
4. Examples based on Brownian motion.
5. Vector-valued examples.

Of course, this is not presented as a deep or all-inclusive classification, and in some cases the categories overlap. We might call attention to type 3, in particular, which seems to merit further study.

3.1. We call process $X(t)$ germ-deterministic if the entire past and future are determined by the germ $G(t)$ at any time. Two types of such processes are as follows.

(a) Random entire functions

$$X(t) = \Sigma_{k=0}^{\infty} \xi_k t^k,$$

where the ξ_k are independent random variables subject to growth conditions. For example, it suffices that $P\{|\xi_k| > (k!)^{-1} \text{ infinitely often}\} = 0$. The autonomy property, and the identification of a K_o in terms of bounds on the derivatives of $X(t)$ (such as $|(d^k/dt^k)f(t)| \leqslant c^k$, $1 \leqslant k$, for some c), are evident.

(b) Random Fourier representations

$$X(t) = \int_a^b e^{i\lambda t} dZ(\lambda), \qquad -\infty < a < b < \infty,$$

where $Z(\lambda)$ is a process with (complex) orthogonal increments and we assume strict stationarity of $X(t)$. Here again $Z(\lambda)$, and hence X, is determined by the derivatives $(d^n/dt^n)X(t) = \int_a^b e^{i\lambda t}(i\lambda)^n dZ(\lambda)$ at a fixed t, in view of the finiteness of $b - a$ and a simple polynomial approximation argument. In the Gaussian case, a more general condition for germ-determinism was given by Levinson and McKean ([11], 8).

Here, the future can be determined by the time-stationary germ quantities ($\int_a^r e^{i\lambda t} dZ(\lambda)$, $a < r < b$, r rational).

3.2. The case of stationary Gaussian germ-Markov processes has been studied in detail by Levinson and McKean [11]. In the "Hardy case" (of imperfect prediction) the necessary and sufficient condition for the germ-Markov property is that h^{-1} be an integral function of minimal exponential type [3] or [11], where h is the outer function for which $|h|^2$ gives the spectral density. There is no question here of going into the details of this deep investigation, which is referenced in [3]. One comment which may be useful concerns the fact that the definition of a germ in [3] appears different than ours in that it is two-sided in time. But in view of [11, 6a] the two definitions are equivalent for stationary Gaussian processes. It should be noted that the Gaussian germs are defined in Hilbert space terms, but (see [12], for example) they reduce to our definition by σ-fields, except for sets of probability zero. In the Gaussian case, construction of our set K_o is in principle quite easy, since the future is obtained by adding to the projection on the germ a fixed independent Gaussian process.

3.3. A potentially quite rich class of examples comes about by using the discreteness of Poisson jumps to bring in a random element without obscuring an underlying continuity. For example, $X(t) = P(t) + tP_o$, where $P(t)$ is a Poisson process and P_o a fixed independent random variable, is of course a germ-Markov process, since $(d^+/dt^+)X(t) = P_o$ is in the germ. We can use the jumps in a more constructive way by considering a "zig-zag" process $X(t)$ which moves at velocity $+v$ in (t_{2n-1}, t_{2n}) and at velocity $-v$ in (t_{2n}, t_{2n+1}), where $0(= t_0) < t_1 < \cdots$ are the jumps times of a Poisson process of intensity λ, and v is a constant. The germ-Markov property is again clear. Let us imagine, moreover, an independent sequence X_n of such processes, with velocities v_n and Poisson intensities $\lambda_n \to \infty$. Posing $Y(t) = \sum_{n=1}^{\infty} X_n(t)$, the "turning points" are now everywhere dense, while it is easy to ensure uniform convergence in t by choosing λ_n large or v_n small. For instance, if $\lambda_n = 2^{3n}$ and $v_n = 2^n$, then routine arguments based on the independent, mean zero differences $X_n(t_{2(n+1)}) - X_n(t_{2n})$ and Chebyshev's inequality, yield such a convergence (one uses the fact that for $\varepsilon > 0$

$$P\{\max_{s \leqslant t} |X_n(s)| > \varepsilon\} \leqslant 2P\{|X_n(t)| > \varepsilon\},$$

as is clear since the maximum must occur either at a turning point or else at t). It is intuitively evident that, given the path of $Y(s)$, $t < s < t + \varepsilon$, one can read off the turning points of all the processes $X_n(s)$ which occur in that interval. But the slopes and values at these points determine $X_n(t)$ itself, whence one can determine $X_n(t + \varepsilon) - X_n(t)$ for all but finitely many X_n's (namely, for those which have turning points in $(t, t + \varepsilon)$). But then the remainder has constant velocity in $(t, t + \varepsilon)$. Assuming that a finite sum $\sum_{k=1}^{n} (\pm 1) v_k$ determines the n choices of sign uniquely, $Y(t)$ is an autonomous germ-Markov process.

This partially heuristic argument is easy to justify rigorously if we make one more "improvement" in the process. We have only to add to each X_n jump discontinuities of small size $\pm\delta_n$ at each turning point $t_{k,n}$, using $+\delta_n$ or $-\delta_n$ according to the sign of $(d^+/dt^+)X_n(t_{k,n})$. Assuming that $\Sigma_{k=n+1}^{\infty}\delta_k < \delta_n$, $1 \leqslant n$, it is clear inductively that all of these jumps and jump times are measurable functions of $Y(t) = \Sigma_{n=1}^{\infty}X_n(t)$. Since they determine the turning points, we do not need to worry further about the nondifferentiability of Y (which was already present in the first construction).

A precise definition of a set K_o for which Y satisfies Definition 2.2 is easy, since the germ at time t determines the right velocities $\pm v_n$ of all of the component processes, which together with $Y(t)$ itself determine the conditional futures.

3.4. Turning to examples based on the Brownian motion $B(t)$, we remark first that if $Y(t)$ is a stationary solution of stable n^{th} order linear equation with constant coefficients

$$\Sigma_{k=0}^{n}a_k\frac{d^{n-k}}{dt^{n-k}}Y(t) = \frac{d}{dt}B(t),$$

then $X(t) = \int_0^t Y(s)\,ds$ has the autonomous germ-Markov property. This is clear by writing

$$\Sigma_{k=0}^{n}a_k\frac{d^{n-k}}{ds^{n-k}}(X(t+s) - X(t)) = B(t+s) - B(t),$$

and appealing to uniqueness of the solution. Actually, of course, these are also Gaussian processes and could be included in 3.2 above, but the reader will find a separate treatment for example in Chapter 6 of [6]. The displacement of a particle starting at 0 and determined by an Ornstein-Uhlenbeck velocity process results from the case $n = 2$. Here it happens that $X' = Y$ is even an ordinary Markov process.

Construction of other germ-Markov processes from $B(t)$ is quite different than from $P(t)$ in 3.3, due to the fact that $B(t)$ plus a "random drift" is absolutely continuous with respect to $B(t)$. Hence the drift cannot be identified in the germ. However, other possibilities open up in terms of random scale factors or random time changes. Thus a process

$$X(t) = YB(t), \qquad \text{or } X(t) = B(tY^2),$$

where $Y > 0$ is a fixed random variable independent of B, is an autonomous germ-Markov process. This can be seen, for example, by applying Lévy's definition of the quadratic variation to prove for $\varepsilon > o$ that

$$Y = \lim_{n\to\infty}\varepsilon^{-1}\left(\Sigma_{k=1}^{n}\left(X\left(t + \varepsilon\frac{k}{n}\right) - X\left(t + \varepsilon\left(\frac{k-1}{n}\right)\right)\right)^2\right)^{\frac{1}{2}}$$

a.s., showing that Y is in the germ at t. Similar considerations apply, for example, to

$$X(t) = B_1\left(\int_0^t |B_2(s)|ds\right),$$

where B_1 and B_2 are independent Brownian motions. Since $\int_t^{t+\varepsilon}|B_2(s)|ds$ is in the (completed) field generated by X from t to $t + \varepsilon$, $|B_2(t)|$ itself is in the germ, and the germ-Markov property follows for $X(t)$.

A somewhat more elaborate example is furnished by the stochastic integral equation

$$X(t) = x_0 + \int_0^t\left(\int_0^s X(\tau)d\tau\right)dB(s); \qquad x_0 \neq 0,$$

where the outer integral is in the sense of Itô (see [13]). We may show by successive approximations that this equation has a unique nonanticipative solution $X(t)$. We set $X_0(t) = x_0$, and define inductively

$$X_{n+1}(t) = x_o + \int_0^t\left(\int_0^s X_n(\tau)d\tau\right)dB(s), \qquad n \geqslant 0.$$

Then one has for $n \geqslant 1$,

$$\begin{aligned} E(X_{n+1}(t) - X_n(t))^2 &= \int_0^t E\left(\int_0^s (X_n(\tau) - X_{n-1}(\tau)d\tau\right)^2 ds \\ &\leqslant \int_0^t sE\int_0^s (X_n(\tau) - X_{n-1}(\tau))^2 d\tau ds \\ &\leqslant \int_0^t s^2 \max_{\tau \leqslant s} E(X_n(\tau) - X_{n-1}(\tau))^2 ds. \end{aligned}$$

Also we have $E(X_1(s) - x_0)^2 = \frac{1}{3}x_0^2 s^3$, hence if we assume that

$$E\left(X_n(s) - X_{n-1}(s)\right)^2 \leqslant (n!)^{-1} x_0^2 3^{-n} s^{3n},$$

the induction step is valid, and this inequality holds for all n. Since each X_n is a continuous martingale, the classical L^2-martingale inequality of Doob [2, VII, Theorem 3.4] yields for $m < n$

$$E^{\frac{1}{2}}\left(\max_{s \leqslant t}(X_n(s) - X_m(s))^2\right) \leqslant \Sigma_{k=m+1}^n \frac{2x_0 3^{-k/2} t^{3k/2}}{(k!)^{\frac{1}{2}}}.$$

Since the sum on the right from $m + 1$ to $n = \infty$ is even square summable over m, it follows by Chebyshev's inequality and the first Borel-Cantelli lemma that $\lim_{n\to\infty} X_n(s) = X(s)$ exists with probability 1, uniformly in $(0, t)$ for every t, and satisfies the defining equation. Moreover, the difference Y of two solutions would satisfy the equation

$$Y(t) = \int_0^t\left(\int_0^s Y(\tau)d\tau\right)dB(s),$$

from which as above

$$EY^2(t) \leqslant \frac{t^3}{3}\max_{s \leqslant t} EY^2(s),$$

implying the uniqueness $Y \equiv 0$.

It is not hard to see that $X(t)$ is not a Markov process, for knowledge of $X(s)$, $s \leqslant t$, would determine $\int_0^t X(s)ds$, which is the "rate" of the process at time t. On the other hand, we can show that X is an autonomous germ-Markov process.

To see this, we write

$$X(t+t_1) - X(t) = \int_t^{t+t_1}\left(\int_0^t X(\tau)d\tau + \int_t^s X(\tau)d\tau\right)dB(s).$$

Forming the quadratic variation in $(t, t+\varepsilon)$ and letting $\varepsilon \to 0+$, we see that $|\int_0^t X(\tau)d\tau|$ is in the germ. But then $(d^+/dt^+)|\int_0^t X(\tau)d\tau| = X(t)$ (sign $\int_0^t X(\tau)d\tau$) is in the germ. Along with $X(t)$, this determines $\int_0^t X(\tau)d\tau$, and then the germ-Markov property is clear (one could also appeal to the symmetry of $dB(s)$ to avoid considering the signs).

3.5. If we permit processes of several components, the possibilities for a germ-Markov property evidently become comparatively unlimited. We will be content to present one nice example which was mentioned to us by D. L. Burkholder. Let B_1, B_2, and B_3 be independent Brownian motions, and consider the process $X = (B_1B_3, B_2B_3)$, as a vector process in the plane. If we form the quadratic variation in $(t, t+\varepsilon)$ of B_1B_3 we can obtain the limits in probability

$$\lim_{n\to\infty}\Sigma_{k=1}^n\left(B_1B_3\left(t+\frac{\varepsilon k}{n}\right) - B_1B_3\left(t+\varepsilon\frac{(k-1)}{n}\right)\right)^2$$

$$= \lim_{n\to\infty}\left[\Sigma_{k=1}^n B_3^2\left(t+\varepsilon\frac{k-1}{n}\right)\left(B_1\left(t+\frac{\varepsilon k}{n}\right) - B_1\left(t+\varepsilon\frac{k-1}{n}\right)\right)^2\right.$$

$$\left. + \Sigma_{k=1}^n B_1^2\left(t+\frac{\varepsilon k}{n}\right)\left(B_3\left(t+\frac{\varepsilon k}{n}\right) - B_3\left(t+\varepsilon\frac{(k-1)}{n}\right)\right)^2\right]$$

$$\sim \varepsilon\left(B_3^2(t) + B_1^2(t)\right).$$

Similarly, from the second component it follows that $B_3^2(t) + B_2^2(t)$ is in the germ, so that the difference $B_1^2(t) - B_2^2(t)$ is likewise. But since $B_1^2B_3^2$ and $B_2^2B_3^2$ are determined, so is $B_3^2(B_1^2 - B_2^2)$. Hence finally B_3^2 is in the germ, and we see that X has the autonomous germ-Markov property.

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